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Investigating the holomorph of a group with GAP

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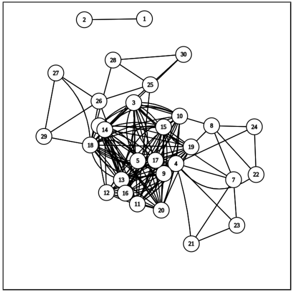
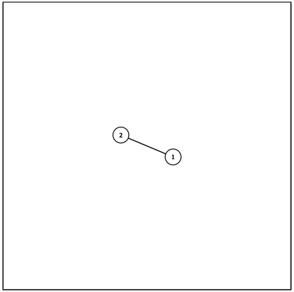
Advisor: *Ok, Mr. Spaggiari, the project of your Master's Thesis is the following.*

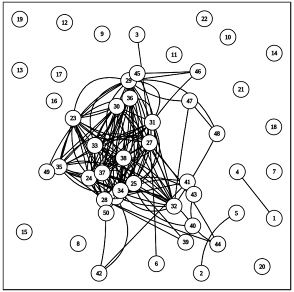
Advisor: *Take any group, consider the holomorph, and draw the normalizing graph. You will find a very regular, symmetric and satisfying picture.*

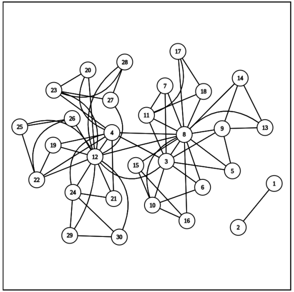
Advisor: *You can use* GAP *to draw it, but, please, look how pleasant and smooth that graph is. Describe that pattern and formulate a conjecture.*

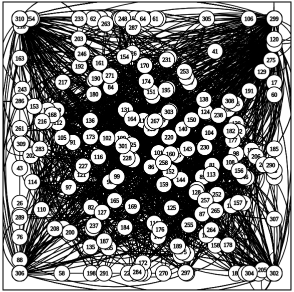
The satisfying, regular, graphs I found

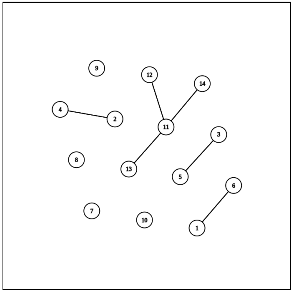












Conjecture. *There is something wrong.*

Let’s take a step back

The holomorph

Let us introduce some fundamental notions.

Definition

Let G be a group. The **holomorph** of G is

$$\mathsf{Hol}(G) = \langle \mathsf{Aut}(G), \rho(G) \rangle \leq \mathsf{Sym}(G)$$

where $\rho(G) = \{\sigma_g \colon x \mapsto xg \mid g \in G\}$ is the subgroup of right multiplication maps.

Thus, the holomorph of a group is a very large subset of bijective maps.

What are these normalizing graphs?

Definition

The **normalizing graph** of a group G is a graph where

1. The *vertices* are the regular subgroups of $\mathrm{Hol}(G)$.
2. An *edge* represents a mutual normalization in $\mathrm{Sym}(G)$.

Recall that N *normalizes* M if

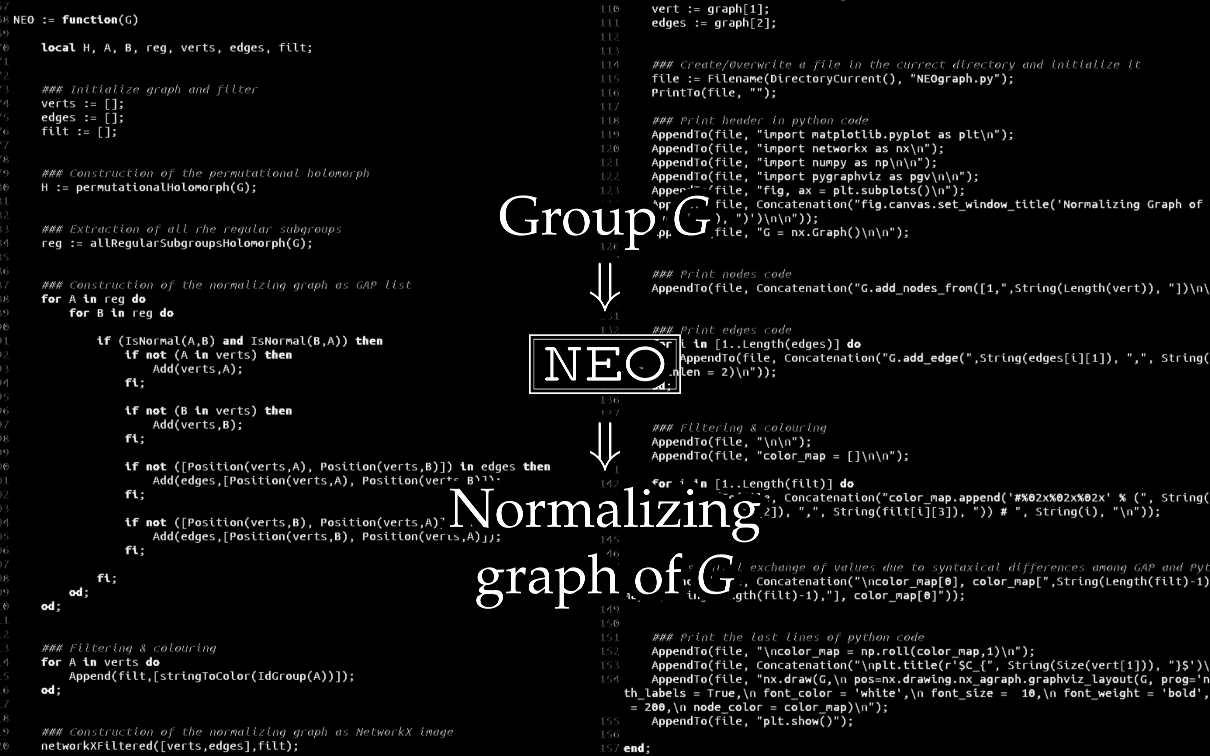
$$n^{-1}Mn = M \qquad \forall n \in N.$$

Motivation: It has several connections with the recent theory of *skew braces* and the *Yang-Baxter equation*.

GAP is a programming language for **computational discrete algebra**, with particular emphasis on Computational Group Theory.

GAP was fundamental in the understanding of the behaviour of the normalizing graphs.

The *pièce de résistance* of the coding part of this work is certainly the GAP function `NEO`.


Group G
NEO
Normalizing graph of G

The function NEO

1. *Input:* **finite group** G.
2. Construction of the **holomorph** $\mathsf{Hol}(G)$.
3. Look for the **regular subgroups**.
4. Look for **mutual normalizations**.
5. Write a **python code** containing the graph.
6. **Paint the vertices** with respect to the isomorphism type.
7. Run the python code and **display the graph**.

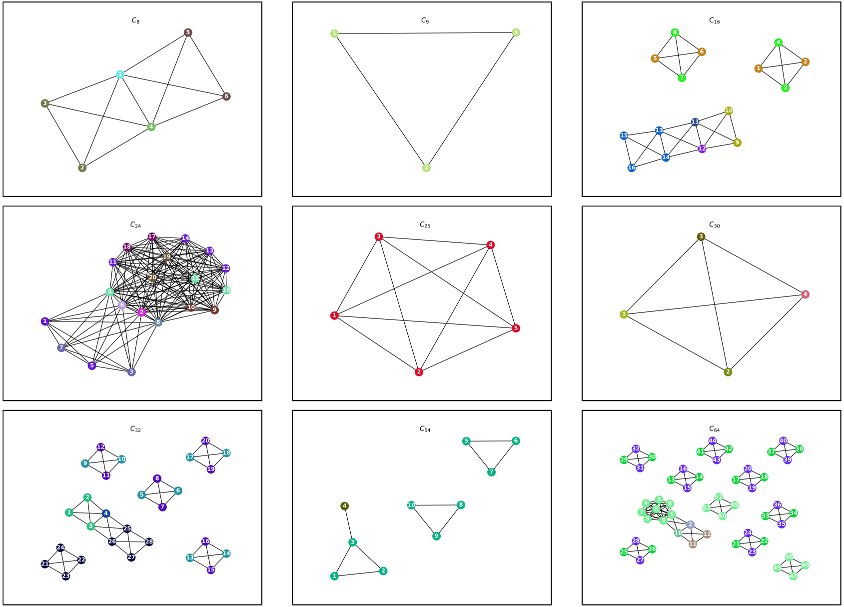
What is the meaning of the colors?
Same color $\leftrightarrow$ isomorphic regular subgroups.

Let’s do some experiments!

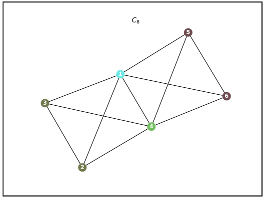
(Show video 1)

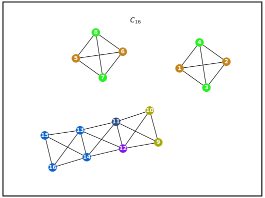
Can you spot the pattern?

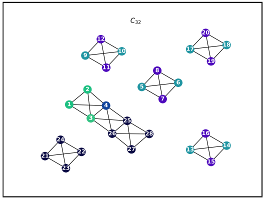
Can you spot the pattern?

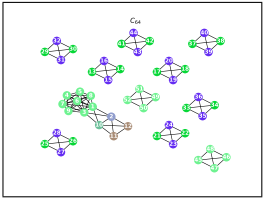


Can you spot the pattern?

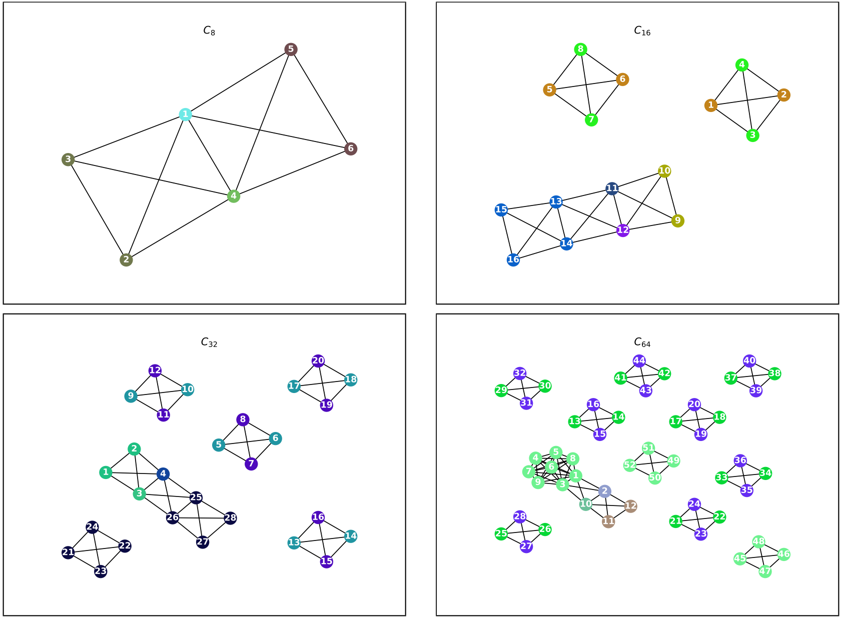








Can you spot the pattern?



Problem

Find and prove the normalizing graph of C_{pn}

Different notions, same concept

Notation. For $x \in G$ and $\varphi \in \mathsf{Sym}(G)$ we denote by $x^{\varphi} = \varphi(x)$.

Theorem (A. Caranti, 2020 [1])

Let $(G, \cdot)$ be a finite group. The following data are equivalent.

1. *A regular subgroup $N \leq \mathsf{Hol}(G, \cdot)$.*
2. *A **gamma function** $\gamma\colon (G, \cdot) \to \mathsf{Aut}(G, \cdot)$, i.e. such that*

$$\gamma(x^{\gamma(y)} \cdot y) = \gamma(x)\gamma(y) \qquad \forall x, y \in G.$$

3. *A group operation $\circ$ on G such that $x \circ y = x^{\gamma(y)}$ for every $x, y \in G$.*

Expected question(s). How is N connected with γ and $\circ$?
Why are we introducing gamma functions?

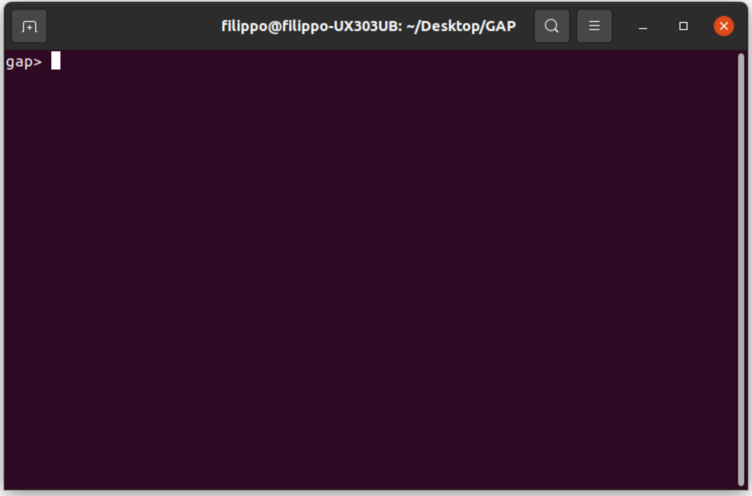
The case $p = 2$

"two is the oddest prime number"

Approach

1. Use GAP to obtain **raw information**.
2. **Guess** some important gamma functions and **prove their existence**.
3. **Generate all the others**.
4. Prove the **uniqueness** of all the gamma functions found.
5. Find a good idea to approach the **mutual normalization problem**.

(1) Use GAP to obtain raw information



(2) Guess some gamma functions...

In C_{16} we have

x	0; 8	1; 9	2; 10	3; 11	4; 12	5; 13	6; 14	7; 15
$\gamma(x)$	σ_1	σ_3	σ_5	σ_7	σ_9	σ_{11}	σ_{13}	σ_{15}

Guess:

$$\begin{aligned} \gamma\colon G &\to \mathsf{Aut}(G) \\ x &\mapsto \sigma_{2x+1} \end{aligned}$$

In the same way, we obtain the following gamma functions

Gamma function	**Isomorphism class**
$\gamma_1(x) = \sigma_1$	C_{2^n}
$\gamma_2(x) = \sigma_{2^{n-1}+1}^x$	C_{2^n}
$\gamma_3(x) = \sigma_{2^{n-1}-1}^x$	Q_{2^n}
$\gamma_4(x) = \sigma_{2^n-1}^x$	D_{2^n}
$\gamma_{\mathsf{p}}(x) = \sigma_{2x+1}$	$\mathsf{C}_2 \times \mathsf{C}_{2^{n-1}}$
$\gamma_{\mathsf{c},u}(x) = \sigma_{2^u x+1} \qquad u = 2, \ldots, n$	C_{2^n}

Gamma function	Isomorphism class
$\gamma_5(x) = \begin{cases} \sigma_1 & x \equiv 0 \pmod 4 \\ \sigma_{2^{n-1}-1} & x \equiv 1 \pmod 4 \\ \sigma_{2^{n-1}+1} & x \equiv 2 \pmod 4 \\ \sigma_{2^n-1} & x \equiv 3 \pmod 4 \end{cases}$	SD_{2^n}
$\gamma_6(x) = \begin{cases} \sigma_1 & x \equiv 0 \pmod 4 \\ \sigma_{2^n-1} & x \equiv 1 \pmod 4 \\ \sigma_{2^{n-1}+1} & x \equiv 2 \pmod 4 \\ \sigma_{2^{n-1}-1} & x \equiv 3 \pmod 4 \end{cases}$	SD_{2^n}
$\gamma_{\mathsf{m}}(x) = \begin{cases} \sigma_{2x+1} & x \equiv 0 \pmod 2 \\ \sigma_{2x+2^{n-2}+1} & x \equiv 1 \pmod 2 \end{cases}$	M_{2^n}

(3) Generate the others via conjugation

Roughly speaking, to **conjugate a gamma function** γ **by an automorphism** means simply to permute the elements of image of γ.

Notation. For a gamma function γ and $\sigma_{2k+1} \in \mathrm{Aut}(G)$ we denote by $\gamma^k = \gamma^{\sigma_{2k+1}^{-1}}$.

γ	γ_1	γ_2	γ_3	γ_4	γ_5	γ_6	γ_{p}	γ_{m}	$\gamma_{\mathrm{c},u}$
$\lvert\gamma^{\mathrm{Aut}(G)}\rvert$	1	1	1	1	2	2	2^{n-2}	2^{n-2}	2^{n-u-1}

Proposition

There are **at least** $3 \cdot 2^{n-2} + 4$ regular subgroups in $\mathrm{Hol}(G)$.

Expected question. Why is this procedure called *conjugation*?

This was the most difficult part of the entire work. Proofs are long, technical and boring (at least, the proofs I found are so).

Mutual normalization problem

Idea: Translate the mutual normalization

Theorem

Let $(G, \cdot)$ be a group such that $\mathsf{Aut}(G)$ is abelian, and let $N, M \leq \mathsf{Hol}(G)$ be regular subgroups. Denote by

$$\gamma\colon (G, \circ) \to \mathsf{Aut}(G), \qquad \delta\colon (G, \bullet) \to \mathsf{Aut}(G)$$

respectively the gamma functions associated with N and M. Then N and M mutually normalize each other if and only if

$$\begin{cases} \gamma(x) = \gamma\left(x \cdot (y \circ x)^{-1} \cdot (x \bullet y)\right) \\ \delta(x) = \delta\left(x \cdot (y \bullet x)^{-1} \cdot (x \circ y)\right) \end{cases} \qquad \forall x, y \in G.$$

Remark. This is a general result. In particular, for cyclic groups, this is a pair of equation in modular arithmetic, since $C_{2^n} \cong \mathbb{Z}/2^n\mathbb{Z}$.

(Show video 2)

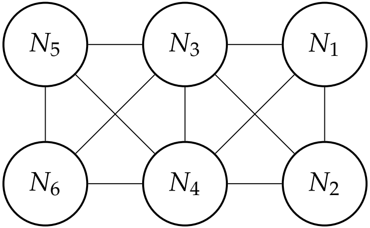
Mutual normalization of γ_i

Those conditions trivially hold for $\gamma_1, \dots, \gamma_6$ in the following sense.

Corollary

$$\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\} \qquad \text{and} \qquad \{\gamma_3, \gamma_4, \gamma_5, \gamma_6\}$$

are mutually normalizing families of gamma functions.



Mutual normalization of γ_{c}

For a gamma function γ and $\sigma_{2k+1} \in \mathsf{Aut}(G)$ we denote by $\gamma^k = \gamma^{\sigma_{2k+1}^{-1}}$.

Proposition

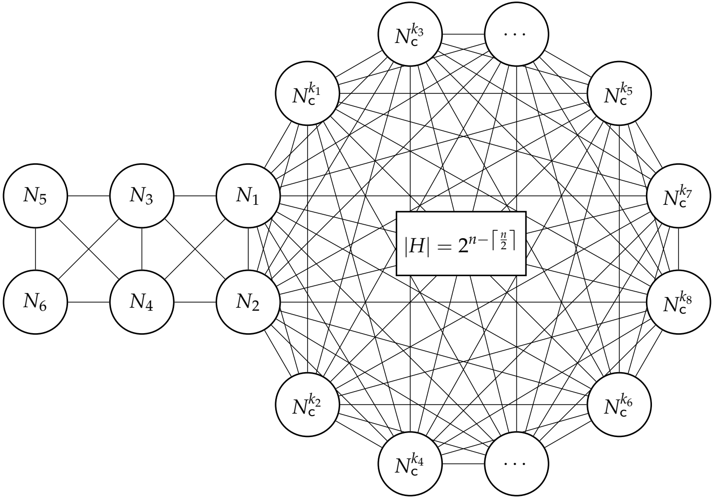
$$\gamma^k_{\mathsf{c},u} \rightleftharpoons \gamma^h_{\mathsf{c},v} \quad \Longleftrightarrow \quad \begin{cases} 2^u(2k+1) \equiv 2^v(2h+1) \pmod{2^{n-u}} \\ 2^u(2k+1) \equiv 2^v(2h+1) \pmod{2^{n-v}} \end{cases}$$

Corollary

$$H = \left\{ \gamma^k_{\mathsf{c},u} \; : \; \left\lceil \frac{n}{2} \right\rceil \leq u \leq n \right\}$$

is composed by $2^{n-\lceil \frac{n}{2} \rceil}$ mutually normalizing gamma functions.

The comet



Fun fact. This picture was made in LaTeX in ≈ 20 hours. Are you faster than me? :)

Mutual normalization of γ_{c}

Corollary

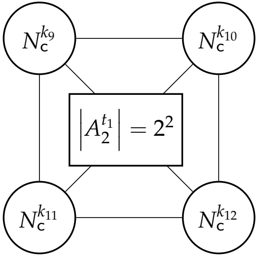
For every $2 \leq u < \left\lceil \frac{n}{2} \right\rceil$ and $0 \leq t < 2^{n-2u-1}$, the family

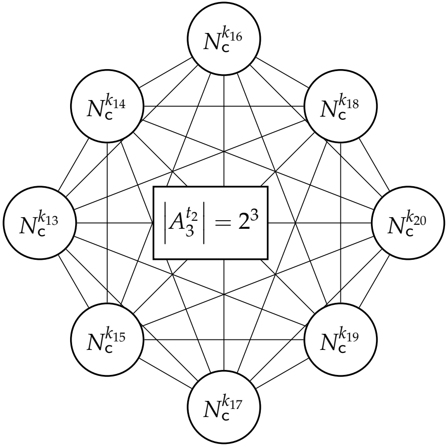
$$A_u^t = \left\{ \gamma_{\mathsf{c},u}^k : k \equiv t \pmod{2^{n-2u-1}} \right\}$$

is composed by 2^u mutually normalizing gamma functions. In total, there are

$$\frac{1}{3} \left(2^{n-3} - 2^{n-2\left\lceil \frac{n}{2} \right\rceil + 1} \right)$$

distinct A_u^t.


$N_c^{k_9}$
$N_c^{k_{10}}$
$N_c^{k_{11}}$
$N_c^{k_{12}}$
$\left|A_2^{t_1}\right| = 2^2$


$N_c^{k_{16}}$
$N_c^{k_{14}}$
$N_c^{k_{18}}$
$N_c^{k_{13}}$
$\left|A_3^{t_2}\right| = 2^3$
$N_c^{k_{20}}$
$N_c^{k_{15}}$
$N_c^{k_{19}}$
$N_c^{k_{17}}$

Mutual normalization of γ_{p} and γ_{m}

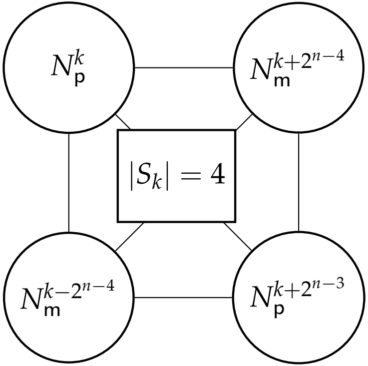
Proposition

$$\gamma_{\mathsf{p}}^{k} \rightleftharpoons \gamma_{\mathsf{p}}^{h} \iff k \equiv h \pmod{2^{n-3}}$$
$$\gamma_{\mathsf{m}}^{k} \rightleftharpoons \gamma_{\mathsf{m}}^{h} \iff k \equiv h \pmod{2^{n-3}}$$
$$\gamma_{\mathsf{p}}^{k} \rightleftharpoons \gamma_{\mathsf{m}}^{h} \iff k - h \equiv 2^{n-4} \pmod{2^{n-3}}$$

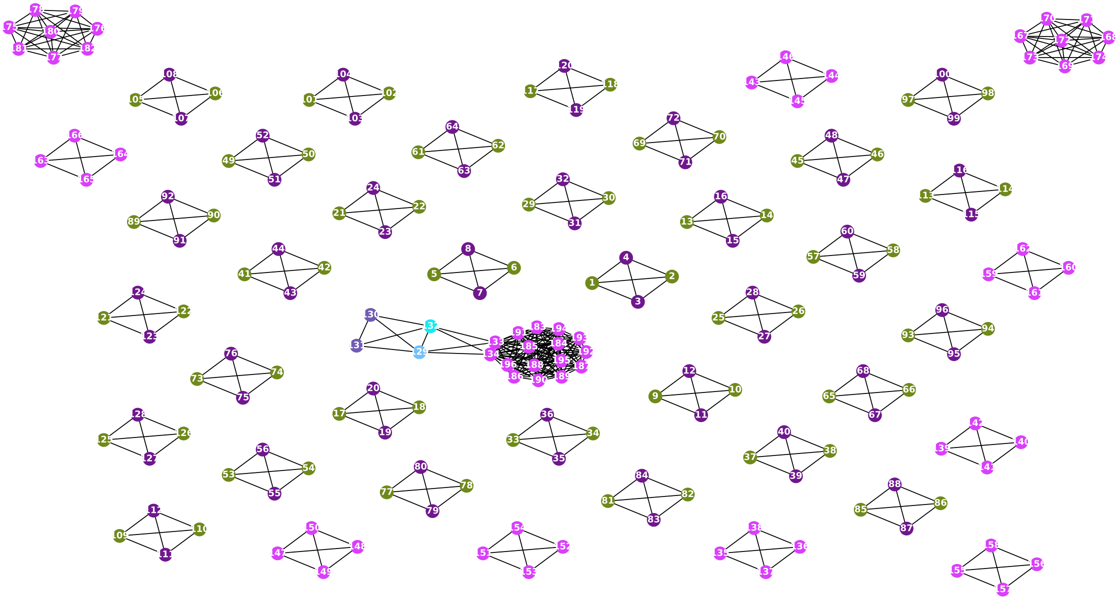
Corollary

$$S_k = \left\{ \gamma_{\mathsf{p}}^{k}, \gamma_{\mathsf{m}}^{k+2^{n-4}}, \gamma_{\mathsf{p}}^{k+2^{n-3}}, \gamma_{\mathsf{m}}^{k+2^{n-3}+2^{n-4}} \right\}$$

is composed by 4 mutually normalizing gamma functions. In total, there are 2^{n-3} distinct S_k.


N_{p}^{k}
$N_{\mathsf{m}}^{k+2^{n-4}}$
$|S_k| = 4$
$N_{\mathsf{m}}^{k-2^{n-4}}$
$N_{\mathsf{p}}^{k+2^{n-3}}$

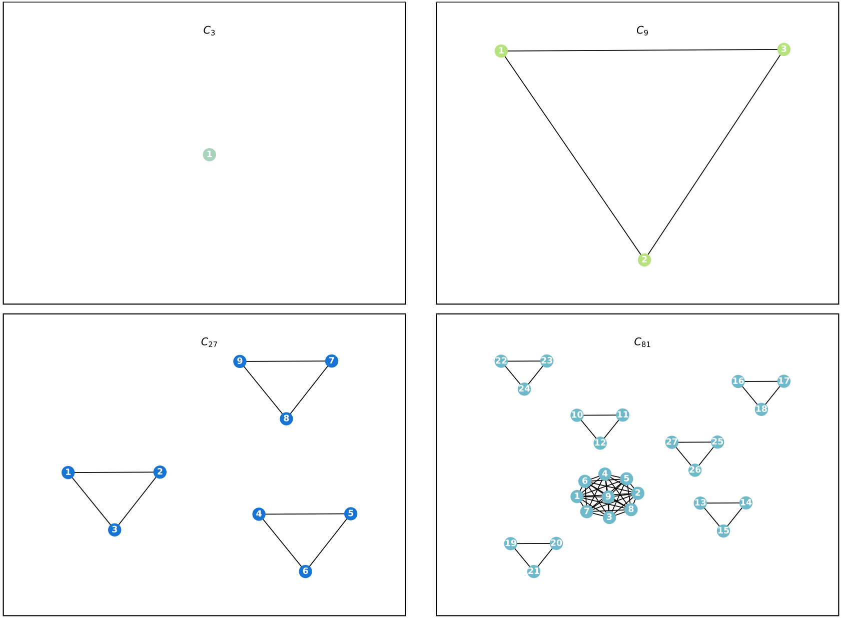
The local normalizing graph of C_{2n}



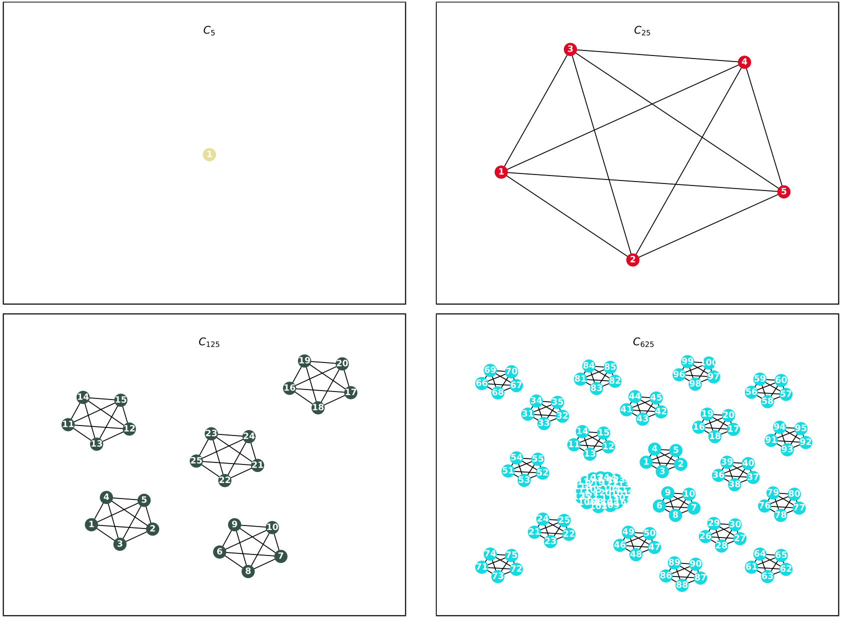
The case p odd

(A very quick look)

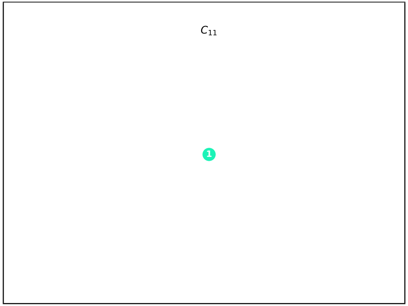
$p = 3$

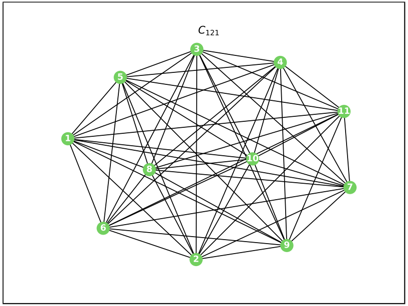


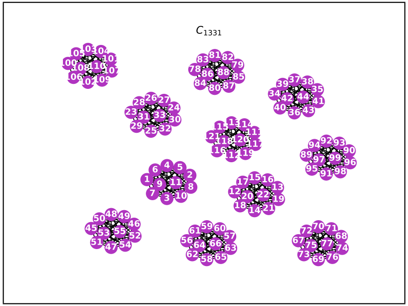
$p = 5$



$p = 11$





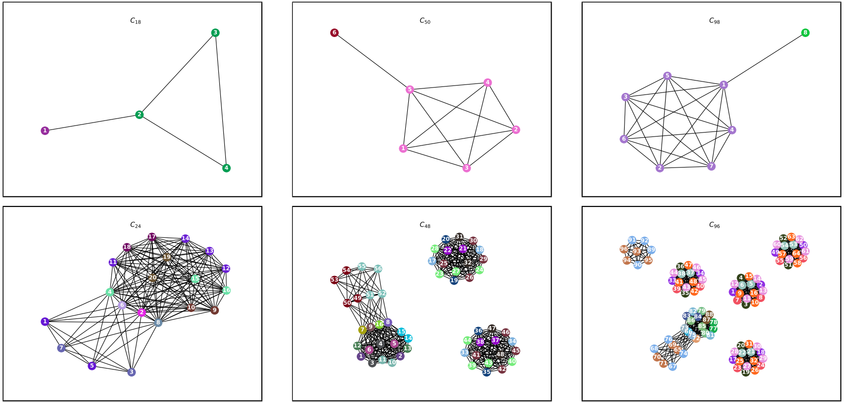


RAM MEMORY

NOT ENOUGH

Conclusion?

The mutually normalizing regular subgroups of $\mathrm{Hol}(C_{p^n})$ have been completely classified. Is it really time to be satisfied?



Ambition: We know that cyclic groups are the building blocks of *abelian groups*...

That’s all, thanks!

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