$\mathcal{U}_q(\mathfrak{g})$, $\mathfrak{g} = \mathfrak{sl}(2)$, $q^n \neq 1$ $\forall n \in \mathbb{N}$, $\mathcal{U}_q(\mathfrak{g}) = \langle E, F, K, K^{-1} \rangle / I$

K ... a field, $\dim_K Z(\mathcal{U}_q(\mathfrak{g})) = 1$, generated by quadratic Casimir

$\uparrow$ centrum of $\mathcal{U}_q(\mathfrak{g})$

$$Cas := \pm \left(FE + \frac{qK + q^{-1}K^{-1}}{(q-q^{-1})^2} \right)$$

$\uparrow$ character on $\mathbb{Z}_2$

Let V be a represent. for $\mathcal{U}_q(\mathfrak{g})$, $\dim_K V = 1$

$V = K\langle u \rangle$, $u \neq 0$ $\quad \exists c \in K^*$: $Ku = cu$

$\Downarrow$

$K(Eu) = q^2 c(Eu) \Rightarrow Eu$ is lin. indep. of u

$\Rightarrow$ $Eu = 0$

$Fu = 0$

Then $0 = [E, F]u = \dfrac{K - K^{-1}}{q - q^{-1}} u = \dfrac{c - c^{-1}}{q - q^{-1}} u \underset{u \neq 0}{\Rightarrow} c^2 = 1 \Rightarrow c = \pm 1$

$\exists$ 2-non isom. classes of 1-dim repr. :

1/ V_+ $\quad K\langle v_+ \rangle$ $\quad Kv_+ = v_+$

$Fv_+ = 0 = Ev_+$

2/ V_- $\quad K\langle v_- \rangle$ $\quad Kv_- = -v_-$

$Fv_- = 0 = Ev_-$

For $\mathrm{char}(K) = 2$, $V_+ \simeq V_-$.

Weyl group $\rightsquigarrow$ Hecke algebra

$q \in \mathbb{C}$, $\mathcal{H}_n = K\langle \sigma_1, \dots, \sigma_{n-1} \rangle / I$

$I =$ (A_n-series)

$$\sigma_k \sigma_{k+1} \sigma_k = \sigma_{k+1} \sigma_k \sigma_{k+1} \qquad 1 \leq k \leq n-2$$

$$(\sigma_k - 1)(\sigma_k + q^2) = 0$$

$$\sigma_k \sigma_l = \sigma_l \sigma_k \qquad |k - l| \geq 2$$

$P_n := \sum_{\sigma \in S_n} q^{-2\ell(\sigma)} \sigma \in \mathcal{H}_n$, $\ell(\sigma)$ ... minimal length of σ

$(\sigma_k - 1) P_n = 0$, symmetric tensors when acting on $V^{\otimes n}$

$\forall k = 1, \dots, n-1$

Quantized function algebras of generalized flag manifolds

Hopf $*$-algebra on $\mathbb{C}_q[G] = U_q(\mathfrak{g})^*_{fin} \subseteq U_q(\mathfrak{g})^*$

$\varphi, \psi \in \mathbb{C}_q[G]$
$X, Y \in U_q(\mathfrak{g})$

$$(\varphi\psi)(X) := (\phi \otimes \psi)(\Delta X)$$
$$1(X) := \epsilon(X)$$
$$(\Delta\phi)(X \otimes Y) := \phi(XY)$$
$$\epsilon(\varphi) := \varphi(1) \qquad 1 \in \mathbb{C}$$
$$(S(\varphi))(X) := \varphi(S(X))$$
$$\varphi^*(X) := \overline{\varphi(S(X)^*)}$$

$*$-invariant subalgebra $= \mathbb{C}_q[U]$

$\mathbb{C}_q[U]$ is a $U_q(\mathfrak{g})$-bimodule:

$X, Y \in U_q(\mathfrak{g})$

$$(X \cdot \varphi)(Y) := \varphi(YX),$$
$$(\varphi \cdot X)(Y) := \varphi(XY).$$

$V(\lambda)$, $\lambda \in P_+$ – fin. dim. (unit.) $U_q(\mathfrak{g})$-module

$(\ ,\ )$ – scalar $U_q(\mathfrak{g})$-inv. product, ON-basis

$$\{ v_\mu^{(i)} \mid \mu \in P(\lambda),\ i \in \{1, \dots, \dim V(\lambda)_\mu\} \}$$

$$C^\lambda_{\mu,i\,;\,\nu,j}(X) := (X \cdot v_\nu^{(j)}, v_\mu^{(i)}), \quad X \in U_q(\mathfrak{g}) \qquad \mu, \nu \in P(\lambda)$$

basis of $U_q(\mathfrak{g})^*_{fin} = \mathbb{C}_q[G]$

$$(\Delta C^\lambda_{\mu,i\,;\,\nu,j}) = \sum_{\sigma,s} C^\lambda_{\mu,i\,;\,\sigma,s} \otimes C^\lambda_{\sigma,s\,;\,\nu,j},$$

$$\cancel{\epsilon(C^\lambda_{\mu,i\,;\,\nu,j}) = \delta_{\mu\nu}\delta_{i,j}}, \quad (C^\lambda_{\mu,i\,;\,\nu,j})^* = S(C^\lambda_{\nu,j\,;\,\mu,i})$$

$$\sum_{\sigma,s} (C^\lambda_{\sigma,s;\,\mu,i})^* C^\lambda_{\sigma,s;\,\nu,j} = \delta_{\mu,\nu}\delta_{i,j}$$

$(C^\lambda_{\mu,i\,;\,\nu,j})^*$ – matrix coefficients of dual representation $V(\lambda)^* := V(-w_0 \cdot \lambda)$

$w_0 \in W$ longest element

②

$(\pi^*, V(\lambda)^*)$ contragradient $\mathcal{U}_q(\mathfrak{g})$-mod : $\pi^*: \mathcal{U}_q(\mathfrak{g}) \to \mathrm{End}(V(\lambda)^*)$

$\pi^*(X)\varphi = \varphi \circ \pi(S(X))$ for $X \in \mathcal{U}_q(\mathfrak{g})$, $\varphi \in V(\lambda)^*$

$u \in V(\lambda) \longrightarrow u^* := (-, u) \in V(\lambda)^*$

$\Rightarrow (u^*, v^*) := (\pi(K^{-2\rho}) v, u)$, $u, v \in V(\lambda)$

$\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha \in \mathfrak{h}^*$

$\{ \phi^{(i)}_{-\mu} := q^{(\mu, \rho)} (v^{(i)}_\mu)^* \}$ ON-base of $V(\lambda)^*$

wt $\phi^{(i)}_{-\mu} = -\mu$

and $(-,-)$, $V(\lambda)^*$ is unitarizable

(use $S^2 u = K^{-2\rho} u K^{2\rho}$)

In particular, $\left(C^{\lambda}_{\mu, i\,;\, \nu, j}\right)^* = q^{(\mu - \nu, \rho)} C^{-w_0 \lambda}_{-\mu, i\,;\, -\nu, j}$

For $\lambda \in P_+$,

$B_\lambda := \langle C^\lambda_{v; v_\lambda} \mid v \in V(\lambda) \rangle$ right $\mathcal{U}_q(\mathfrak{g})$-submodule of $\mathbb{C}_q[\mathcal{U}]$

$\cong V(\lambda)$

Set $A^+ := \bigoplus_{\lambda \in P_+} B_\lambda$, $A^{++} := \bigoplus_{\lambda \in P_{++}} B_\lambda$

$\mathbb{C}_q[\mathcal{U}] \supseteq A^+ \simeq$ algebra of (left) $\mathcal{U}^+ = \mathcal{U}_q(\mathfrak{n}^+)$-invariant elements

__Th:__ The multiplication map $m: A^{++} \otimes A^{++} \to \mathbb{C}_q[\mathcal{U}]$ is surjective.

$(B_\lambda)^* B_\mu \underset{\text{right } \mathcal{U}_q(\mathfrak{g})\text{-mod}}{\simeq} V(\lambda)^* \otimes V(\mu)$

Non-commutative ~~[illegible]~~ algebra structure on $\mathbb{C}_q[\mathcal{U}]$:

$\lambda_1, \lambda_2 \in P_+$, $\mu \in P(\lambda_1)$
$\nu \in P(\lambda_2)$

$N(\mu, \lambda_1; \nu, \lambda_2) := \langle C^{\lambda_1}_{v; v_{\lambda_1}} C^{\lambda_2}_{w; v_{\lambda_2}} \mid (v, w) \in sN \rangle$

where sN = set of pairs $(v, w) \in V(\lambda_1)_{\mu'} \times V(\lambda_2)_{\nu'}$
with $\mu' > \mu$ and $\nu' < \nu$ and $\mu' + \nu' = \mu + \nu$.

③

$$M(\mu,\lambda_1;\nu,\lambda_2) := \langle (C^{\lambda_1}_{v,v_{\lambda_1}})^* C^{\lambda_2}_{w;v_{\lambda_2}} \mid (v,w) \in {}_sO \rangle$$

${}_sO$ = set of pairs $(v,w) \in V(\lambda_1)_{\mu'} \times V(\lambda_2)_{\nu'}$
with $\mu' < \mu$, $\nu' < \nu$ and $\mu - \mu' = \nu - \nu'$

Lemma: $\lambda_1, \lambda_2 \in P_+$, $v \in V(\lambda_1)_\mu$, $w \in V(\lambda_2)_\nu$

1/ Then

$$C^{\lambda_1}_{v;v_{\lambda_1}} C^{\lambda_2}_{w;v_{\lambda_2}} = q^{(\lambda_1,\lambda_2)-(\mu,\nu)} C^{\lambda_2}_{w;v_{\lambda_2}} C^{\lambda_1}_{v;v_{\lambda_1}} \mod N(\mu,\lambda_1;\nu,\lambda_2)$$

2/ Then

$$\left(C^{\lambda_1}_{v;v_{\lambda_1}}\right)^* C^{\lambda_2}_{w;v_{\lambda_2}} = q^{(\mu,\nu)-(\lambda_1,\lambda_2)} C^{\lambda_2}_{w;v_{\lambda_2}} \left(C^{\lambda_1}_{v,v_{\lambda_1}}\right)^* \mod O(\mu,\lambda_1;\nu,\lambda_2)$$

1/, 2/ $\Leftarrow$ 1/ PBW theorem for $U_q(g)$ acting on $\mathbb{C}_q[U]$
2/ R-matrix

Passing to generalized flag manifold

$S \subseteq \Delta$, $S \equiv \{i \mid \alpha_i \in S\}$, $\mathfrak{p}_S \subseteq g$, $U_q(\mathfrak{p}_S)$
$\cup$ $\mathfrak{l}_S$

fin. dim $U_q(\mathfrak{l}_S)$-mod, $Z(\mathfrak{l}_S) = \bigcap_{i \in S} \mathrm{Ker}(\alpha_i) \subseteq \mathfrak{h}$

$\mathring{\mathfrak{l}}_S$ ... semisimple part of $\mathfrak{l}_S$, $U_q(\mathring{\mathfrak{l}}_S)$

Lemma: $\forall$ fin. dim. $U_q(\mathfrak{l}_S)$-mod V is completely reducible as $U_q(\mathfrak{h})$-mod, is compl. red. as $U_q(\mathfrak{l}_S)$-mod.

Hopf $*$-algebra embedding $i_S: U_q(\mathfrak{l}_S) \hookrightarrow U_q(g)$
induces $i_S^*: U_q^*(g) \to U_q(\mathfrak{l}_S)^*$

$$\mathbb{C}_q[L_S] := i_S^*(\mathbb{C}_q[G]) = \{\varphi \circ i_S \mid \varphi \in \mathbb{C}_q[G]\}$$

and i_S^* is Hopf $*$-algebra map.

(4) $\mathbb{C}_q[K_S] \subseteq \mathbb{C}_q[L_S]$ $\quad$ $*$-invariant elements in $\mathbb{C}_q[L_S]$

Define: $*$-subalgebra $\mathbb{C}_q[U/K_S] \subseteq \mathbb{C}_q[U]$ by

$$\mathbb{C}_q[U/K_S] := \{ \varphi \in \mathbb{C}_q[U] \mid (\mathrm{Id} \otimes i_S^*)(\Delta\varphi) = \varphi \otimes 1 \}$$
$$= \{ \varphi \in \mathbb{C}_q[U] \mid X \cdot \varphi = \epsilon(X)\varphi \quad \forall X \in \mathcal{U}_q(\mathfrak{l}_S) \}$$

because:

$$\Delta\varphi\,(X \otimes Y) = \varphi(XY)$$
$$(\mathrm{Id} \otimes i_S^*)(\Delta\varphi)\,(X \otimes Y) = \Delta\varphi\,(X \otimes i_S Y) = \varphi\,(X\, i_S(Y))$$
$$(\varphi \otimes 1)\,(X \otimes Y) = \varphi(X)\,1(Y) = \varphi\,(1(Y)X) = \varphi\,(\epsilon(Y)X)$$
$$\forall X \in \mathcal{U}_q(\mathfrak{g})$$

The algebra $\mathbb{C}_q[U/K_S]$ is a left $\mathbb{C}_q[U]$-subcomodule of $\mathbb{C}_q[U]$

Recall: K a field, C/K coalgebra, a left C-~~coalg~~ comodule over K is K-vector space M: $\quad \rho : M \to C \otimes M$

1/ $(\Delta \otimes \mathrm{id}) \circ \rho = (\mathrm{id} \otimes \rho) \circ \rho$ $\quad$ on $C \otimes C \otimes M$

2/ $(\epsilon \otimes \mathrm{Id}) \circ \rho = \mathrm{Id}$ $\quad$ on M.

and analogously for right C-comodules