1 The Hopf Algebra $U(\mathfrak{sl}_2)$

In this lecture we introduce the basic definitions and results of Hopf algebra theory. Throughout, the classical enveloping algebra $U_q(\mathfrak{sl}_2)$ will be taken as the motivating example. Indeed, this example is sufficiently rich as to be able to demonstrate many of the fundamental properties of Hopf algebras.

1.1 Conventions and Basic Facts About Tensor Products

We begin with some basic conventions: All algebras discussed here will be assumed to be unital, and all algebra, and anti-algebra, maps will be assumed to be unital. For sake of simplicity, all vector spaces V will be assumed to be over $\mathbf{C}$. Unless stated otherwise, all tensor products are taken over $\mathbf{C}$, and we will tacitly identify $\mathbf{C} \otimes V$, $V \otimes \mathbf{C}$, and V. Recall that every element of $V \otimes V$ can (by construction) be written as a sum $\sum_{i=1}^{m} v_i \otimes w_i$, for some $m \in \mathbf{N}$. However, such presentations are not unique.

For an algebra A, the *algebra tensor product* $A \otimes A$ is the vector space tensor product of A with itself, endowed with the multiplication

$$(a \otimes b).(a' \otimes b') := (aa') \otimes (bb'), \qquad (a, a', b, b' \in A).$$

Note that $A \otimes A$ is again a unital algebra, with unit $1 \otimes 1$. The *flip map* for $A \otimes A$ is defined by

$$\tau : A \otimes A \to A \otimes A, \qquad \tau(a \otimes b) = b \otimes a.$$

If m is the multiplication of A, then the multiplication of $A \otimes A$ is given by the map $(m \otimes m) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id})$.

We will denote by $\mathbf{C}\langle x_i, \ldots, x_m \rangle$, the free unital associative noncommutative algebra generated by the symbols x_i, and denote its multiplication by m. For any two-sided ideal $I \subseteq \mathbf{C}\langle x_i, \ldots, x_m \rangle$, one can define a unique algebra (or anti-algebra) map

$$f : \mathbf{C}\langle x_i, \ldots, x_m \rangle / I \to B$$

for some algebra B, by specifying the action of f on each x_i, and then verifying that the map vanishes on I. In what follows we will make free use of this fact.

1.2 $U(\mathfrak{sl}_2)$ as a Bialgebra

Let us recall that

$$U(\mathfrak{sl}_2) = \mathbf{C}\langle E, F, H\rangle / I,$$

where I is the two-sided ideal of $\mathbf{C}\langle E, F, H\rangle$ generated by the elements

$$[E, F] - H, \qquad [H, E] - 2E, \qquad [H, F] + 2F.$$

Lemma 1.1 *There exist algebra maps*

$$\Delta : U(\mathfrak{sl}_2) \to U(\mathfrak{sl}_2) \otimes U(\mathfrak{sl}_2); \qquad \varepsilon : U(\mathfrak{sl}_2) \to \mathbf{C},$$

uniquely defined by

$$\Delta(Z) = 1 \otimes Z + Z \otimes 1, \qquad \varepsilon(Z) = 0, \qquad (\textit{for } Z \in \mathfrak{sl}_2).$$

Moreover, it holds that

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta, \qquad (\varepsilon \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \varepsilon) \circ \Delta = \mathrm{id}. \tag{1}$$

Proof. We first need to confirm that the map Δ is well-defined, by showing that it vanishes on I. For the element $[E, F]$, we calculate that

$$\begin{aligned}\Delta(EF) &= (1 \otimes E + E \otimes 1)(1 \otimes F + F \otimes 1)\\ &= 1 \otimes EF + F \otimes E + E \otimes F + EF \otimes 1,\end{aligned}$$

and similarly that

$$\Delta(FE) = 1 \otimes FE + E \otimes F + F \otimes E + FE \otimes 1.$$

This gives us that

$$\begin{aligned}\Delta([E, F] - H) &= \Delta(EF) - \Delta(FE) - \Delta(H)\\ &= 1 \otimes (EF - FE) + (EF - FE) \otimes 1 - (1 \otimes H + H \otimes 1)\\ &= 1 \otimes [E, F] + [E, F] \otimes 1 - (1 \otimes H + H \otimes 1)\\ &= 1 \otimes H + H \otimes 1 - (1 \otimes H + H \otimes 1)\\ &= 0.\end{aligned}$$

Similarly, it is easy to show that Δ vanishes on the elements $[H, E] - 2E$, and $[H, F] + 2F$. Hence, Δ vanishes on I and extends to a well-defined map on $U(\mathfrak{sl}_2)$. (Note that since ε obviously vanishes on I, we do not need to worry about proving that it is well-defined.)

Let us next consider the identity $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$. For any $Z \in \mathfrak{sl}_2$, we have equality between

$$\begin{aligned}(\Delta \otimes \mathrm{id}) \circ \Delta(Z) &= (\Delta \otimes \mathrm{id})(1 \otimes Z + Z \otimes 1) \\ &= 1 \otimes 1 \otimes Z + 1 \otimes Z \otimes 1 + Z \otimes 1 \otimes 1,\end{aligned}$$

and

$$\begin{aligned}(\mathrm{id} \otimes \Delta) \circ \Delta(Z) &= (\mathrm{id} \otimes \Delta)(1 \otimes Z + Z \otimes 1) \\ &= 1 \otimes 1 \otimes Z + 1 \otimes Z \otimes 1 + Z \otimes 1 \otimes 1.\end{aligned}$$

Moreover, since it is obvious that we also have equality for 1, we can see that the identity holds for all generators. The fact that Δ is an algebra map now implies that the identity holds for all elements of $U(\mathfrak{sl}_2)$.

Finally, we come to the identity $(\varepsilon \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \varepsilon) \circ \Delta = \mathrm{id}$: For any $Z \in \mathfrak{sl}_2$, we have equality between

$$\begin{aligned}(\varepsilon \otimes \mathrm{id}) \circ \Delta(Z) &= (\varepsilon \otimes \mathrm{id})(1 \otimes Z + Z \otimes 1) \\ &= \varepsilon(1)Z + \varepsilon(Z)1 = Z,\end{aligned}$$

and

$$\begin{aligned}(\mathrm{id} \otimes \varepsilon) \circ \Delta(Z) &= (\mathrm{id} \otimes \varepsilon)(1 \otimes Z + Z \otimes 1) \\ &= \varepsilon(Z)1 + \varepsilon(1)Z = Z.\end{aligned}$$

Moreover, since it is obvious that we also have equality for 1, we have that the identity holds for all generators, and hence for all elements of $U(\mathfrak{sl}_2)$. □

Excercise: For any Lie algebra $\mathfrak{g}$, with universal enveloping algebra $U(\mathfrak{g})$, show that the algebra maps

$$\Delta : U(\mathfrak{sl}_2) \to U(\mathfrak{sl}_2) \otimes U(\mathfrak{sl}_2); \qquad \varepsilon : U(\mathfrak{sl}_2) \to \mathbf{C},$$

uniquely defined by

$$\Delta(Z) = 1 \otimes Z + Z \otimes 1, \qquad \varepsilon(Z) = 0, \qquad (\text{for } Z \in \mathfrak{g}),$$

satisfy (1).

1.3 Coalgebras and Bialgebras

We shall now see that this structure is not an isolated example:

Definition 1.2. A *coalgebra* is a triple (C, Δ, ε), where C is a vector space, and

$$\Delta : C \to C \otimes C; \qquad\qquad \varepsilon : C \to \mathbf{C},$$

are linear maps (called the *coproduct* and *counit* respectively), satisfying the following axioms:

1. $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$, (*coassociativity axiom*);
2. $(\varepsilon \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \varepsilon) \circ \Delta = \mathrm{id}$, (*counit axiom*).

These two axioms are sometimes presented in the form of the commutative diagrams:

$$\begin{array}{ccc} V & \xrightarrow{\Delta} & V \otimes V \\ {\scriptstyle \Delta}\downarrow & & \downarrow{\scriptstyle \mathrm{id}\otimes\Delta} \\ V \otimes V & \xrightarrow[\Delta\otimes\mathrm{id}]{} & V \otimes V \otimes V. \end{array} \qquad \begin{array}{ccc} V & \xrightarrow{\Delta} & V \otimes V \\ {\scriptstyle \Delta}\downarrow & \searrow^{\mathrm{id}} & \downarrow{\scriptstyle \mathrm{id}\otimes\varepsilon} \\ V \otimes V & \xrightarrow[\varepsilon\otimes\mathrm{id}]{} & V. \end{array}$$

For $(D, \Delta', \varepsilon')$ a coalgebra , a *coalgebra morphism* $f : C \to D$ is a linear map for which

$$\Delta' \circ f = (f \otimes f) \circ \Delta, \qquad \text{and} \qquad \varepsilon = \varepsilon' \circ f.$$

As is evident, $(U(\mathfrak{sl}_2), \Delta, \varepsilon)$ is a coalgebra. In what follows, we will, by abuse of notation, usually denote the triple of a coalgebra (C, Δ, ε) by C.

Excercise Let C be a coalgebra, and c an element of C. Show that there exist presentations of $\Delta(c)$ of the form $1 \otimes c + \sum_{i=1}^{m} a_i \otimes b_i$, and of the form $c \otimes 1 + \sum a_i' \otimes b_i'$, where $a_i, b_i' \in \ker(\varepsilon)$.

We finish this section with the definition of a bialgebra. (Again, $U(\mathfrak{sl}_2)$ is our motivating example here.)

Definition 1.3. A coalgebra (A, Δ, ε) is called a bialgebra if A is an algebra, and Δ and ε are algebra maps with respect to the algebra tensor product $A \otimes A$.

A morphism between two bialgebras is simultaneously a coalgebra morphism and an algebra morphism.

1.4 $U(\mathfrak{sl}_2)$ as a Hopf algebra

In addition to the maps Δ and ε presented above, we have an equally important third map:

Lemma 1.4 *There exists an anti-algebra map* $S : U(\mathfrak{sl}_2) \to U(\mathfrak{sl}_2)$, *uniquely defined by*

$$S(Z) = -Z, \qquad\qquad (\textit{for } Z \in \mathfrak{sl}_2).$$

Moreover, it holds that

$$m \circ (S \otimes \mathrm{id}) \circ \Delta = m \circ (\mathrm{id} \otimes S) \circ \Delta = \varepsilon 1. \tag{2}$$

Proof. We begin by showing that S is well-defined. For $[E, F] - H$, we have

$$\begin{aligned} S([E,F] - H) &= S(F)S(E) - S(E)S(F) - S(H) \\ &= FE - EF + H = -[E,F] + H = 0. \end{aligned}$$

One can similarly show that S vanishes on $[H, E] + 2E$, and $[H, F] - 2F$. Hence, S vanishes on I, and as a result, has a well-defined anti-algebra extension to $U(\mathfrak{sl}_2)$. Moving on to the identity in (2), we see that since it clearly holds for 1, we shall only need to verify it on a general element of $\mathfrak{sl}_2$: this amounts to the calculations

$$\begin{aligned} m \circ (S \otimes \mathrm{id}) \circ \Delta(Z) &= m \circ (S \otimes \mathrm{id})(1 \otimes Z + Z \otimes 1) \\ &= Z - Z = 0 = \varepsilon(Z), \end{aligned}$$

and

$$\begin{aligned} m \circ (\mathrm{id} \otimes S) \circ \Delta(Z) &= m \circ (\mathrm{id} \otimes S)(1 \otimes Z + Z \otimes 1) \\ &= -Z + Z = 0 = \varepsilon(Z). \end{aligned}$$

□

Just as the properties of Δ and ε led us to the definition of a coalgebra and a bialgebra, the properties of S lead us to the definition of a new algebraic object:

Definition 1.5. A *Hopf algebra* is a quadruple $(H, \Delta, \varepsilon, S)$, where (H, Δ, ε) is a bialgebra, and S is a linear map (called the *antipode*) for which

$$m \circ (S \otimes \mathrm{id}) \circ \Delta = m \circ (\mathrm{id} \otimes S) \circ \Delta = \varepsilon 1.$$

For H, G two Hopf algebras with antipodes S and S' respectively, a *Hopf algebra morphism* between H and G is a bialgebra map $f : H \to G$, such that $f \circ S = S' \circ f$.

Excercise Write the anti-pode axiom in the form of a commutative diagram.

Exercise: Show that, just as for the bialgebra construction, the definition of S generalises to give a Hopf algebra structure on $U(\mathfrak{g})$, for any Lie algebra $\mathfrak{g}$.

We finish this section with a lemma, which gives some of basic properties of the antipode. (For a proof, see [1, 2], or alternatively attempt it as a more challenging excercise.)

Lemma 1.6 *For any Hopf algebra H, with antipode S, then it holds that:*

1. *S is the unique map on H satisfying the antipode axiom;*
2. *S is an anti-algebra map;*
3. *$S(1) = 1$;*
4. *$\Delta \circ S = (S \otimes S) \circ \tau \circ \Delta$.*

1.5 Sweedler Notation

We will now introduce a special type of notation for dealing with Hopf algebras that proves very useful in practice. For a coalgebra C, and an element $c \in C$, one very often needs to consider presentations

$$\Delta(c) = \sum_{i=1}^{m} c'_i \otimes c''_i.$$

Dealing with summations and indices on a regular basis tends to be quite tiresome, so one adopts the shorthand

$$\Delta(c) = \sum_{i=1}^{m} c'_i \otimes c''_i =: c_{(1)} \otimes c_{(2)}.$$

This is known as *Sweedler notation.*

Let us now consider the coassociativity axiom in terms of Sweedler notation. For $c \in C$, we have by definition that

$$(\Delta \otimes \mathrm{id}) \circ \Delta(c) = (\mathrm{id} \otimes \Delta) \circ \Delta(c).$$

Using Sweedler notation, this is equivalent to

$$\Delta(c_{(1)}) \otimes c_{(2)} = c_{(1)} \otimes \Delta(c_{(2)}),$$

which is in turn equivalent to

$$(c_{(1)})_{(1)} \otimes (c_{(1)})_{(2)} \otimes c_{(2)} = c_{(1)} \otimes (c_{(2)})_{(1)} \otimes (c_{(2)})_{(2)}.$$

This allows to extend Sweedler notation by denoting

$$c_{(1)} \otimes c_{(2)} \otimes c_{(3)} := (c_{(1)})_{(1)} \otimes (c_{(1)})_{(2)} \otimes c_{(2)} = c_{(1)} \otimes (c_{(2)})_{(1)} \otimes (c_{(2)})_{(2)}.$$

In fact, as is easy to see, one can can iterate coassociativity and attach a unique meaning to

$$c_{(1)} \otimes c_{(2)} \otimes \cdots \otimes c_{(k-1)} \otimes c_{(k)}, \qquad \text{(for any } k \in \mathbf{N}\text{)}.$$

Let us now look at the counit axiom in terms of Sweedler notation. By definition we have

$$(\varepsilon \otimes \mathrm{id}) \circ \Delta(c) = (\mathrm{id} \otimes \varepsilon) \circ \Delta(c).$$

In Sweedler notation this is equivalent to

$$\varepsilon(c_{(1)})c_{(2)} = \varepsilon(c_{(2)})c_{(1)}.$$

Similarly, the antipode axiom is given in Sweedler notation by

$$S(c_{(1)})c_{(2)} = c_{(1)}S(c_{(2)}) = \varepsilon(c)1.$$

References

[1] A. Klimyk, K. Schmüdgen, *Quantum Groups and their Representations*, Springer Verlag, Heidelberg–New York, 1997

[2] C. Kassel, *Quantum Groups*, Springer–Verlag, New York–Heidelberg-Berlin, 1995