1) $F(x,y) = \arctan(x^2y+x) - \log(x+y)$

$[x_0, y_0] = [0,1]$

- $F \in C^\infty(G)$ pro vhodné G otevřenou obsahující $[0,1]$ +1
- $F(0,1) = \arctan(0^2 1+0) - \log(0+1) = 0 - 0 = 0$ +1
- $\frac{\partial F}{\partial y}(0,1) = \left(\frac{1}{1+(x^2y+x)^2} x^2 - \frac{1}{x+y} \right)\Big|_{(x,y)=(0,1)} = -1 \neq 0$ +1

$\Rightarrow$ ex. $U \ni 0$, ex. $V \ni 1$, že: $\forall x \in U\ \exists! y \in V: F(x,y) = 0$

Navíc je tato funkce třídy C^∞ na U. +2

- $\operatorname{arctg}(x^2y+x) - \log(x+y) = 0 \quad //'$

$$\frac{1}{1+(x^2y+x)^2}(2xy + x^2y' + 1) - \frac{1}{x+y}(1+y') = 0 \quad (*)$$

$x = 0, y(0) = 1$: $\frac{1}{1+0^2} \cdot 1 - \frac{1}{1}(1+y') = 0$

$1 - 1 - y' = 0$

$y' = 0 \Rightarrow y'(0) = 0$

+3

derivujeme $(*)$:

$$\left(\frac{1}{1+(x^2y+x)^2}\right)'(2xy + x^2y'+1) + \frac{1}{1+(x^2y+x)^2} \cdot (2xy + x^2y'+1)' - \left(\frac{1}{x+y}\right)'(1+y') - \frac{1}{\cancel{x+y}} y'' = 0$$

$$-\frac{1}{(1+(x^2y+x)^2)^2}(2(x^2y+x))(2xy+x^2y'+1)^2 + \frac{1}{1+(x^2y+x)^2}(2y + 2xy' + 2xy' + x^2y'')$$

$$-\frac{-1}{(x+y)^2}(1+y')^2 - \frac{y''}{x+y} = 0$$

$x=0, y(0)=1$, $y'(0)=0$: $0 + \frac{1}{1+0}(2+0) + \frac{1}{1^2}(1+0)^2 - \frac{y''(0)}{1} = 0$

$2 + 1 - y''(0) = 0$

$y''(0) = 3$

+5

$f \in C^\infty(U) \Rightarrow f'' > 0$ na nějakém okolí $0 \Rightarrow f$ konvexní na okolí 0 +2

(2) $f(x,y,z) = x - y$, $M = \{x^2+y^2+z^2 \le 1,\ x+y+z \ge 0\}$

$f \in C^\infty(\mathbb{R}^3)$, M kompakt +2

$M_1 = \{x^2+y^2+z^2 < 1,\ x+y+z > 0\}$: $Df = (1,-1,0) \ne 0$ +1

M_2 : $\{x^2+y^2+z^2 = 1,\ x+y+z > 0\}$, vazební podmínka je $g(x,y,z) = x^2+y^2+z^2-1$

$G = \{x+y+z > 0\}$

$\cdot\ Dg = 2(x,y,z) \ne 0$ na M_2

$$\Rightarrow \begin{array}{ll} 1 + \lambda x = 0 & \overset{\lambda \ne 0}{\Rightarrow} x = -1/\lambda \\ -1 + \lambda y = 0 & y = 1/\lambda \\ 0 + \lambda z = 0 & \Rightarrow z = 0 \\ x^2+y^2+z^2 = 1 & \Rightarrow \frac{1}{\lambda^2} + \frac{1}{\lambda^2} = 1 \end{array}$$

$2 = \lambda^2$

$\lambda = \pm\sqrt{2}$

$\Rightarrow \left[-\frac{2}{\sqrt{2}}, \frac{2}{\sqrt{2}}, 0\right]$, $\left[\frac{2}{\sqrt{2}}, -\frac{2}{\sqrt{2}}, 0\right] \notin G$

+3

$M_3 = \{x^2+y^2+z^2 < 1,\ x+y+z = 0\}$, vazební podmínka $g(x,y,z) = x+y+z$

$G = \{x^2+y^2+z^2 < 1\}$

$Dg = (1,1,1) \ne 0$

$$\Rightarrow \begin{array}{l} 1 + \lambda = 0 \\ -1 + \lambda = 0 \\ 0 + \lambda = 0 \\ x+y+z = 0 \end{array}$$

nemá řešení +3

$M_4 = \{x+y+z = 0,\ x^2+y^2+z^2 = 1\}$ $\Rightarrow$ $z = -x-y$, dosadíme

$N = \{x^2+y^2+(x+y)^2 = 1\}$, $\tilde{f}(x,y) = x-y$, vazební podmínka

$g(x,y) = x^2+y^2+(x+y)^2-1$

$Dg = (2x+2(x+y),\ 2y+2(x+y)) = 0 \iff \begin{array}{l} x+x+y = 0 \\ y+x+y = 0 \end{array}$ $\quad \begin{array}{l} 2x+y = 0 \\ x+2y = 0 \end{array}$ $\Rightarrow x = y = 0$ není v N

$$\begin{array}{l} 1 + \lambda(2x+2(x+y)) = 0 \\ -1 + \lambda(2y+2(x+y)) = 0 \\ x^2+y^2+(x+y)^2 = 1 \end{array}$$

$\overset{\lambda \ne 0}{\Rightarrow}$ $2\lambda(2x+y+2y+x) = 0$

$3x+3y = 0$

$y = -x$

$\Rightarrow 2x^2 = 1$ +5

$x = \pm\frac{2}{\sqrt{2}}$ $y = \mp\frac{1}{\sqrt{2}}$, $z = 0$

Maximum: $\left[\frac{2}{\sqrt{2}}, -\frac{2}{\sqrt{2}}, 0\right]$, Minimum: $\left[-\frac{2}{\sqrt{2}}, \frac{2}{\sqrt{2}}, 0\right]$ +1

3

$$\begin{pmatrix} 2 & 3 & 1 & 4 \\ -3 & 5 & -2 & 1 \\ 4 & -2 & 3 & -3 \\ 3 & 6 & 2 & 2 \end{pmatrix} \sim \begin{pmatrix} -1 & 8 & -1 & 5 \\ -3 & 5 & -2 & 1 \\ 4 & -2 & 3 & -3 \\ 3 & 6 & 2 & 2 \end{pmatrix}$$

$$\sim \begin{pmatrix} -1 & 8 & -1 & 5 \\ 0 & -19 & 1 & -14 \\ 0 & 30 & -1 & 17 \\ 0 & 30 & -1 & 17 \end{pmatrix} \sim \begin{pmatrix} -1 & 8 & -1 & 5 \\ 0 & -570 & 30 & -420 \\ 0 & 570 & -19 & 323 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} -1 & 8 & -1 & 5 \\ 0 & -570 & 30 & -420 \\ 0 & 0 & 11 & -97 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow h(A) = 3$$

③ $\sum_{n=1}^{\infty} (x-1)^n \sin 1/n$

$$\frac{|x-1|^{n+1} \sin \frac{1}{n+1}}{|x-1|^n \sin \frac{1}{n}} = |x-1| \frac{\sin \frac{1}{n+1}}{\frac{1}{n+1}} \frac{1/n}{\sin 1/n} \frac{1}{n+1} \cdot \frac{1}{1/n} \to |x-1| \quad +3$$

$\Rightarrow$ $x \in (0, 2)$: řada konverguje absolutně $+2$

$x \in (-\infty, 0) \cup (2, \infty)$: řada diverguje

$\Rightarrow$ $x = 0$: $\sum_{n=1}^{\infty} (-1)^n \sin 1/n$ konverguje dle Leibnize : $\sin \frac{1}{n} \searrow 0$ $+3$

$\Rightarrow$ $x = 2$: $\sum \sin 1/n$ diverguje ze srovnání $\frac{\sin 1/n}{1/n} \to 1$, protože

$+5$

$\sum \frac{1}{n}$ diverguje

$\Rightarrow$ $x = 0$: $\sum (-1)^n \sin 1/n$ nekonverguje absolutně $+2$