LECTURE 7

G ... a Lie group, $e \in G$ neutral element
$\mathfrak{g}$ ... the Lie algebra of G

Recall that $\exp : \mathfrak{g} \to G$ is a local diffeomorphism, so there is local smooth inverse as a map from G to $\mathfrak{g}$. We define log on a sufficiently small neighbourhood U of $e \in G$ by

$$\exp \circ \log = Id_G$$

We consider a map

$$\mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$$
$$X, Y \longmapsto \log(\exp(X) \cdot \exp(Y))$$

and observe:

- this map is locally around $(0,0) \in \mathfrak{g} \times \mathfrak{g}$ infinitely different. (because it is the composition of exp, "$\cdot$", log.)
- the value of $\log(\exp(-) \cdot \exp(-))$ at $(0,0)$ is 0.
- the tangent map is given by composition of tangent map of exponential, tangent map of the group law and ~~tangent~~ tangent map of log $\left(T_e(\log) = (T_0(\exp))^{-1} \right)$

$$T_{(0,0)} \log(\exp(X) \cdot \exp(Y)) = X + Y,$$ and so

$$\log(\exp(X) \cdot \exp(Y)) = X + Y + \rho(X,Y),$$ where

$$\rho(X,Y) = o(\|X\| + \|Y\|) \text{ for } (X,Y) \to (0,0).$$

($\| \ \|$ is any norm on $\mathfrak{g}$.)

<u>Question</u>: What is "?" in the formula

$$\exp(X) \cdot \exp(Y) = \exp(?)$$

As we have seen, $\mathrm{ad}(X)Y = [X, Y] = 0 \ \forall X, Y \in \mathfrak{g}$ in the case of G commutative (abelian.) Consequently, $\mathrm{ad}(X) = 0$ $\forall X \in \mathfrak{g}$, because abelian Lie groups have abelian Lie algebra.

For G connected, $\mathfrak{g}$ abelian $\Rightarrow$ G abelian, so the Lie bracket $[\ ,\ ]$ on $\mathfrak{g}$ measures the non-commutativity of G.

How to characterize it explicitly?

$X_e \in \mathfrak{g}$, X the left-invariant vector field on G, $X|_e = X_e$:

$$(Xf)(a) = X_a f = \underset{\overset{\shortparallel}{(L_a)_*}}{(dL_a)} X_e f = X_e(f \circ L_a) = \frac{d}{dt}\Big|_{t=0} f(a \exp(tX_e))$$

for $\forall f \in C^\infty(G)$, $a \in G$. Taking $a\exp(tX_e) \in G$ instead of $a \in G$, we get

$$(Xf)(a \exp(tX_e)) = \frac{d}{ds}\Big|_{s=0} f(a \exp(tX_e)\exp(sX_e))$$

$$= \frac{d}{ds}\Big|_{s=0} f(a \exp((t+s)X_e)) = \frac{d}{dt} f(a \exp(tX_e)).$$

By induction, $(X^k f)(a \exp(tX_e)) = \frac{d^k}{dt^k}\left(f(a \exp(tX_e))\right)$,

so for $t=0$ $$(X^k f)(a) = \frac{d^k}{dt^k}\Big|_{t=0} f(a \exp(tX_e)), \quad k \in \mathbb{N}.$$

There is a multivariable generalization: $X_{1e}, X_{2e}, \ldots, X_{ke} \in \mathfrak{g}$

$$(X_1 \ldots X_k f)(a) = \frac{\partial^k}{\partial t_1 \ldots \partial t_k}\Bigg|_{\substack{t_1=0\\ t_2=0\\ \vdots\\ t_k=0}} f(a \exp(t_1 X_{1e}) \ldots \exp(t_k X_{ke}))$$

As a consequence, we have Taylor-type expansion

<u>Lemma 1</u>: If $f \in C^\infty(G)$, then for $t \to 0$

$$f(\exp(tX_1)\dots\exp(tX_n)) = f(e) + t\sum_{i=1}^{n}(X_i f)(e) + \frac{t^2}{2}\Big(\sum_i (X_i^2 f)(e) + 2\sum_{i<j}(X_i X_j f)(e)\Big) + O(t^3).$$

This fomula works for any $f \in C^\infty(G, V)$, V a vector space.

<u>Theorem 2</u>: Let $n \in \mathbb{N}$ and $X_{1e}, \dots, X_{ne} \in \mathfrak{g}$. Then for all t sufficiently small ($t \to 0$),

$$\exp(tX_{1e})\cdot\ldots\cdot\exp(tX_{ne}) = \exp\Big(t\sum_{i=1}^{n}X_i + \frac{t^2}{2}\sum_{1\le i<j\le n}[X_{ie}, X_{je}] + O(t^3)\Big).$$

Pf: Recall from Lecture 6 (around Theorem 6) that $\log(\exp(tX)) = tX$ for $X \in \mathfrak{g}$ and t sufficiently small. Then $\log(e) = 0$,

$$\underbrace{(X\log)(e)}_{X_e f} = \frac{d}{dt}\Big|_{t=0}(\log\exp(tX_e)) = \frac{d}{dt}\Big|_{t=0}(tX_e) = X_e,$$

$$(X^n\log)(e) = \frac{d^n}{dt^n}\Big|_{t=0}(\log\exp(tX_e)) = \frac{d^n}{dt^n}\Big|_{t=0}(tX_e) = 0 \text{ for } n > 1.$$

Notice $\sum_i X_{ie}^2 + 2\sum_{i<j}X_{ie}X_{je} = (X_{1e} + \dots + X_{ne})^2 + \sum_{i<j}[X_{ie}, X_{je}]$,

which implies (via lemma 1)

$$\log(\exp(tX_{1e})\cdot\ldots\cdot\exp(tX_{ne})) = t\sum_i X_{ie} + \frac{t^2}{2}\sum_{i<j}[X_{ie}, X_{je}] + O(t^3).$$

Because $\exp(tX_{1e})\cdots\exp(tX_{ne}) = \exp(\log(\exp(tX_{1e})\dots\exp(tX_{ne})))$

for t sufficiently small, the proof follows. ∎

In particular, $\exp(tX)\cdot\exp(tY) = \exp(tX + tY + \frac{t^2}{2}[X,Y] + O(t^3))$

for t small, hence $[\ ,\ ]$ measures (first order) non-commutativity of group multiplication.

By Lecture 6 (around Theorem 6), $\log(\exp(X)\cdot\exp(Y))$ is well-defined and exists for X, Y close to $0 \in \mathfrak{g}$. What are the higher order contributions to $\log(\exp(X_e)\exp(Y_e))$?

<u>Theorem 3</u>: (Baker-Campbell-Hausdorff formula) For $X_e, Y_e \in \mathfrak{g}$ sufficiently small, $\log(\exp(X_e)\cdot\exp(Y_e)) = X_e + Y_e + \sum_{m\geq 2} P_m(X_e, Y_e)$, with $P_m(X_e, Y_e)$ a combination of iterated commutators (Lie brackets) in X_e, Y_e involving $m-1$ Lie brackets.

In fact, we prove the following explicit formula:

<u>Theorem 4</u> (Dynkin's formula) For $X_e, Y_e \in \mathfrak{g}$ sufficiently small,

$$\log(\exp(X_e)\cdot\exp(Y_e)) = X_e + Y_e + \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \sum_{\substack{l_1,\dots,l_k \geq 0 \\ m_1,\dots,m_k \geq 0 \\ l_i + m_i > 0\ \forall i}} \frac{(-1)^{\sum_i (l_i+m_i)}}{l_1 + \dots + l_k + 1} \frac{(\operatorname{ad} Y_e)^{l_1}}{l_1!} \circ \frac{(\operatorname{ad} X_e)^{m_1}}{m_1!} \circ \dots \circ \frac{(\operatorname{ad} Y_e)^{l_k}}{l_k!} \circ \frac{\operatorname{ad}(X_e)^{m_k}}{m_k!} (Y_e).$$

For small $m \in \{2, 3, \dots\}$

<u>$m=2$</u> $\quad P_2(X_e, Y_e) \overset{!}{=} \frac{1}{2}[X_e, Y_e]$

<u>$m=3$</u>

$$\left.\begin{array}{ll} k=1,\ l_1=1,\ m_1=1 & -\frac{1}{2}\frac{1}{2}[Y_e,[X_e,Y_e]] \\ k=1,\ l_1=1,\ m_1=2 & -\frac{1}{2}\frac{1}{2}[X_e,[X_e,Y_e]] \\ k=2,\ l_1=1,\ m_1=0,\ l_2=0,\ m_2=1 & \frac{1}{3}\cdot\frac{1}{2}[Y_e,[X_e,Y_e]] \\ k=2,\ l_1=0,\ m_1=0,\ l_2=0,\ m_2=1 & \frac{1}{3}[X_e,[X_e,Y_e]] \end{array}\right\} \begin{array}{l} P_3(X_e,Y_e) = \\ \frac{1}{12}[X_e,[X_e,Y_e]] \\ -\frac{1}{12}[Y_e,[X_e,Y_e]] \end{array}$$

Homework: Compute $P_4(X_e, Y_e)$!

(it is a mutiple of $[X_e,[Y_e,[Y_e,X_e]]]$)

The proof of Dynkin formula:

<u>Lemma 5</u>: $\forall X_e \in \mathfrak{g}$, $(d\exp)_{X_e} = (dL_{\exp X_e})_e \circ \varphi(\operatorname{ad} X_e)$,

where $\varphi(z) = \dfrac{1-e^{-z}}{z} = \displaystyle\sum_{m=0}^{\infty} \frac{(-1)^m}{(m+1)!} z^m$.

<u>Remark</u>: φ is entire fcion of z; $(d\exp)_{X_e}$ denotes tangent map at X_e of exp-map: $(d\exp)_{X_e}(Y_e) = \frac{d}{dt}\Big|_{t=0} \exp(X_e + tY_e)$,

$t \mapsto X_e + tY_e$ being affine curve in $\mathfrak{g}$.

<u>Proof of Dynkin formula</u>:

Consider the elements $Z(t) := \log(\exp(X_e)\cdot\exp(tY_e))$,

then the Lemma 5 gives

$$\underbrace{\frac{d}{dt}(\exp Z(t))}_{*} = (dL_{\exp(X_e)})\frac{d}{dt}(\exp(tY_e)) = dL_{\exp(X_e)}\, dL_{\exp(tY_e)}$$

$$\varphi(\operatorname{ad}(tY_e))\, Y_e = dL_{\exp(X_e)}\, dL_{\exp(tY_e)} = dL_{\exp(Z(t))}(Y_e),$$

Because $\varphi(\operatorname{ad}(tY_e))(Y_e) = Y_e$ $(\operatorname{ad}(tY_e)(Y_e) = 0.)$

On the other hand, again by Lemma 5 $(Z(t) \in \mathfrak{g})$

$$\frac{d}{dt}(\exp(Z(t)) = dL_{\exp(Z(t))}\, \varphi(\operatorname{ad}(Z(t)))\, \frac{dZ(t)}{dt},$$

and the substitution of $*$ on the left hand side gives

$$\frac{dZ(t)}{dt} = \frac{\operatorname{ad}(Z(t))}{Id - \exp(-\operatorname{ad}(Z(t)))} Y_e = \sum_{k\geq 0} \frac{1}{k+1}\left(Id - \exp(-\operatorname{ad}(Z(t)))\right)^k Y_e$$

By properties of ad, Ad, we have

$$\begin{aligned}\exp(-\operatorname{ad}(Z(t))) &= \operatorname{Ad}(\exp(-Z(t))) = \operatorname{Ad}(\exp(-tY)\exp(-X)) \\ &= \operatorname{Ad}(\exp(-tY)) \circ \operatorname{Ad}(\exp(-X)) \\ &= \exp(-t\operatorname{ad} Y)) \circ \exp(-\operatorname{ad} X),\end{aligned}$$

therefore

$$\frac{dZ(t)}{dt} = \sum_{k\geq 0} \frac{\left(\mathrm{Id} - \exp(-t\,\mathrm{ad}\,Y_e)\circ \exp(-\mathrm{ad}\,X_e)\right)^k}{k+1}(Y_e)$$

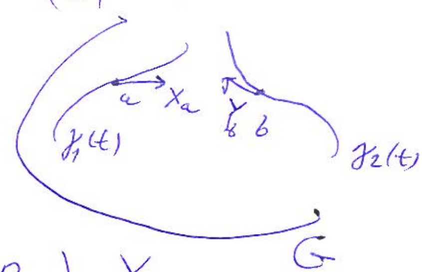
$$= \sum_{k\geq 0} \frac{(-1)^k}{k+1} \sum_{\substack{l_1,\dots,l_k\geq 0\\ m_1,\dots,m_k\geq 0\\ l_i+m_i>0}} t^{|l|}(-1)^{|l|+|m|} \frac{(\mathrm{ad}\,Y_e)^{l_1}}{l_1!}\circ\frac{(\mathrm{ad}\,X_e)^{m_1}}{m_1!}\circ\dots\frac{(\mathrm{ad}\,Y_e)^{l_k}}{l_k!}\frac{(\mathrm{ad}\,X_e)^{m_k}}{m_k!} Y_e\,,$$

where we used $\mathrm{ad}\,X\in \mathrm{End}(\mathfrak{g})$ so that the exponential map is exactly the matrix exponential. Now integrate the last formula over t on the interval $\langle 0,1\rangle$. $\boxtimes$

Recall that for G there is the group law $\mu : G\times G \to G$, $(a,b)\mapsto ab$.
It is easy to see the tangent map $d\mu$ to μ:

$\gamma_1 : t\to G$, $a\in G$, $a\in\gamma_1$, $\frac{d\gamma_1}{dt} = X$

$\gamma_2 : t\to G$, $b\in G$, $b\in\gamma_2$, $\frac{d\gamma_2}{dt} = Y$



$$d\mu_{(a,b)}(X_a, Y_b) = (dL_a)_b(Y_a) + (dR_b)_a X_b\,,$$

implies (or, is equivalent) to

$$\dot\gamma(t) = dL_{\gamma_1(t)}(\dot\gamma_2(t)) + dR_{\gamma_2(t)}(\dot\gamma_1(t))\,,\quad \gamma(t)=\gamma_1(t)\cdot\gamma_2(t).$$

$\dot\gamma(t) = \frac{d\gamma}{dt}(t)$ for $\gamma(t) = \gamma_1(t)\gamma_2(t)$.

Iteration gives for $\gamma(t) = \gamma_1(t)\cdot\dots\cdot\gamma_m(t)$, $\gamma_1,\dots,\gamma_m$ smooth curves in G

$$\dot\gamma(t) = \sum_{k=1}^{m} (dL_{\gamma_1(t)})\dots(dL_{\gamma_{k-1}(t)})(dR_{\gamma_{k+1}(t)})\dots(dR_{\gamma_m(t)})(\dot\gamma_k(t))$$

<u>Proof of Lemma 5:</u> We have $(d\exp)_{X_e}(Y_e) := \frac{d}{dt}\Big|_{t=0}\exp(X_e + tY_e)$,

and so we have to prove

$$\frac{d}{dt}\Big|_{t=0}\exp(X_e+tY_e) = (dL_{\exp(X)})_e\circ\varphi(\mathrm{ad}\,X)(Y).$$

This is a lengthy combinatorial calculation, based on the result from lecture 6 (Lie product formula): $e^{X_e+Y_e} = \lim_{m\to\infty}\left(e^{\frac{X_e}{m}}e^{\frac{Y_e}{m}}\right)^m$ for all $X_e, Y_e \in Mat(n\times n, \mathbb{C})$.

Exercises for LECTURE 7

Example: let $X = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \in Mat(2\times 2, \mathbb{C})$ $x\in\mathbb{C}$

$Y = \begin{pmatrix} y & 1 \\ 0 & 0 \end{pmatrix} \in Mat(2\times 2, \mathbb{C})$ $y\in\mathbb{C}$

Compute $\log(e^X\cdot e^Y)$, and find $a, b \in \mathbb{C}$ such that $\log(e^X\cdot e^Y) = aX + bY$!

Notice $Y^n = y^{n-1}Y$, so

$$e^Xe^Y = \begin{pmatrix} e^x & 0 \\ 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} e^y & \frac{e^y-1}{y} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{x+y} & \frac{e^x(e^y-1)}{y} \\ 0 & 1 \end{pmatrix},$$

and (for $Z = aX+bY$ holds $Z^n = (ax+by)^{n-1}Z$)

$$e^{aX+bY} = e^Z = \exp\begin{pmatrix} (ax+by) & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} e^{ax+by} & \frac{b(e^{ax+by}-1)}{ax+by} \\ 0 & 1 \end{pmatrix},$$

hence $x+y = ax+by$

$(ax+by)e^x(e^y-1) = by(e^{ax+by}-1)$

$$a = \frac{(x+y)(e^x-1)}{x(e^{x+y}-1)}, \quad b = \frac{(x+y)(e^y-1)e^x}{y(e^{x+y}-1)}$$

<u>Example</u>: In the proof of Dynkin formula, we needed the identity

$$\frac{ad(Z(t))}{Id - \exp(-ad\, Z(t))} = \sum_{k\geq 0} \frac{1}{k+1}\left(Id - \exp(-ad(Z(t)))\right)^k.$$

For example, consider a variable w and prove

$$\frac{w}{1-e^{-w}} = \sum_{k\geq 0} \frac{1}{k+1}\left(1-e^{-w}\right)^k.$$

<u>Example</u>: Consider basis of $sl(2)$ $(\mathbb{R}, \mathbb{C})$ of the form $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

(these generators fulfill matrix Lie algebra relations $[X, Y] = H$, $[H, X] = 2X$, $[H, Y] = -2Y$)

Then we can compute

$$\exp(X+Y)\exp(-\cancel{Y})\exp(-X) = \begin{pmatrix} \cosh 1 & \sinh 1 \\ \sinh 1 & \cosh 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 2e^{-1} & e-3e^{-1} \\ -2e^{-1} & e+3e^{-1} \end{pmatrix} =$$

$$= \frac{e+e^{-1}}{4}\mathbb{1}_{2\times 2} + \frac{1}{4}\begin{pmatrix} -(e+e^{-1}) & 2e-6e^{-1} \\ -4e^{-1} & e+e^{-1} \end{pmatrix}.$$

We can normalize (rescale) the second matrix to be D such that $D^2 = \mathbb{1}_{2\times 2}$; we have

$$\frac{1}{4}\begin{pmatrix} -(e+e^{-1}) & 2e-6e^{-1} \\ -4e^{-1} & e+e^{-1} \end{pmatrix} =: \frac{\sqrt{e^2+25e^{-2}-6}}{4} D$$

Define $\sinh\theta := \frac{\sqrt{e^2+25e^{-2}-6}}{4}$,

we easily verify the identity

$$\exp(A+B)\exp(-B)\exp(-A) = \exp(\theta D).$$

Example: Assume $A, B \in Mat(n\times n, \mathbb{C})$ such that $[A,B]=C$ and $[A,C]=0=[B,C]$. Then

$$\cos(A+B) = e^{\frac{1}{2}[A,B]}(\cos A\cos B - \sin A \sin B),$$

where $\cos X := \frac{e^{iX}+e^{-iX}}{2}$ (and similarly for $\sin X$)

$X \in Mat(n\times n, \mathbb{C})$

Example: (Another version of the previous example) Suppose $X, Y \in Mat(n\times n, \mathbb{C})$, such that $[X,[X,Y]] = [Y,[X,Y]] = 0$. Then $e^X \cdot e^Y = e^{X+Y+\frac{1}{2}[X,Y]}$

Example: Assume X, Y elements of a Lie algebra satisfy $[X,Y]=sY$, $s\in\mathbb{C}$. Show that

$$\log(e^X e^Y) = X + \frac{s}{1-e^{-s}}Y,$$

$$e^X \cdot e^Y \cdot e^{-X} = e^{e^s Y},$$

and discuss corresponding differential equations satisfied by $e^X e^{tY}$ and $e^{X + t\frac{s}{1-e^{-s}}Y}$.

We have $e^X e^Y e^{-X} = Ad_{e^X}(e^Y) = e^{ad(X)} e^Y$

$$= e^{ad(X)}\left(Id + Y + \frac{Y^2}{2} + \dots\right) = Id + e^s Y + e^{2s}\frac{Y^2}{2} + \dots = e^{e^s Y},$$

since $ad(X)$ acts on Y s, i.e. $ad(X)\, Y^n = ns\, Y^n$, $n \in \mathbb{N}$.

There is a matrix realization:

$$X = \begin{pmatrix} s & 0 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

We compute $e^X = \begin{pmatrix} e^s & 0 \\ 0 & 1 \end{pmatrix}$, $e^Y = Id + Y = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$,

$$e^X e^Y = \begin{pmatrix} e^s & e^s \\ 0 & 1 \end{pmatrix},$$

$$\exp\left(X + \frac{s}{1-e^{-s}} Y\right) = \exp\begin{pmatrix} s & se^s(e^s-1)^{-1} \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} s & se^s(e^s-1)^{-1} \\ 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} s^2 & s^2e^s(e^s-1)^{-1} \\ 0 & 0 \end{pmatrix} + \dots$$

$$= \begin{pmatrix} e^s & e^s \\ 0 & 1 \end{pmatrix}.$$

The basic differential equations: $W := e^X e^{tY}$

$\Rightarrow \frac{d}{dt} W = W \cdot Y$

On the other hand, $\tilde{W} = e^V = e^{\overbrace{X + tY\frac{s}{1-e^{-s}}}^{=V}}$,

$$\cancel{\frac{d}{dt}(\tilde{W})}\ \frac{d}{dt}\tilde{W} = \tilde{W}\,\frac{Id - e^{-ad(V)}}{ad(V)}\left(\frac{d}{dt}V\right) = \tilde{W}\,\frac{(Id - e^{-ad(X)})}{ad(X)}\,\frac{s}{1-e^{-s}}\,Y$$

$$= \tilde{W} Y, \text{ because } [V,Y] = [X,Y] = sY.$$