DESCRIPTIVE PROPERTIES OF VECTOR-VALUED AFFINE FUNCTIONS

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Abstract. Let X be a compact convex set, $\operatorname{ext} X$ stand for the set of extreme points of X, F be a Fréchet space and $f\colon X \to F$ be a strongly affine mapping. The aim of our paper is to investigate transfer of descriptive properties of $f|_{\operatorname{ext} X}$ to f, generalizing thus results from [18] and [12] to the vector-valued context. As a corollary of our results we obtain a vector-valued analogue of a result of J. Lindenstrauss and D.E. Wulbert on L_1-preduals and answer positively Questions 10.6 and 10.7 from [7].

1. Introduction

If X is a compact convex set in a locally convex space and $\operatorname{ext} X$ is the set of all extreme points of X, the Krein-Milman theorem asserts that $X = \overline{\operatorname{co}} \operatorname{ext} X$ (if A is a subset of a locally convex space, symbol $\overline{\operatorname{co}}A$ denotes the closed convex hull of A, whereas $\overline{\operatorname{aco}}A$ stands for the closed absolute convex hull of A). Hence if $f\colon X \to \mathbb{R}$ is affine and continuous, it follows that $f(X) \subset \overline{\operatorname{co}} f(\operatorname{ext} X)$ and thus behaviour of f is in a way determined by the properties of $f|_{\operatorname{ext} X}$. This easy observation leads to a natural question whether it is possible for an affine function f on X to transfer some properties of $f|_{\operatorname{ext} X}$ to f. It has turned out that such a transfer is possible in case of strongly affine functions (see Section 1.1 for the definition). In [12] we investigated transfer of descriptive properties of real (or complex) strongly affine functions on X. The main goal of the present paper is to generalize these results to the case of functions with values in Fréchet spaces. As a corollary we obtain a positive answer to Questions 10.6 and 10.7 in [7].

The paper is organized as follows. The rest of the first section is devoted to definitions and basic facts on compact convex sets, Baire functions and vector integration. Main results are collected in Section 2. The remaining parts provide the proofs of the main results.

1.1. **Compact convex sets.** We will deal both with real and complex spaces. To shorten the notation we will use the symbol $\mathbb{F}$ to denote the respective field $\mathbb{R}$ or $\mathbb{C}$.

If X is a compact Hausdorff space, we denote by $\mathcal{C}(X, \mathbb{F})$ the Banach space of all $\mathbb{F}$-valued continuous functions on X equipped with the sup-norm. The dual of $\mathcal{C}(X, \mathbb{F})$ will be identified (by the Riesz representation theorem) with $\mathcal{M}(X, \mathbb{F})$, the space of $\mathbb{F}$-valued Radon measures on X equipped with the total variation norm and the respective weak* topology. Let $\mathcal{M}^+(X)$ and $\mathcal{M}^1(X)$ stand for the set of all

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positive and probability measures, respectively. If $B \subset X$ is a Borel set, we write $\mathcal{M}^1(B)$ for the set of all $\mu \in \mathcal{M}^1(X)$ with $\mu(X \setminus B) = 0$.

Let X be a convex subset of a (real or complex) vector space E and F be another (real or complex) vector space. Recall that a mapping $f\colon X \to F$ is said to be *affine* if $f(tx + (1-t)y) = tf(x) + (1-t)f(y)$ whenever $x, y \in X$ and $t \in [0,1]$. We stress that the notion of an affine function uses only the underlying structure of real vector spaces.

Let X be a compact convex set in a locally convex topological vector space. We write $\mathfrak{A}(X, \mathbb{F})$ for the space of all $\mathbb{F}$-valued continuous affine functions on X. This space is a closed subspace of $\mathcal{C}(X, \mathbb{F})$ and is equipped with the inherited sup-norm. Given a Radon probability measure μ on X, we write $r(\mu)$ for the *barycenter of* μ, i.e., the unique point $x \in X$ satisfying $a(x) = \int_X a \, \mathrm{d}\mu$ for each affine continuous function on X (see [1, Proposition I.2.1] or [10, Chapter 7, § 20]; note that it does not matter whether we consider real or complex affine functions). Conversely, for a point $x \in X$, we denote by $\mathcal{M}_x(X)$ the set of all Radon probability measures on X with barycenter x (i.e., the set of all probabilities *representing* x).

The usual dilation order $\prec$ on the set $\mathcal{M}^1(X)$ of Radon probability measures on X is defined as $\mu \prec \nu$ if and only if $\mu(f) \leq \nu(f)$ for any real-valued convex continuous function f on X. (Recall that $\mu(f)$ is a shortcut for $\int f \, \mathrm{d}\mu$.) A measure $\mu \in \mathcal{M}^1(X)$ is said to be *maximal* if it is maximal with respect to the dilation order. In case X is metrizable, maximal measures are exactly the probabilities carried by the G_δ set $\operatorname{ext} X$ of extreme points of X (see, e.g., [1, p. 35] or [13, Corollary 3.62]). By the Choquet representation theorem, for any $x \in X$ there exists a maximal representing measure (see [10, p. 192, Corollary] or [1, Theorem I.4.8]). A compact convex set X is termed *simplex* if this maximal measure is uniquely determined for each $x \in X$.

1.2. **Vector integration.** We will deal with vector-valued strongly affine mappings. To be able to do that we need some vector integral. We will use the Pettis approach.

Let μ be an $\mathbb{F}$-valued σ-additive measure defined on an abstract measurable space $(X, \mathcal{A})$ (i.e., X is a set and $\mathcal{A}$ is a σ-algebra of subsets of X) and F a locally convex space over $\mathbb{F}$. (To avoid confusion we stress that we will consider only finite measures.) A mapping $f\colon X \to F$ is said to be *μ-measurable* if $f^{-1}(U)$ is μ-measurable for any $U \subset F$ open. The map f is called *weakly μ-measurable* if $\tau \circ f$ is μ-measurable for each $\tau \in F^*$.

A mapping $f\colon X \to F$ is said to be *μ-integrable* over a μ-measurable set $A \subset X$ if

- $\tau \circ f \in L^1(|\mu|)$ for each $\tau \in F^*$,
- for each $B \subset A$ μ-measurable there exists an element $x_B \in F$ such that

$$\tau(x_B) = \int_B \tau \circ f \, \mathrm{d}\mu, \quad \tau \in F^*.$$

It is clear that the element x_B is uniquely determined, we denote it as $\int_B f \, \mathrm{d}\mu$. If μ is clearly determined, we say only that f is integrable.

Lemma 1.1. *Let μ be an $\mathbb{F}$-valued measure defined on a measurable space $(X, \mathcal{A})$ and F be a Fréchet space over $\mathbb{F}$. Suppose that $f\colon X \to F$ is a bounded weakly μ-measurable mapping with (essentially) separable range. Then the following assertions hold.*

(a) *The mapping f is μ-integrable.*
(b) *If μ is a probability measure and $L \subset F$ is a closed convex set such that $f(X) \subset L$, then $\mu(f) \in L$.*
(c) *If $\|\mu\| \leq 1$ and $L \subset F$ is a closed absolutely convex set such that $f(X) \subset L$, then $\mu(f) \in L$.*
(d) *If ρ is any continuous seminorm on F, then $\rho \circ f$ is μ-integrable and $\rho\left(\int_X f \,\mathrm{d}\mu\right) \leq \int_X \rho \circ f \,\mathrm{d}\,|\mu|$.*
(e) *Let $f_n, g\colon X \to F$ be mappings such that*
   - *f_n are weakly μ-measurable and have separable range,*
   - *the sequence $\{f_n\}$ is bounded in F (i.e., $\bigcup_{n=1}^{\infty} f_n(X)$ is bounded in F),*
   - *$f_n(x) \to g(x)$ in F for $x \in X$.*

   Then g is bounded and μ-measurable. Moreover, all the involved functions are μ-integrable and $\int_X f_n \,\mathrm{d}\mu \to \int_X g \,\mathrm{d}\mu$ in F.

Proof. See [7, Lemma 3.8 and Theorem 3.10]. □

An important class of integrable functions are Baire measurable functions. We recall that if X is a topological space, a *zero set* in X is the inverse image of a closed set in $\mathbb{R}$ under a continuous function $f\colon X \to \mathbb{R}$. The complement of a zero set is a *cozero set*. If X is normal, it follows from Tietze's theorem that a closed set is a zero set if and only if it is also a G_δ set, i.e., a countable intersection of open sets. The complement of a G_δ set is called an F_σ set. We recall that *Baire sets* are members of the σ-algebra generated by the family of all cozero sets in X.

Lemma 1.2. *Let X be a compact space, $\mu \in \mathcal{M}(X, \mathbb{F})$ and $f\colon X \to F$ be a bounded Baire measurable mapping from X to a Fréchet space F over $\mathbb{F}$. Then the mapping f is μ-integrable.*

Proof. See [7, Lemma 3.9]. □

1.3. **Baire mappings.** Given a set K, a topological space L and a family of mappings $\mathcal{F}$ from K to L, we define the *Baire classes* of mappings as follows. Let $(\mathcal{F})_0 = \mathcal{F}$. Assuming that $\alpha \in [1, \omega_1)$ is given and that $(\mathcal{F})_\beta$ have been already defined for each $\beta < \alpha$, we set

$$(\mathcal{F})_\alpha = \{f\colon K \to L;\ \text{there exists a sequence } (f_n) \text{ in } \bigcup_{\beta<\alpha} (\mathcal{F})_\beta \text{ such that } f_n \to f \text{ pointwise}\}.$$

Among other hierarchies (see Section 1.5) we will use the following ones:

- If K and L are topological spaces, by $\mathcal{C}_\alpha(K, L)$ we denote the set $(\mathcal{C}(K, L))_\alpha$, where $\mathcal{C}(K, L)$ is the set of all continuous functions from K to L.
- If K is a compact convex set and L is a convex subset of a locally convex space, by $\mathfrak{A}_\alpha(K, L)$ we denote $(\mathfrak{A}(K, L))_\alpha$, where $\mathfrak{A}(K, L)$ is the set of all affine continuous functions defined on K with values in L.

1.4. **Strongly affine mappings.** If X is a compact convex set and F is a Fréchet space, a mapping $f\colon X \to F$ is called *strongly affine* if, for any measure $\mu \in \mathcal{M}^1(X)$, f is μ-integrable and $\int_X f \,\mathrm{d}\mu = f(r(\mu))$. Note that this is a strengthening of the notion of an affine mapping. Indeed, f is affine if and only if the formula holds for any finitely supported probability μ.

By [7, Fact 1.2], the mapping f is strongly affine if and only if $\tau \circ f$ is strongly affine for each $\tau \in F^*$. It is known that any affine function $f \in \mathcal{C}_1(X, \mathbb{F})$ is strongly affine (see e.g., [1, Theorem I.2.6], [14, Section 14], [19] or [13, Corollary 4.22]) and, moreover, $f \in \mathfrak{A}_1(X, \mathbb{F})$ by a result of Mokobodzki (see, e.g., [15, Théorème 80] or [13, Theorem 4.24]).

If F is a Fréchet space and $f \in \mathcal{C}_1(X, F)$ is affine then it is strongly affine (see [7, Theorem 2.1]). If F is a Banach space with a bounded approximation property, any function $f \in \mathcal{C}_1(X, F)$ is in $\mathfrak{A}_1(X, F)$. However, this does not hold in general. Indeed, if F is a separable reflexive Banach space which fails the compact approximation property, $X = (B_F, \text{weak})$ and $f\colon X \to F$ is the identity embedding, then f is affine, $f \in \mathcal{C}_1(X, F)$ and $f \notin \bigcup_{\alpha<\omega_1} \mathfrak{A}_\alpha(X, F)$ (see [7, Example 2.3].)

For affine functions of higher Baire classes the situation is different even in the scalar case. Firstly, an affine function of the second Baire class need not be strongly affine even if X is simplex (the example is due to Choquet, see, e.g., [1, Example I.2.10], [14, Section 14] or [13, Proposition 2.63]). Further, by [24] there is a compact convex set X and a strongly affine function $f\colon X \to \mathbb{R}$ of the second Baire class which does not belong to $\bigcup_{\alpha<\omega_1} \mathfrak{A}_\alpha(X, \mathbb{R})$.

1.5. **Descriptive classes of sets and functions.** Further we need to recall descriptive classes of functions in topological spaces. We follow the notation of [20]. If X is a set and $\mathcal{F}$ is a family of subsets of X, $\mathcal{F}$ is an *algebra* if $\emptyset, X \in \mathcal{F}$ and $\mathcal{F}$ is closed with respect to complements and finite unions.

If X is a topological space, we recall that *Borel sets* are members of the σ-algebra generated by the family of all open subset of X. We write $\operatorname{Bos}(X)$ and $\operatorname{Bas}(X)$ for the algebras generated by open or cozero sets in X, respectively.

A set $A \subset X$ is *resolvable* (or an *H-set*) if for any nonempty $B \subset X$ (equivalently, for any nonempty closed $B \subset X$) there exists a relatively open $U \subset B$ such that either $U \subset A$ or $U \cap A = \emptyset$. It is easy to see that the family $\operatorname{Hs}(X)$ of all resolvable sets in X is an algebra, see e.g. [9, § 12, VI]. Let $\Sigma_2(\operatorname{Bas}(X))$, $\Sigma_2(\operatorname{Bos}(X))$ and $\Sigma_2(\operatorname{Hs}(X))$ denote countable unions of sets from the respective algebras.

Let F be a topological space and let

$$\operatorname{Baf}_1(X, F) = \{f\colon X \to F;\ f^{-1}(U) \in \Sigma_2(\operatorname{Bas}(X)), U \subset F \text{ open}\}.$$

Analogously we define families $\operatorname{Bof}_1(X, F)$ and $\operatorname{Hf}_1(X, F)$.

Now we use pointwise limits to create higher hierarchies of functions as in Section 1.3. More precisely, if F is a topological space and Φ is a family of functions from X to F, we define $\Phi_1 = \Phi$ and, for each countable ordinal $\alpha > 1$, Φ_α consists of all pointwise limits of sequences from $\bigcup_{\beta<\alpha} \Phi_\beta$. Starting the procedure with $\operatorname{Baf}_1(X, F)$ and creating higher families $\operatorname{Baf}_\alpha(X, F)$ as pointwise limits of sequences contained in $\bigcup_{1\le\beta<\alpha} \operatorname{Baf}_\beta(X, F)$, we obtain the hierarchy of *Baire measurable* functions. Analogously we define, for $\alpha \in [1, \omega_1)$, families $\operatorname{Bof}_\alpha(X, F)$ and $\operatorname{Hf}_\alpha(X, F)$ of *Borel measurable* functions and *resolvably measurable* functions.

The algebras $\operatorname{Bas}(X)$, $\operatorname{Bos}(X)$ and $\operatorname{Hs}(X)$ serve as a starting point of an inductive definition of descriptive classes of sets. More precisely, if $\mathcal{F}$ is any of the families above, $\Sigma_2(\mathcal{F})$ consists of all countable unions of sets from $\mathcal{F}$ and $\Pi_2(\mathcal{F})$ of all countable intersections of sets from $\mathcal{F}$. Proceeding inductively, for any $\alpha \in (2, \omega_1)$ we let $\Sigma_\alpha(\mathcal{F})$ to be made of all countable unions of sets from $\bigcup_{1\le\beta<\alpha} \Pi_\beta(\mathcal{F})$ and $\Pi_\alpha(\mathcal{F})$ is made of all countable intersections of sets from $\bigcup_{1\le\beta<\alpha} \Sigma_\beta(\mathcal{F})$. The algebra $\Pi_\alpha(\mathcal{F}) \cap \Sigma_\alpha(\mathcal{F})$ is denoted as $\Delta_\alpha(\mathcal{F})$. The union of all created additive (or multiplicative) classes is then the σ-algebra generated by $\mathcal{F}$.

In case X is metrizable, all the resulting classes coincide (see [20, Proposition 3.4]). These classes characterize in terms of measurability the classes $\mathrm{Baf}_\alpha(X,F)$, $\mathrm{Bof}_\alpha(X,F)$ and $\mathrm{Hf}_\alpha(X,F)$ defined above. (We recall that, given a family $\mathcal{F}$ of sets in X, a mapping $f\colon X \to F$ is called *$\mathcal{F}$-measurable* if $f^{-1}(U) \in \mathcal{F}$ for every $U \subset F$ open.) Precisely, the following proposition is proved in [20, Theorem 5.2].

Proposition 1.3. *Let $f\colon X \to F$ be a function on a Tychonoff space X with values in a separable metrizable space F and $\alpha \in [1,\omega_1)$. Then the following assertions hold.*

- (a) *$f \in \mathrm{Baf}_\alpha(X)$ if and only if f is $\Sigma_{\alpha+1}(\mathrm{Bas}(X))$-measurable.*
- (b) *$f \in \mathrm{Bof}_\alpha(X)$ if and only if f is $\Sigma_{\alpha+1}(\mathrm{Bos}(X))$-measurable.*
- (c) *$f \in \mathrm{Hf}_\alpha(X)$ if and only if f is $\Sigma_{\alpha+1}(\mathrm{Hs}(X))$-measurable.*

If we take $\Phi_0 = \mathcal{C}(X,F)$, i.e., the family of all continuous mapping from X to F, and create the hierarchy of functions using pointwise limits, we have the following result (see [26, Theorem 3.7(i)]).

Proposition 1.4. *If X is a normal topological space and L is a convex subset of a separable Fréchet space, then $\mathcal{C}_1(X,L) = \mathrm{Baf}_1(X,L)$. Thus $\mathcal{C}_\alpha(X,L) = \mathrm{Baf}_\alpha(X,L)$ for each $\alpha \in [1,\omega_1)$.*

Next we need to recall a characterization of resolvable sets that asserts that *a subset H of a topological space X is resolvable if and only if there exist an ordinal κ and an increasing sequence of open sets $\emptyset = U_0 \subset U_1 \subset U_2 \subset \cdots \subset U_\gamma \subset \cdots \subset U_\kappa = X$ and $I \subset [0,\kappa)$ such that, for a limit ordinal $\gamma \in [0,\kappa]$, we have $\bigcup\{U_\lambda;\, \lambda < \gamma\} = U_\gamma$ and $H = \bigcup\{U_{\gamma+1} \setminus U_\gamma;\, \gamma \in I\}$* (see [4, Section 2] and references therein). We call such a transfinite sequence of open sets *regular* and such a description of a resolvable set a *regular representation* (this notion of regular representation is slightly more useful for us than the one used in [4, Section 2]).

2. Main results

Now we can formulate our main results. The first one is a vector-valued generalization of [12, Theorem 1.1] and [18, Corollaire 8].

Theorem 2.1. *Let X be a compact convex set, F be a Fréchet space and $f\colon X \to F$ be strongly affine. Then the following assertions hold.*

- (a) *If $\alpha \in [1,\omega_1)$ and F is separable, then $f|_{\overline{\mathrm{ext}\, X}} \in \mathrm{Hf}_\alpha(\overline{\mathrm{ext}\, X}, F)$ impliesf $\in \mathrm{Hf}_\alpha(X,F)$.*
- (b) *If $\alpha \in [1,\omega_1)$ and F is separable, $f|_{\overline{\mathrm{ext}\, X}} \in \mathrm{Bof}_\alpha(\overline{\mathrm{ext}\, X}, F)$ implies $f \in \mathrm{Bof}_\alpha(X,F)$.*
- (c) *If $\alpha \in [0,\omega_1)$ and $f|_{\overline{\mathrm{ext}\, X}} \in \mathcal{C}_\alpha(\overline{\mathrm{ext}\, X}, F)$, then $f(X)$ is separable and $f \in \mathcal{C}_\alpha(X,F)$.*

Theorem 2.2 is a vector-valued generalization of [12, Theorem 1.2]. We recall that a topological space is Lindelöf, if its any open cover has a countable subcover.

Theorem 2.2. *Let X be a compact convex set with $\mathrm{ext}\, X$ being Lindelöf and let F be a Fréchet space. Let $f\colon X \to F$ be a strongly affine function satisfying $f|_{\mathrm{ext}\, X} \in \mathcal{C}_\alpha(\mathrm{ext}\, X, F)$ for some $\alpha \in [0,\omega_1)$. Then $f \in \mathcal{C}_{1+\alpha}(X,F)$.*

The following Corollary 2.3 answers [7, Question 10.6] for the case (R) and (C); the case (S) was solved in [21]. It can be regarded as a vector analogue of [11, Theorem 1] and the result from [5].

We recall that a Banach space E is an L_1-predual if its dual is isometric to the space $L^1(Z,\mathcal{S},\mu)$ for some measure space $(Z,\mathcal{S},\mu)$. The families $\mathfrak{A}_{\mathrm{odd},\alpha}(X,F)$ and $\mathfrak{A}_{\mathrm{hom},\alpha}(X,F)$, where $X=(B_{E^*},w^*)$, are created from the set $\mathfrak{A}_{\mathrm{odd}}(X,F)$ of all continuous odd affine mappings from X to F in case E is real or from the set $\mathfrak{A}_{\mathrm{hom}}(X,F)$ of all continuous homogeneous affine mappings from X to F in case E and F are complex; see [7, Section 1.2].

Corollary 2.3. *Let X be a compact convex set with $\operatorname{ext} X$ being Lindelöf and let F be a Fréchet space. Let $f\colon \operatorname{ext} X\to F$ be a bounded continuous mapping.*

- (R) *If $X=(B_{E^*},w^*)$, where E is a real L_1-predual and f is odd, then f can be extended to a mapping from $\mathfrak{A}_{\mathrm{odd},1}(X,F)$.*
- (C) *If $X=(B_{E^*},w^*)$, where E is a complex L_1-predual, F is complex and f is homogeneous, then f can be extended to a mapping from $\mathfrak{A}_{\mathrm{hom},1}(X,F)$.*

The next result further improves the transfer of the class in Theorem 2.2 in the case when $\operatorname{ext} X$ is moreover a resolvable set.

Theorem 2.4. *Let X be a compact convex set with $\operatorname{ext} X$ being a Lindelöf resolvable set and let F be a Fréchet space. Let $f\colon X\to F$ be a strongly affine function satisfying $f|_{\operatorname{ext} X}\in\mathcal{C}_\alpha(\operatorname{ext} X,F)$ for some $\alpha\in[1,\omega_1)$. Then $f\in\mathcal{C}_\alpha(X,F)$.*

Using Theorem 2.4 we can answer [7, Question 10.7] affirmatively.

Corollary 2.5. *Let X be a compact convex set with $\operatorname{ext} X$ being Lindelöf, $\alpha\ge 1$, F a Fréchet space and $f\colon \operatorname{ext} X\to F$ a bounded mapping from $\mathcal{C}_\alpha(\operatorname{ext} X,F)$.*

- (S) *If X is a simplex, then f can be extended to a mapping in $\mathfrak{A}_\alpha(X,F)$.*
- (R) *If $X=(B_{E^*},w^*)$, where E is a real L_1-predual and f is odd, then f can be extended to a mapping from $\mathfrak{A}_{\mathrm{odd},\alpha}(X,F)$.*
- (C) *If $X=(B_{E^*},w^*)$, where E is a complex L_1-predual, F is complex and f is homogeneous, then f can be extended to a mapping from $\mathfrak{A}_{\mathrm{hom},\alpha}(X,F)$.*

3. Auxiliary results on compact convex sets with $\operatorname{ext} X$ being Lindelöf

The aim of this section is to provide auxiliary results on compact convex sets that are needed throughout the paper.

Lemma 3.1. *Let K be a compact topological space, F be a Fréchet space and $f\colon K\to F$ be a bounded function in $\mathcal{C}_\alpha(K,F)$ for some $\alpha\in[0,\omega_1)$. Then the function $\widetilde{f}\colon \mathcal{M}^1(K)\to F$ defined as $\widetilde{f}(\mu)=\mu(f)$, $\mu\in\mathcal{M}^1(K)$, is well defined and contained in $\mathcal{C}_\alpha(\mathcal{M}^1(K),F)$.*

Proof. Since $f\in\mathcal{C}_\alpha(K,F)$, its range is separable. Hence we may assume that F itself is separable. Lemma 1.1(a) now implies that $\widetilde{f}$ is well defined.

We consider first the case $\alpha=0$, i.e., the case when f is continuous. We want to show that $\widetilde{f}$ is continuous. Since $L=\overline{\operatorname{co}}f(K)$ is a compact convex subset of F and $\widetilde{f}(\mu)\in L$ for each $\mu\in\mathcal{M}^1(K)$ (see Lemma 1.1(b)), the original topology coincides with the weak topology on L. For any $\tau\in F^*$, the function $\tau\circ f$ is continuous on K, and thus the mapping

$$\mu\mapsto\mu(\tau\circ f)=\tau(\mu(f)),\quad \mu\in\mathcal{M}^1(K),$$

is continuous. Hence $\widetilde{f}\colon \mathcal{M}^1(K) \to (L, \text{weak})$ is continuous, and thus continuous with respect to the original topology.

If $\alpha > 0$, we can finish the proof by transfinite induction using Lemma 1.1(e) and Proposition 1.4. □

Lemma 3.2. *Let X be a compact convex set, F be a Fréchet space and $f\colon X \to F$ be a Baire measurable mapping. Then the following assertions hold.*

(a) *$f(X)$ is separable and there exists $\alpha \in [0, \omega_1)$ such that $f \in \mathcal{C}_\alpha(X, F)$.*
(b) *There exist a metrizable compact convex set Y, a continuous affine surjection $\varphi\colon X \to Y$ and a unique mapping $h \in \mathcal{C}_\alpha(Y, F)$ such that $f = h \circ \varphi$.*
(c) *If f is strongly affine, h is strongly affine as well.*

Proof. (a) See [7, Lemma 3.3].

(b) Let $E = \mathfrak{A}(X, \mathbb{F})$ and $\kappa\colon X \to E^*$ be the canonical evaluation embedding. Using [7, Lemma 6.2] there exists a separable space $E_1 \subset E$ and $h \in \mathcal{C}_\alpha(\pi(\kappa(X)), F)$ such that $f \circ \kappa^{-1} = h \circ \pi$ (here $\pi\colon E^* \to E_1^*$ is the restriction mapping). By setting $Y = \pi(\kappa(X))$ we obtain the desired metrizable compact convex set, whereas $\varphi = \pi \circ \kappa$.

(c) See [7, Lemma 6.4]. □

Lemma 3.3. *Let X be a compact convex set with $\operatorname{ext} X$ being Lindelöf, F be a Fréchet space and let $f\colon \operatorname{ext} X \to F$ be a bounded mapping in $\mathcal{C}_\alpha(\operatorname{ext} X, F)$ for some $\alpha \in [1, \omega_1)$. Let $L = \overline{\operatorname{co}} f(\operatorname{ext} X)$. Then there exists a Baire set $B \supset \operatorname{ext} X$ and a bounded mapping $g \in \mathcal{C}_\alpha(B, L)$ such that*

- *$g = f$ on $\operatorname{ext} X$,*
- *the function $\widetilde{g}\colon \mathcal{M}^1(B) \to F$ defined $\widetilde{g}(\mu) = \mu(g)$, $\mu \in \mathcal{M}^1(B)$, is in $\mathcal{C}_\alpha(\mathcal{M}^1(B), L)$.*

Proof. We will prove the result by transfinite induction on α. Suppose first that $\alpha = 1$, i.e., that $f \in \mathcal{C}_1(\operatorname{ext} X, F)$. Since L is separable and completely metrizable, by [6, Theorem 30 and Proposition 28] there is an extension $g\colon X \to L$ which is $\Sigma_2(\operatorname{Bas}(X))$-measurable. Proposition 1.4 now implies that $g \in \mathcal{C}_1(X, L)$. By setting $B = X$ we obtain from Lemma 3.1 that $\widetilde{g} \in \mathcal{C}_1(\mathcal{M}^1(B), L)$.

Assume now that $\alpha > 1$ and the assertion is valid for all $\beta \in [1, \alpha)$. Suppose that $f \in \mathcal{C}_\alpha(\operatorname{ext} X, F)$ is a bounded mapping and let $L = \overline{\operatorname{co}} f(\operatorname{ext} X)$. Then $f \in \mathcal{C}_\alpha(\operatorname{ext} X, L)$ by Proposition 1.4 (note that $\operatorname{ext} X$ is normal, being Lindelöf and regular), and thus there exist mappings $f_n \in \bigcup_{\beta<\alpha} \mathcal{C}_\beta(\operatorname{ext} X, L)$, $n \in \mathbb{N}$, converging pointwise to f on $\operatorname{ext} X$. For each $n \in \mathbb{N}$, let $B_n \supset \operatorname{ext} X$ be a Baire set and

$$g_n \in \mathcal{C}_{\beta_n}(B_n, \overline{\operatorname{co}} f_n(\operatorname{ext} X)) \subset \mathcal{C}_{\beta_n}(B_n, L)$$

for some $\beta_n < \alpha$ be such that

- $g_n = f_n$ on $\operatorname{ext} X$,
- the function $\widetilde{g_n}\colon \mathcal{M}^1(B_n) \to F$ defined by $\widetilde{g_n}(\mu) = \mu(g_n)$, $\mu \in \mathcal{M}^1(B_n)$, is in $\mathcal{C}_{\beta_n}(\mathcal{M}^1(B_n), L)$.

Let

$$B = \{x \in \bigcap_{n=1}^{\infty} B_n;\ (g_n(x)) \text{ converges}\}.$$

Let ρ be a compatible complete metric on F. Then

$$B=\{x\in\bigcap_{n=1}^{\infty}B_n;\ \forall k\in\mathbb{N}\ \exists l\in\mathbb{N}\ \forall m_1,m_2\geq l\colon \rho(g_{m_1}(x),g_{m_2}(x))<\frac{1}{k}\},$$

which gives that B is a Baire subset of X. Obviously, $B\supset \operatorname{ext}X$ and the function $g(x)=\lim g_n(x)$, $x\in B$, satisfies $g=f$ on $\operatorname{ext}X$. Let $\mu\in\mathcal{M}^1(B)$ be arbitrary. Since $g_n\to g$ on B, from Lemma 1.1(e) we obtain $\widetilde{g}(\mu)=\lim\widetilde{g_n}(\mu)$. By the inductive assumption, $\widetilde{g_n}\in\mathcal{C}_{\beta_n}(\mathcal{M}^1(B_n),L)$. Thus $\widetilde{g}\in\mathcal{C}_\alpha(\mathcal{M}^1(B),L)$. □

Lemma 3.4. *Let X be a compact convex set with $\operatorname{ext}X$ being Lindelöf, F be a Fréchet space and f be a bounded function in $\mathcal{C}_\alpha(\operatorname{ext}X,F)$ for some $\alpha\in[1,\omega_1)$. Let $L=\overline{\operatorname{co}}f(\operatorname{ext}X)$. Then there exist a Baire set $B\supset\operatorname{ext}X$ and a mapping $g\colon B\to L$ such that*

- *$g\in\mathcal{C}_\alpha(B,L)$,*
- *$g=f$ on $\operatorname{ext}X$,*
- *for each $\mu\in\mathcal{M}^1(B)$ with $r(\mu)\in B$ holds $g(r(\mu))=\mu(g)$,*
- *the function $\widetilde{g}\colon\mathcal{M}^1(B)\to F$ defined as $\widetilde{g}(\mu)=\mu(g)$, $\mu\in\mathcal{M}^1(B)$, is in $\mathcal{C}_\alpha(\mathcal{M}^1(B),L)$.*

Proof. Without loss of generality we may assume that F is real because otherwise we would consider on F only multiplication by real numbers.

Since $f\in\mathcal{C}_\alpha(\operatorname{ext}X,F)$ and $\operatorname{ext}X$ is Lindelöf, $f(\operatorname{ext}X)$ is separable. Thus we may assume that F itself is separable. Let (τ_n) in F^* be a sequence separating points of F (see [17, Chapter 3, Exercise 28]).

By [12, Lemma 4.6], for each $n\in\mathbb{N}$ there exist a Baire set $B_n\supset\operatorname{ext}X$ and a bounded Baire function $f_n\colon B_n\to\mathbb{R}$ such that

- $f_n=\tau_n\circ f$ on $\operatorname{ext}X$,
- for each $\mu\in\mathcal{M}^1(B_n)$ with $r(\mu)\in B_n$ it holds $f_n(r(\mu))=\mu(f_n)$.

By Lemma 3.3 there exist a Baire set $C\supset\operatorname{ext}X$ and a Baire function $h\in\mathcal{C}_\alpha(C,L)$ extending f such that the function $\widetilde{h}\colon\mathcal{M}^1(C)\to F$ defined by $\widetilde{h}(\mu)=\mu(h)$, $\mu\in\mathcal{M}^1(C)$, is in $\mathcal{C}_\alpha(\mathcal{M}^1(C),L)$.

Set

$$B=\{x\in C\cap\bigcap_{n=1}^{\infty}B_n;\ \tau_n(h(x))=f_n(x),n\in\mathbb{N}\}\quad\text{and}\quad g=h|_B.$$

Then B is a Baire set containing $\operatorname{ext}X$. Let $\mu\in\mathcal{M}^1(B)$ with $r(\mu)\in B$ be given. Then for each $n\in\mathbb{N}$ we have

$$\begin{aligned}\tau_n(\mu(g))&=\int_B\tau_n(h(x))\,\mathrm{d}\mu(x)=\int_B f_n(x)\,\mathrm{d}\mu(x)=f_n(r(\mu))\\&=\tau_n(h(r(\mu)))=\tau_n(g(r(\mu))).\end{aligned}$$

Thus $\mu(g)=g(r(\mu))$. Since $\widetilde{g}(\mu)=\widetilde{h}(\mu)$ for $\mu\in\mathcal{M}^1(B)$, we obtain that $\widetilde{g}\in\mathcal{C}_\alpha(\mathcal{M}^1(B),L)$. This finishes the proof. □

Before the proof of the following lemma we recall that a topological space is *K-analytic* if it is an image of a Polish space under an upper semicontinuous compact valued mapping (see [16, Section 2.1]).

Lemma 3.5. *Let X be a compact convex set with $\operatorname{ext} X$ being Lindelöf, F be a Fréchet space and $f\colon X \to F$ be strongly affine such that $f|_{\operatorname{ext} X} \in \mathcal{C}_\alpha(\operatorname{ext} X, F)$ for some $\alpha \in [1, \omega_1)$. Then f is a Baire measurable mapping.*

Proof. Let $L = \overline{\operatorname{co}} f(X)$. Let $B \supset \operatorname{ext} X$, $g\colon B \to L$ and $\widetilde{g}\colon \mathcal{M}^1(B) \to L$ be as in Lemma 3.4. We claim that $g = f$ on B. To this end, let $x \in B$ be fixed. We choose a maximal measure $\mu \in \mathcal{M}_x(X)$. Then μ is supported by B and $f = g$ μ-almost everywhere.

(Indeed, the set $\{y \in B;\ f(y) = g(y)\}$ is μ-measurable and contains $\operatorname{ext} X$. Since $\operatorname{ext} X$ is Lindelöf, the assertion follows from [12, Lemma 4.4].)

Thus

$$g(x) = \mu(g) = \mu(f) = f(x).$$

Now we consider the barycentric mapping $r\colon \mathcal{M}^1(B) \to X$. Then r is a continuous surjection. We want to show that f is Baire measurable. To this end, let $U \subset F$ be an open set. By strong affinity and the fact that $f = g$ on B, $\widetilde{g} = f \circ r$. Hence

$$f^{-1}(U) = r(\widetilde{g}^{-1}(U)) \quad \text{and} \quad f^{-1}(F \setminus U) = r(\widetilde{g}^{-1}(F \setminus U)).$$

Since $\widetilde{g}$ is Baire and $\mathcal{M}^1(B) = \{\mu \in \mathcal{M}^1(K);\ \mu(B) = 1\}$, as a Baire subset of a K-analytic space, is a K-analytic space, both the sets $\widetilde{g}^{-1}(U)$ and $\widetilde{g}^{-1}(F \setminus U)$ are K-analytic as well. Since r is continuous, both the sets $f^{-1}(U)$ and $f^{-1}(F \setminus U)$ are K-analytic and thus, being disjoint, they are Baire (see [16, Theorem 3.3.1]). □

4. Proof of Theorem 2.1

We start the proof of Theorem 2.1 by the following simple approximation lemma. If $\mathcal{A}$ is a family of sets in a set X, we recall that $\Sigma_2(\mathcal{A})$ stands for the family of all countable unions of elements from $\mathcal{A}$.

Lemma 4.1. *Let X be a set with an algebra $\mathcal{A}$ and let (F, ρ) be a separable pseudometric space. Let $f\colon X \to F$ be a $\Sigma_2(\mathcal{A})$-measurable mapping. Then for each $\varepsilon > 0$ there exists a disjoint partition $\{A_n;\ n \in \mathbb{N}\}$ of X containing sets from $\Sigma_2(\mathcal{A})$ and elements $\{y_n;\ n \in \mathbb{N}\}$ in F such that the function*

$$g(x) = y_n, \quad x \in A_n, n \in \mathbb{N},$$

satisfies $\sup_{x \in X} \rho(f(x), g(x)) < \varepsilon$.

Proof. If $\{y_n;\ n \in \mathbb{N}\}$ is a dense set in F, we consider the family $\{B_\rho(y_n, \varepsilon);\ n \in \mathbb{N}\}$ (here $B_\rho(y, \varepsilon)$ denotes the open ball centered at y with radius ε). Then it covers F and thus $\{f^{-1}(B_\rho(y_n, \varepsilon));\ n \in \mathbb{N}\}$ is a cover of X consisting of sets from $\Sigma_2(\mathcal{A})$. Now it is enough to use the standard reduction lemma to find sets $A_n \in \Sigma_2(\mathcal{A})$, $n \in \mathbb{N}$, such that they form a disjoint partition of X and $A_n \subset f^{-1}(B_\rho(y_n, \varepsilon))$, $n \in \mathbb{N}$ (see e.g. [20, Proposition 2.3(f)]). This finishes the proof. □

Lemma 4.2. *Let X be a set with an algebra $\mathcal{A}$ and let F be a separable Fréchet space over $\mathbb{F}$. Then the following assertions hold.*

(a) *The family of all $\Sigma_2(\mathcal{A})$-measurable functions is a vector space.*
(b) *Let ρ be a continuous seminorm on F and let $f_n, f\colon X \to F$ be such that $\rho \circ f_n$ is $\Sigma_2(\mathcal{A})$ measurable and $\sup_{x \in X} \rho(f_n(x) - f(x)) \to 0$. Then $\rho \circ f$ is $\Sigma_2(\mathcal{A})$-measurable.*
(c) *If $f\colon X \to \mathbb{F}$ is $\Sigma_2(\mathcal{A})$-measurable and $y \in F$, then the function $g\colon X \to F$ defined as $g(x) = f(x)y$, $x \in X$, is $\Sigma_2(\mathcal{A})$-measurable.*

Proof. (a) Let f, g be $\Sigma_2(\mathcal{A})$-measurable functions from X to F. Since any open set in $F \times F$ is a countable union of open rectangles, the mapping $f \times g\colon X \to F \times F$ defined as $(f \times g)(x) = (f(x), g(x))$, $x \in X$, is $\Sigma_2(\mathcal{A})$-measurable. From the continuity of the operation $+\colon F \times F \to F$ we infer the $\Sigma_2(\mathcal{A})$-measurability of $f + g$.

If f is $\Sigma_2(\mathcal{A})$-measurable and $\lambda \in \mathbb{F} \setminus \{0\}$, for an open set $U \subset F$ we have

$$(\lambda f)^{-1}(U) = f^{-1}(\lambda^{-1}U) \in \Sigma_2(\mathcal{A}).$$

If $\lambda = 0$, λf is clearly $\Sigma_2(\mathcal{A})$-measurable.

(b) Let ρ be a continuous seminorm on F and let $f_n, f\colon X \to F$ be a sin the premise. Without loss of generality we may assume that $\sup_{x\in X} \rho(f_n(x) - f(x)) < \frac{1}{2^n}$, $n \in \mathbb{N}$.

Let $U \subset \mathbb{R}$ be open and let

$$U_k = \{\lambda \in \mathbb{R}; \operatorname{dist}(\lambda, \mathbb{R} \setminus U) > \frac{1}{2^k}\}, \quad k \in \mathbb{N}.$$

Then

$$(\rho \circ f)^{-1}(U) = \bigcup_{k=1}^{\infty} \bigcup_{n=k+1}^{\infty} (\rho \circ f_n)^{-1}(U_k) \in \Sigma_2(\mathcal{A})$$

and $\rho \circ f$ is $\Sigma_2(\mathcal{A})$-measurable.

(c) Consider the mapping $m\colon \mathbb{F} \to F$ defined as $m(\lambda) = \lambda y$, $\lambda \in \mathbb{F}$. Then m is continuous and $g = m \circ f$. Hence g is $\Sigma_2(\mathcal{A})$-measurable. ☐

The key Lemma 4.3 uses the following notation. If $\mathcal{A}$ is a class of sets in topological spaces and K is a topological space, the symbol $\mathcal{A}(K)$ stands for the family of all subsets in K that belong to $\mathcal{A}$.

Lemma 4.3. *Let $\mathcal{A}$ be an algebra of sets in topological spaces and let K be a compact space. Assume that the following properties are satisfied.*

1. *Any element of $\mathcal{A}(K)$ is μ-measurable for each $\mu \in \mathcal{M}^1(K)$.*
2. *If $f\colon K \to \mathbb{R}$ is a bounded $\Sigma_2(\mathcal{A}(K))$-measurable function, then the function $\widetilde{f}\colon \mathcal{M}^1(K) \to \mathbb{R}$ defined as*

$$\widetilde{f}(\mu) = \mu(f), \quad \mu \in \mathcal{M}^1(K),$$

is $\Sigma_2(\mathcal{A}(\mathcal{M}^1(K)))$-measurable.

Let F be a separable Fréchet space and $f\colon K \to F$ be a bounded $\Sigma_2(\mathcal{A}(K))$-measurable mapping. Then the function $\widetilde{f}\colon \mathcal{M}^1(K) \to F$ defined as

$$\widetilde{f}(\mu) = \mu(f), \quad \mu \in \mathcal{M}^1(K),$$

is well defined and $\Sigma_2(\mathcal{A}(\mathcal{M}^1(K)))$-measurable.

Proof. Without loss of generality we may assume that F is a real vector space. Let $L = \overline{\text{co}} f(K)$. Then L is a bounded set. For each $\mu \in \mathcal{M}^1(K)$, the mapping f is by (1) weakly μ-measurable, and thus by Lemma 1.1(a) it is μ-integrable. Hence $\widetilde{f}$ is a well defined mapping with $\widetilde{f}(\mathcal{M}^1(K)) \subset L$ (see Lemma 1.1(b)).

By the separability of F it is enough to prove that $\rho \circ \widetilde{f}$ is $\Sigma_2(\mathcal{A}(\mathcal{M}^1(K)))$-measurable for any continuous seminorm ρ on F. Let ρ be a continuous seminorm.

Let $\varepsilon > 0$ be arbitrary. By Lemma 4.1 there exist a partition $\{A_n;\, n \in \mathbb{N}\}$ of X consisting of elements of $\Sigma_2(\mathcal{A}(K))$ and vectors $\{y_n;\, n \in \mathbb{N}\} \subset L$ such that the mapping

$$g(x) = \sum_{n=1}^{\infty} y_n \chi_{A_n}(x), \quad x \in K, \tag{4.1}$$

satisfies $\sup_{x\in K} \rho(f(x) - g(x)) < \varepsilon$. Let $\widetilde{g}\colon \mathcal{M}^1(K) \to Y$ be defined as

$$\widetilde{g}(\mu) = \mu(g) = \sum_{n=1}^{\infty} y_n \mu(A_n), \quad \mu \in \mathcal{M}^1(K).$$

By Lemma 1.1(d), we have for $\mu \in \mathcal{M}^1(K)$ estimate

$$\rho\left(\widetilde{f}(\mu) - \widetilde{g}(\mu)\right) \le \int_K \rho(f(x) - g(x))\, \mathrm{d}\mu(x) \le \varepsilon.$$

By choosing $\varepsilon_n = \frac{1}{n}$ we construct a sequence $\{g_n\}$ of mappings of type (4.1) such that $\widetilde{g_n}$ converges ρ-uniformly to $\widetilde{f}$. By Lemma 4.2(b), the $\Sigma_2(\mathcal{A}(\mathcal{M}^1(K)))$-measurability of $\rho \circ \widetilde{f}$ will be thus ensured by the $\Sigma_2(\mathcal{A}(\mathcal{M}^1(K)))$-measurability of mappings $\rho \circ \widetilde{g_n}$.

Let g be a mapping of the form (4.1). To check the $\Sigma_2(\mathcal{A}(\mathcal{M}^1(K)))$-measurability of $\rho \circ \widetilde{g}$, it is enough to verify the $\Sigma_2(\mathcal{A}(\mathcal{M}^1(K)))$-measurability of the mapping $\widetilde{g}\colon \mathcal{M}^1(K) \to (F, \tau_\rho)$, where τ_ρ is the topology generated by the seminorm ρ.

Let $\{U_k;\, k \in \mathbb{N}\}$ be an open basis of the topology τ_ρ. (Such a basis exists because F, and thus also (F, τ_ρ), is separable.) For each $m \in \mathbb{N}$ set

$$\widetilde{g}_m(\mu) = \sum_{n=1}^{m} y_n \mu(A_n) \quad \text{and} \quad h_m(\mu) = \mu(A_1 \cup \cdots \cup A_m), \qquad \mu \in \mathcal{M}^1(K).$$

For each $k, m, j \in \mathbb{N}$ we consider the set

$$A_{k,m,j} = \widetilde{g}_m^{-1}(U_k) \cap h_m^{-1}\left((1 - \frac{1}{j}, \infty)\right).$$

Any τ_ρ-open set is originally open, and thus by Lemma 4.2(a),(c) and assumption (2), each set $A_{k,m,j} \in \Sigma_2(\mathcal{A}(\mathcal{M}^1(K)))$.

Let now $U \subset F$ be a τ_ρ-open set. We set

$$A = \bigcup_{k,m,j=1}^{\infty} \left\{A_{k,m,j};\, A_{k,m,j} \subset \widetilde{g}^{-1}(U)\right\}.$$

Then A is in $\Sigma_2(\mathcal{A}(\mathcal{M}^1(K)))$. We need to show that $A = \widetilde{g}^{-1}(U)$.

Clearly, $A \subset \widetilde{g}^{-1}(U)$. To check the converse inclusion, let $M > 0$ satisfy $\rho(y) \le M$ for $y \in L$ and let $\mu \in \widetilde{g}^{-1}(U)$ be given. Choose $\delta > 0$ such that

$$B_\rho(\widetilde{g}(\mu), 4\delta) = \{y \in F;\, \rho(\widetilde{g}(\mu) - y) < 4\delta\} \subset U$$

and let $k \in \mathbb{N}$ be such that

$$\mu \in U_k \subset B_\rho(\widetilde{g}(\mu), \delta).$$

Let $j \in \mathbb{N}$ be chosen such that $\frac{L}{j} < \delta$ and let $m \in \mathbb{N}$ satisfy $\mu(A_1 \cup \cdots \cup A_m) > 1 - \frac{1}{j}$. Since

$$\widetilde{g}_m(\mu) \to \widetilde{g}(\mu)$$

by Lemma 1.1(e), we can further enlarge m in such a way that $\widetilde{g}_m(\mu) \in U_k$. Then

$$\mu \in A_{k,m,j}.$$

We need to verify that $A_{k,m,j} \subset \widetilde{g}^{-1}(U)$. To this end, let $\nu \in A_{k,m,j}$ be given. Then we have due to Lemma 1.1(d)

$$\rho\left(\sum_{n=m+1}^{\infty} a_n \nu(A_n)\right) \leq \int_{\bigcup_{n=m+1}^{\infty} A_n} L \,\mathrm{d}\nu < \frac{L}{j} < \delta,$$

and analogously

$$\rho\left(\sum_{n=m+1}^{\infty} a_n \mu(A_n)\right) < \delta.$$

Since $\widetilde{g}_m(\mu), \widetilde{g}_m(\nu) \in U_k$ and $\operatorname{diam}_\rho U_k < 2\delta$, we obtain

$$\begin{aligned}
&\rho\left(\widetilde{g}(\mu) - \widetilde{g}(\nu)\right) = \\
&= \rho\left(\sum_{n=1}^{m} a_n \mu(A_n) - \sum_{n=1}^{m} a_n \nu(A_n) + \sum_{n=m+1}^{\infty} a_n \mu(A_n) - \sum_{n=m+1}^{\infty} \nu(A_n)\right) \\
&\leq \rho\left(\sum_{n=1}^{m} a_n \mu(A_n) - \sum_{n=1}^{m} a_n \nu(A_n)\right) + 2\delta < 4\delta,
\end{aligned}$$

which implies

$$\widetilde{g}(\nu) \in B(\widetilde{g}(\mu), 4\delta) \subset U.$$

Hence $A_{k,m,j} \subset \widetilde{g}^{-1}(U)$.

Thus A is a set in $\Sigma_2(\mathcal{A}(\mathcal{M}^1(K)))$ and equals $\widetilde{g}^{-1}(U)$. Hence $\rho\circ\widetilde{g}$ is $\Sigma_2(\mathcal{A}(\mathcal{M}^1(K)))$-measurable and the proof is finished. □

Lemma 4.4. *Let K be a compact topological space, F be a separable Fréchet space and $f\colon K \to F$ be bounded.*

(a) *If $\alpha \in [1, \omega_1)$ and $f \in \mathrm{Hf}_\alpha(K, F)$, then the function $\widetilde{f}\colon \mathcal{M}^1(K) \to F$ defined as $\widetilde{f}(\mu) = \mu(f)$, $\mu \in \mathcal{M}^1(K)$, is well defined and belongs to the family* $\mathrm{Hf}_\alpha(\mathcal{M}^1(K), F)$.

(b) *If $\alpha \in [1, \omega_1)$ and $f \in \mathrm{Bof}_\alpha(K, F)$, then the function $\widetilde{f}\colon \mathcal{M}^1(K) \to F$ defined as $\widetilde{f}(\mu) = \mu(f)$, $\mu \in \mathcal{M}^1(K)$, is well defined and belongs to the family* $\mathrm{Bof}_\alpha(\mathcal{M}^1(K), F)$.

Proof. (a) Let $f\colon K \to F$ be a bounded function with values in a separable Fréchet space F such that $f \in \mathrm{Hf}_\alpha(K, F)$ for some $\alpha \in [1, \omega_1)$. Then $\tau \circ f$ is μ-measurable for each $\mu \in \mathcal{M}^1(K)$ and $\tau \in F^*$ (see [8, Lemma 4.4]), i.e., f is weakly μ-measurable for each $\mu \in \mathcal{M}^1(K)$. Since F is separable, $\widetilde{f}$ is well defined by Lemma 1.1(a).

By [20, Theorem 5.2], f is $\Sigma_{\alpha+1}(\mathrm{Hs}(K))$-measurable. Let $\mathcal{A}$ denote the algebra $\Delta_{\alpha+1}(\mathrm{Hs})$. By [20, Proposition 2.3(e)], f is $\Sigma_2(\mathcal{A}(K))$-measurable. It follows from [12, Lemma 3.3(a)] that assumption (2) of Proposition 4.3 is satisfied for $\mathcal{A}$. Hence the function $\widetilde{f}\colon \mathcal{M}^1(K) \to F$ defined as $\widetilde{f}(\mu) = \mu(f)$, $\mu \in \mathcal{M}^1(K)$, is $\Sigma_2(\mathcal{A}(\mathcal{M}^1(K)))$-measurable. Thus $\widetilde{f} \in \mathrm{Hf}_\alpha(\mathcal{M}^1(K), F)$, again by [20, Theorem 5.2].

(b) The proof is analogous to the proof of (a), the only difference is that we use the algebra $\Delta_{\alpha+1}(\mathrm{Bos})$ and [12, Lemma 3.3(b)]. □

Proof of Theorem 2.1. (a) Let $f\colon X \to F$ be a strongly affine function such that $f|_{\overline{\operatorname{ext} X}} \in \mathrm{Hf}_\alpha(\overline{\operatorname{ext} X}, F)$ for some $\alpha \in [1, \omega_1)$. By Lemma 4.4(a), the function $\widetilde{f}\colon \mathcal{M}^1(\overline{\operatorname{ext} X}) \to F$ defined as $\widetilde{f}(\mu) = \mu(f)$, $\mu \in \mathcal{M}^1(\overline{\operatorname{ext} X})$, is in the family $\mathrm{Hf}_\alpha(\mathcal{M}^1(\overline{\operatorname{ext} X}), F)$.

Consider now the barycentric mapping $r\colon \mathcal{M}^1(\overline{\operatorname{ext} X}) \to X$. By the strong affinity of f, we have $\widetilde{f} = f \circ r$ on $\mathcal{M}^1(\overline{\operatorname{ext} X})$. By [4, Theorem 4], $f \in \mathrm{Hf}_\alpha(X, F)$.

(b) The proof is analogous to the proof of (a), we just use Lemma 4.4(b) and [4, Theorem 10].

(c) We first notice that $f(\overline{\operatorname{ext} X})$ is separable by Lemma 3.2(a). Thus $\widetilde{f}$ is well defined by Lemma 1.1(a). Now we use [16, Theorem 5.9.13] to finish the proof as in the cases (a) and (b)

□

5. Proof of Theorem 2.2 and Corollary 2.3

The first step in the proof of Theorem 2.2 is the following result on metrizable reduction. Its analogue appeared e.g. in [3], [16, Theorem 5.9.13], [25, Theorem 1], [2] or [13, Theorem 9.12]. The argument is modelled along the lines of the proofs of [21, Lemma 3.1] and [12, Lemma 5.1].

Lemma 5.1. *Let X be a compact convex set, F be a Fréchet space and $f\colon X \to F$ be a Baire measurable mapping such that $f|_{\operatorname{ext} X} \in \mathcal{C}_\alpha(\operatorname{ext} X, F)$ for some $\alpha \in [0, \omega_1)$. Then there exist a metrizable compact convex set Y, an affine continuous surjection $\varphi\colon X \to Y$ and a Baire measurable mapping $h\colon Y \to F$ such that $f = h \circ \varphi$ and $h|_{\operatorname{ext} Y} \in \mathcal{C}_\alpha(\operatorname{ext} Y, F)$.*

Proof. Since f is Baire measurable, $L = \overline{\operatorname{aco}} f(X)$ is separable by Lemma 3.2(a). Hence we may assume that F itself is separable. Let $\beta \in [0, \omega_1)$ be such that $f \in \mathcal{C}_\beta(X, F)$ (see Lemma 3.2(a)). By Lemma 3.2(b), there exist a metrizable compact convex set X_1, a continuous surjection $\varphi_1\colon X \to X_1$ and $h_1 \in \mathcal{C}_\beta(X_1, F)$ such that $f = h_1 \circ \varphi_1$.

Let $\{U_n;\, n \in \mathbb{N}\}$ be a countable basis of the topology of F. For each $n \in \mathbb{N}$ we select a continuous function $g_n\colon F \to [0, 1]$ such that $U_n = g_n^{-1}((0, 1])$. Let $\mathcal{G} \subset \mathcal{C}(\operatorname{ext} X, L)$ be a countable family satisfying $f|_{\operatorname{ext} X} \in \mathcal{G}_\alpha$, see the definition of Φ_α in Section 1.5.

For each $g \in \mathcal{G}$ and $n \in \mathbb{N}$ we choose using [12, Lemma 4.5] finite families $\mathcal{U}^k_{g,n}, \mathcal{L}^k_{g,n} \subset \mathfrak{A}(X, \mathbb{R})$, $k \in \mathbb{N}$, such that we have for functions $u^k_{g,n} = \inf \mathcal{U}^k_{g,n}$ and $l^k_{g,n} = \sup \mathcal{L}^k_{g,n}$ properties

- $0 \le u^k_{g,n} \le 1$, $0 \le l^k_{g,n} \le 1$,
- $\lim_{k\to\infty} u^k_{g,n}(x) = \lim_{k\to\infty} l^k_{g,n}(x) = g_n(g(x))$ for each $x \in \operatorname{ext} X$,
- $(l^k_{g,n})_{k=1}^\infty$ is increasing and $(u^k_{g,n})_{k=1}^\infty$ is decreasing.

Let

$$\mathcal{F} = \bigcup_{g \in \mathcal{G}, n, k \in \mathbb{N}} \mathcal{U}^k_{g,n} \cup \mathcal{L}^k_{g,n}.$$

Consider the compact convex set

$$\left(\prod_{g \in \mathcal{F}} g(X) \right) \times X_1$$

and the projection ψ on the second coordinate. Let

$$\varphi(x) = ((g(x))_{g\in\mathcal{F}}, \varphi_1(x)), \quad x \in X,$$

and $Y = \varphi(X)$.

For each function $g \in \mathcal{F}$ there exists a unique function $\widetilde{g} \in \mathfrak{A}(Y, \mathbb{R})$ such that $g = \widetilde{g} \circ \varphi$. For each $g \in \mathcal{G}$, $n, k \in \mathbb{N}$, let $\widetilde{\mathcal{U}}^k_{g,n} \subset \mathfrak{A}(Y, \mathbb{R})$ be such that

$$\mathcal{U}^k_{g,n} = \{\widetilde{u} \circ \varphi;\, \widetilde{u} \in \widetilde{\mathcal{U}}^k_{g,n}\}.$$

Analogously we pick $\widetilde{\mathcal{L}}^k_{g,n}$. Then for $\widetilde{u}^k_{g,n} = \inf \widetilde{\mathcal{U}}^k_{g,n}$ and $\widetilde{l}^k_{g,n} = \sup \widetilde{\mathcal{L}}^k_{g,n}$ hold

$$u^k_{g,n} = \widetilde{u}^k_{g,n} \circ \varphi \quad \text{and} \quad l^k_{g,n} = \widetilde{l}^k_{g,n} \circ \varphi.$$

Given $y \in \operatorname{ext} Y$, we pick $x \in \operatorname{ext} X \cap \varphi^{-1}(y)$. Then

$$\begin{aligned} \lim_{k\to\infty} \widetilde{u}^k_{g,n}(y) &= \lim_{k\to\infty} \widetilde{u}^k_{g,n}(\varphi(x)) = \lim_{k\to\infty} u^k_{g,n}(x) = g_n(g(x)) \quad \text{and} \\ \lim_{k\to\infty} \widetilde{l}^k_{g,n}(y) &= \lim_{k\to\infty} \widetilde{l}^k_{g,n}(\varphi(x)) = \lim_{k\to\infty} l^k_{g,n}(x) = g_n(g(x)). \end{aligned}$$

Thus $(\widetilde{u}^k_{g,n})_{k=1}^\infty$ is a decreasing sequence on $\operatorname{ext} Y$, $(\widetilde{l}^k_{g,n})_{k=1}^\infty$ is increasing on $\operatorname{ext} Y$ and both converge to a common limit $\widetilde{g}_{g,n} \colon \operatorname{ext} Y \to \mathbb{R}$ given by

$$\widetilde{g}_{g,n}(y) = \lim_{k\to\infty} \widetilde{u}^k_{g,n}(y), \quad y \in \operatorname{ext} Y.$$

Then $\widetilde{g}_{g,n}$ is a continuous function on $\operatorname{ext} Y$ with values in $[0, 1]$ that satisfies

$$\widetilde{g}_{g,n}(y) = g_n(g(x)), \quad x \in \operatorname{ext} X \cap \varphi^{-1}(y). \tag{5.1}$$

For each $g \in \mathcal{G}$ we now define a function $\widetilde{g} \colon \operatorname{ext} Y \to L$ by the formula

$$\widetilde{g}(y) = g(x), \quad x \in \operatorname{ext} X \cap \varphi^{-1}(y), \quad y \in \operatorname{ext} Y.$$

Firstly we check that the definition is correct. Indeed, if $y = \varphi(x_1) = \varphi(x_2)$ for some $y \in \operatorname{ext} Y$, $x_1, x_2 \in \operatorname{ext} X$ and $g(x_1) \neq g(x_2)$, there exists $n \in \mathbb{N}$ such that $g_n(g(x_1)) \neq g_n(g(x_2))$. But then we have by (5.1) a contradiction since

$$g_n(g(x_1)) = \widetilde{g}_{g,n}(y) = g_n(g(x_2)).$$

Secondly, $\widetilde{g}$ is continuous. To see this, fix $n \in \mathbb{N}$. Then the equalities

$$\begin{aligned} \widetilde{g}^{-1}(U_n) &= \{y \in \operatorname{ext} Y;\, \widetilde{g}(y) \in U_n\} = \{y \in \operatorname{ext} Y;\, g_n(\widetilde{g}(y)) > 0\} \\ &= \{y \in \operatorname{ext} Y;\, \exists x \in \operatorname{ext} X \cap \varphi^{-1}(y) \colon g_n(g(x)) > 0\} \\ &= \{y \in \operatorname{ext} Y;\, \widetilde{g}_{g,n}(y) > 0\} = \widetilde{g}_{g,n}^{-1}(U_n) \end{aligned}$$

implies that $\widetilde{g}^{-1}(U_n)$ is open. Hence $\widetilde{g}$ is continuous.

Denote $\widetilde{\mathcal{G}} = \{\widetilde{g};\, g \in \mathcal{G}\}$.

Claim. Now we claim that for each $\gamma \in [0, \alpha]$ and $h \in \mathcal{G}_\gamma$ there is a function $\widetilde{h} \in \widetilde{\mathcal{G}}_\gamma$ such that $h = \widetilde{h} \circ \varphi$ on $\operatorname{ext} X \cap \varphi^{-1}(\operatorname{ext} Y)$. To verify this we proceed by transfinite induction.

Assume that it holds for all $\gamma' < \gamma$ for some $\gamma \leq \alpha$ and that we are given $h \in \mathcal{G}_\gamma$. Let $\gamma_n < \gamma$ and $h_n \in \mathcal{G}_{\gamma_n}$, $n \in \mathbb{N}$, be such that $h = \lim h_n$. By the inductive assumption, there exist $\widetilde{h}_n \in \widetilde{\mathcal{G}}_{\gamma_n}$ satisfying $h_n = \widetilde{h}_n \circ \varphi$ on $\operatorname{ext} X \cap \varphi^{-1}(\operatorname{ext} Y)$. Then the sequence $(\widetilde{h}_n(y))$ converges for every point $y \in \operatorname{ext} Y$. Hence we may define a function $\widetilde{h} \in \widetilde{\mathcal{G}}_\gamma$ by

$$\widetilde{h}(y) = \lim_{n\to\infty} \widetilde{h}_n(y), \quad y \in \operatorname{ext} Y,$$

and then, for every $y \in \operatorname{ext} Y$ and $x \in \varphi^{-1}(y) \cap \operatorname{ext} X$,

$$\widetilde{h}(y) = \lim_{n\to\infty} \widetilde{h}_n(y) = \lim_{n\to\infty} h_n(x) = h(x).$$

This proves the claim.

Consider now the function $h = h_1 \circ \psi$. Then h is a Baire measurable mapping on Y. We want to prove that $h|_{\operatorname{ext} Y} \in \widetilde{\mathcal{G}}_\alpha$. By the claim there exists $\widetilde{g} \in \widetilde{\mathcal{G}}_\alpha$ such that

$$f(x) = \widetilde{g}(\varphi(x)), \quad x \in \operatorname{ext} X \cap \varphi^{-1}(\operatorname{ext} Y).$$

For $y \in \operatorname{ext} Y$ we select $x \in \operatorname{ext} X \cap \varphi^{-1}(y)$. Then have

$$\widetilde{g}(y) = \widetilde{g}(\varphi(x)) = f(x) = h_1(\varphi_1(x)) = h_1(\psi(\varphi(x))) = h(\varphi(x)) = h(y).$$

Hence $h = \widetilde{g} \in \widetilde{\mathcal{G}}_\alpha$ as required.

□

The following lemma is a variant of [7, Lemma 6.1].

Lemma 5.2. *Let X be a compact space, F a Fréchet space over $\mathbb{F}$ and $U\colon X \to \mathcal{M}^1(X)$. If $U \in \mathcal{C}_\alpha(X, \mathcal{M}^1(X))$ and $f \in \mathcal{C}_\beta(X, F)$ is bounded, then $Uf\colon X \to F$ defined by*

$$Uf(x) = \int_X f \,\mathrm{d}U(x), \quad x \in X,$$

satisfies $Uf \in \mathcal{C}_{\alpha+\beta}(X, \overline{\operatorname{aco}} f(X))$.

Proof. We proceed by transfinite induction along β. Let $\beta = 0$, i.e., f is continuous. If $\alpha = 0$, then $f(X)$ is a compact subset of F and thus $L = \overline{\operatorname{aco}} f(X)$ is an absolutely convex compact set in F. By Lemma 1.1(c), $Uf(X) \subset L$. We want to check the continuity of Uf. To this end, let $\tau \in F^*$ be arbitrary. Then

$$\tau(Uf(x)) = \int_X (\tau \circ f) \,\mathrm{d}U(x) = U(x)(\tau \circ f), \quad x \in X.$$

Since U is continuous and $\tau \circ f \in \mathcal{C}(X, \mathbb{F})$, the mapping $x \mapsto U(x)(\tau \circ f)$ is continuous on X. Hence $Uf\colon X \to (L, \text{weak})$ is continuous. Since L is compact, the original topology of F coincides on L with the weak topology . Hence $Uf \in \mathcal{C}(X, L)$.

Assume now that the assertion is true for each $\alpha < \gamma$, where $\gamma > 0$. Let $U \in \mathcal{C}_\gamma(X, \mathcal{M}^1(X))$ be given. Then there exists a sequence (U_n) in $\bigcup_{\alpha<\gamma} \mathcal{C}_\alpha(X, \mathcal{M}^1(X))$ such that $U_n(x) \to U(x)$ in $\mathcal{M}^1(X)$ for each $x \in X$. Then for any $\tau \in F^*$ we have

$$\tau(U_n f(x)) = U_n(x)(\tau \circ f) \to U(x)(\tau \circ f) = \tau(U(f(x))).$$

Thus $U_n f(x) \to Uf(x)$ weakly in F. Since the sequence is contained in the compact set L, $U_n f(x) \to Uf(x)$ in F. Hence $Uf \in \mathcal{C}_{\gamma+0}(X, F)$.

Let now the assertion hold true for all $\beta < \gamma$, where $\gamma > 0$. Let $U \in \mathcal{C}_\alpha(X, \mathcal{M}^1(X))$ and let $f \in \mathcal{C}_\gamma(X, F)$ be bounded. Then $L = \overline{\operatorname{aco}} f(X)$ is bounded and $f \in \mathcal{C}_\gamma(X, L)$ due to Proposition 1.4. Thus there exists a sequence $(f_n) \in \bigcup_{\beta<\gamma} \mathcal{C}_\beta(X, L)$ pointwise converging to f. Then $Uf_n \in \bigcup_{\beta<\gamma} \mathcal{C}_{\alpha+\beta}(X, F)$ by the induction hypothesis and $Uf_n \to Uf$ by Lemma 1.1(e). Hence $Uf \in \mathcal{C}_{\alpha+\gamma}(X, L)$. □

Proof of Theorem 2.2. Let f be as in the premise. By Lemma 3.5 and Lemma 3.2(a), $f \in \mathcal{C}_\beta(X, F)$ for some $\beta < \omega_1$. Hence we may use Lemma 5.1 to obtain relevant Y, h and φ. By [23, Théorème 1] (see also [13, Theorem 11.41]) there exists a mapping $U\colon y \mapsto U_y$, $y \in Y$, such that

(a) U_y is a maximal measure in $\mathcal{M}_y(Y)$,

(b) the function $y \mapsto U_y(h)$ is Baire-one on Y for every $h \in \mathcal{C}(Y, \mathbb{R})$.

Case $\alpha = 0$. Consider the space

$$\mathcal{M}^1(\operatorname{ext} Y) = \{\mu \in \mathcal{M}^1(Y);\ \mu(Y \setminus \operatorname{ext} Y) = 0\}$$

endowed with the topology τ_{w^*} inherited from $\mathcal{M}^1(Y)$. The property (b) says that $U\colon Y \to (\mathcal{M}^1(\operatorname{ext} Y), \tau_{w^*})$ is F_σ-measurable and thus by Proposition 1.4, $U \in \mathcal{C}_1(Y, (\mathcal{M}^1(\operatorname{ext} Y), \tau_{w^*}))$. Hence there is a sequence (U_n) in $\mathcal{C}(Y, (\mathcal{M}^1(\operatorname{ext} Y), \tau_{w^*}))$ converging pointwise to U.

Assume now that $f|_{\operatorname{ext} X} \in \mathcal{C}_0(X, F)$, i.e., f is continuous on $\operatorname{ext} X$. Then h is continuous on $\operatorname{ext} Y$. For $n \in \mathbb{N}$, consider the function

$$U_n h(y) = \int_{\operatorname{ext} Y} h(x)\, \mathrm{d}(U_n)_y(x), \quad y \in Y.$$

Let

$$Uh(y) = \int_{\operatorname{ext} Y} h(x)\, \mathrm{d}U_y(x), \quad y \in Y.$$

By [21, Lemma 3.4], $(U_n h)$ is a sequence of continuous functions with values in

$$\overline{\operatorname{aco}} h(\operatorname{ext} Y) \subset L$$

which by [21, Lemma 3.3] converges to Uh. Since h is strongly affine by [7, Lemma 6.4], $Uh = h$. Thus $h \in \mathcal{C}_1(Y, L)$. Since $f = h \circ \varphi$, $f \in \mathcal{C}_1(X, L)$.

Case $\alpha > 0$. Let now $f|_{\operatorname{ext} X} \in \mathcal{C}_\alpha(\operatorname{ext} X, F)$ for some $\alpha > 0$. Let $L = \overline{\operatorname{aco}} f(X)$. Then $h|_{\operatorname{ext} Y} \in \mathcal{C}_\alpha(\operatorname{ext} Y, F)$. Since $\operatorname{ext} Y$ is a G_δ set, by [9, §31, VI, Théorème] there exists a function $\widetilde{h} \in \mathcal{C}_\alpha(Y, L)$ extending $h|_{\operatorname{ext} Y}$. By Lemma 5.2, $U\widetilde{h} \in \mathcal{C}_{1+\alpha}(Y, L)$. Since U_y is carried by $\operatorname{ext} Y$ and h is strongly affine,

$$U\widetilde{h} = Uh = h.$$

Since $f = h \circ \varphi$, $f \in \mathcal{C}_{1+\alpha}(X, L)$. □

Proof of Corollary 2.3. (R) Let $f\colon \operatorname{ext} X \to F$ be a bounded continuous mapping. By [7, Theorem 2.7(R)], f can be extended to a function h from $\mathfrak{A}_{\mathrm{odd},2}(X, F)$. By Theorem 2.2, $h \in \mathcal{C}_1(X, F)$. Corollary 3.5 in [7] yields $h \in \mathfrak{A}_1(X, F)$. By [7, Lemma 3.12] we obtain $h \in \mathfrak{A}_{\mathrm{odd},1}(X, F)$.

The proof of (C) is completely analogous, we just use [7, Theorem 2.7(C) and Lemma 3.14(c)] instead.

□

5.1. **Proof of Theorem 2.4 and Corollary 2.5.** The proof of Theorem 2.4 is an adaptation of the proof of [12, Theorem 6.4]. We start it by the following selection lemma.

Lemma 5.3. *Let $\varphi\colon X \to Y$ be a continuous surjective mapping of a compact space X onto a compact space Y, F be a separable metrizable space and let $f\colon X \to F$ be a $\Sigma_\alpha(\mathrm{Bos}(X))$-measurable function for some $\alpha \in [2, \omega_1)$. Then there exists a mapping $\phi\colon Y \to X$ such that*

- $\varphi(\phi(y)) = y$, $y \in Y$,
- $f \circ \phi$ *is a* $\Sigma_\alpha(\mathrm{Bos}(Y))$*-measurable function.*

Proof. Let ρ be a compatible metric on F. Given a $\Sigma_\alpha(\mathrm{Bos}(X))$-measurable function f on X, using a standard approximation technique and [20, Proposition 2.3(f)] we construct a sequence (f_n) of $\Sigma_\alpha(\mathrm{Bos}(X))$-measurable simple functions ρ-uniformly converging to f (see Lemma 4.1). More precisely, each f_n is of the form

$$f_n(x) = y_{n,k}, \quad x \in A_{n,k},$$

where $y_{n,k} \in F$ and $\{A_{n,k};\, k \in \mathbb{N}\}$ is a partition of X formed by sets in $\Delta_\alpha(\mathrm{Bos}(X))$. For every $n, l \in \mathbb{N}$ we consider a countable family $\mathcal{A}_{n,k} \subset \mathrm{Bos}(X)$ satisfying $A_{n,k} \in \Sigma_\alpha(\mathcal{A}_{n,k})$. We include all these families in a single family $\mathcal{A}$.

By [4, Lemma 8], there exists a mapping $\phi\colon Y \to X$ such that $\varphi(\phi(y)) = y$ for every $y \in Y$ and $\phi^{-1}(A) \in \mathrm{Bos}(Y)$ for every $A \in \mathcal{A}$. Then both $\phi^{-1}(A_{n,k})$ and $\phi^{-1}(X \setminus A_{n,k})$ are in $\Sigma_\alpha(\mathrm{Bos}(Y))$ for every set $A_{n,k}$. Thus the functions $f_n \circ \phi$ are $\Sigma_\alpha(\mathrm{Bos}(Y))$-measurable and consequently, since they converge ρ-uniformly to $f \circ \phi$, the function $f \circ \phi$ is $\Sigma_\alpha(\mathrm{Bos}(Y))$-measurable as well. □

Lemma 5.4. *Let X be a compact convex set with $\operatorname{ext} X$ being a resolvable Lindelöf set, F be a Fréchet space and $f\colon X \to F$ be a strongly affine function such that $f|_{\operatorname{ext} X} \in \mathcal{C}_\alpha(\operatorname{ext} X)$ for some $\alpha \in [1, \omega_0)$. Let (ρ_n) be an increasing sequence of seminorms generating topology on F and let $\sigma(y_1, y_2) = \sum_{n=1}^\infty 2^{-n} \min\{\rho_n(y_1 - y_2), 1\}$, $y_1, y_2 \in F$.*

Let $K \subset X$ be a nonempty compact set and $\varepsilon > 0$. Then there exists a nonempty open set U in K and a $\Sigma_{\alpha+1}(\mathrm{Hs}(U))$-measurable function $g\colon U \to F$ such that $\sigma(f, g) < \varepsilon$ on U.

Proof. By Theorem 2.2, $f \in \mathcal{C}_{1+\alpha}(X, F)$, and thus we may assume that F is separable.

Let K be a compact set in X and $\varepsilon \in (0, 1)$. By Lemma 3.3, there exists a Baire set $C \supset \operatorname{ext} X$ and a bounded mapping $g \in \mathcal{C}_\alpha(C, F)$ extending f. Let

$$B = \{x \in X;\, g(x) = f(x)\}.$$

Then B is a Baire set containing $\operatorname{ext} X$ and $f|_B \in \mathcal{C}_\alpha(B, F)$.

We claim that there exists a G_δ set G with

$$X \setminus B \subset G \subset X \setminus \operatorname{ext} X. \tag{5.2}$$

Indeed, if there were no such set, [22, Proposition 11] applied to $X_1 = X \setminus B$ and $X_2 = \operatorname{ext} X$ (observe that $X \setminus B$ is Lindelöf since it is a Baire set; see [16, Theorem 2.7.1]) would provide a nonempty closed set $H \subset X$ satisfying $\overline{H \cap (X \setminus B)} = \overline{H \cap \operatorname{ext} X} = H$. But this would contradict the fact that $\operatorname{ext} X$ is a resolvable set.

We pick a G_δ set G satisfying (5.2) and write $F = X \setminus G = \bigcup_{n=1}^\infty F_n$, where the sets $F_1 \subset F_2 \subset \cdots$ are closed in X. Then $\operatorname{ext} X \subset \bigcup F_n \subset B$.

Let $L = \overline{\operatorname{aco}} f(X)$. Let $n_0 \in \mathbb{N}$ be such that $\sum_{n=n_0+1}^\infty 2^{-n} < \frac{\varepsilon}{2}$ and let $M > 0$ satisfy $\rho_{n_0} \le M$ on L. For each $n \in \mathbb{N}$, we set

$$M_n = \{\mu \in \mathcal{M}^1(X);\, \mu(F_n) \ge 1 - \frac{\varepsilon}{6M}\} \quad \text{and}$$

$$X_n = \{x \in X;\, \text{there exists } \mu \in M_n \text{ such that } r(\mu) = x\} \ (= r(M_n)).$$

Then each X_n is a closed set by the upper semicontinuity of the function $\mu \mapsto \mu(F_n)$ on $\mathcal{M}^1(X)$ and $X = \bigcup_{n=1}^\infty X_n$. Indeed, for any $x \in X$ there exists a maximal measure $\mu \in \mathcal{M}_x(X)$, which is carried by F (see [1, Corollary I.4.12 and the subsequent remark] or [13, Theorem 3.79]), and thus $\mu(F_n) \ge 1 - \frac{\varepsilon}{6M}$ for $n \in \mathbb{N}$ large enough.

Since $K \subset \bigcup X_n$, by the Baire category theorem there exists $m \in \mathbb{N}$ such that $X_m \cap K$ has nonempty interior in K. Let U denote this interior.

Since $f|_{F_m} \in \mathcal{C}_\alpha(F_m, F)$, we can extend $f|_{F_m}$ to a function $h \in \mathcal{C}_\alpha(X, F)$ satisfying $h(X) \subset \overline{\mathrm{co}} f(F_m)$ (see [7, Theorem 2.8]). Let the functions $\widetilde{h}, \widetilde{f} \colon \mathcal{M}^1(X) \to F$ be defined as

$$\widetilde{h}(\mu) = \mu(h),\ \widetilde{f}(\mu) = \mu(f), \quad \mu \in \mathcal{M}^1(X).$$

Then

$$\sigma(\widetilde{f}(\mu), \widetilde{h}(\mu)) < \varepsilon, \quad \mu \in M_m. \tag{5.3}$$

Indeed, for $\mu \in M_m$ and $n \in \{1, \dots, n_0\}$ we have

$$\begin{aligned}\rho_n(\mu(f) - \mu(h)) = \rho_n &\left(\int_{F_m} (f-h)\,\mathrm{d}\mu + \int_{X \setminus F_m} (f-h)\,\mathrm{d}\mu \right) \\ &\le \int_{X\setminus F_m} \rho_n \circ (h-f)\,\mathrm{d}\mu \le \int_{X\setminus F_m} \rho_{n_0} \circ (h-f)\,\mathrm{d}\mu \\ &\le 2M\mu(X \setminus F_m) \le 2M\frac{\varepsilon}{6M} < \frac{\varepsilon}{2}.\end{aligned}$$

Hence

$$\sigma(\widetilde{f}(\mu), \widetilde{h}(\mu)) \le \sum_{n=1}^{n_0} 2^{-n}\rho_n(\mu(f) - \mu(h)) + \sum_{n=n_0+1}^{\infty} 2^{-n} < \varepsilon.$$

By Lemma 3.1, $\widetilde{h} \in \mathcal{C}_\alpha(\mathcal{M}^1(X), F)$, and thus it is a $\Sigma_{\alpha+1}(\mathrm{Bos}(\mathcal{M}^1(X)))$-measurable function on $\mathcal{M}^1(X)$.

We consider the mapping $r \colon M_m \to r(M_m)$ and use Lemma 5.3 to find a selection $\phi \colon r(M_m) \to M_m$ such that

- $r(\phi(x)) = x$, $x \in r(M_m)$,
- $\widetilde{h} \circ \phi$ is $\Sigma_{\alpha+1}(\mathrm{Bos}(r(M_m)))$-measurable on $r(M_m)$.

By setting $g = \widetilde{h} \circ \phi$ we obtain the desired function. Indeed, for a given point $x \in r(M_m)$, the measure $\phi(x)$ is contained in $\mathcal{M}_x(X) \cap M_m$, and hence by (5.3) and the strong affinity of f, we have

$$\sigma(g(x), f(x)) = \sigma(\widetilde{h}(\phi(x)), \widetilde{f}(\phi(x))) < \varepsilon.$$

Thus the function $g|_U$ is the required one because $\Sigma_{\alpha+1}(\mathrm{Bos})$-measurability implies $\Sigma_{\alpha+1}(\mathrm{Hs})$-measurability. □

Proof of Theorem 2.4. Let X, F, f be as in the premise. We may assume that $\alpha \in [1, \omega_0)$ since other cases are covered by Theorem 2.2. We claim that f is $\Sigma_{\alpha+1}(\mathrm{Hs}(X))$-measurable.

Let (ρ_n) be an increasing sequence of seminorms generating topology on F and let $\sigma(y_1, y_2) = \sum_{n=1}^\infty 2^{-n} \min\{\rho_n(y_1 - y_2), 1\}$, $y_1, y_2 \in F$.

Let $\varepsilon > 0$ be arbitrary. We construct a regular sequence $\emptyset = U_0 \subset U_1 \subset \cdots \subset U_\kappa = X$ and functions

$$g_\gamma \in \Sigma_{\alpha+1}(\mathrm{Hs}(U_{\gamma+1} \setminus U_\gamma)), \quad \gamma < \kappa,$$

satisfying $\sigma(g, f) < \varepsilon$ on $U_{\gamma+1} \setminus U_\gamma$ as follows.

Let $U_0 = \emptyset$. Using Lemma 5.4 we select a nonempty open set U of X along with a $\Sigma_{\alpha+1}(\mathrm{Hs}(U))$-measurable function $g \colon U \to F$ with $\sigma(g, f) < \varepsilon$ on U. We set $U_1 = U$ and $g_0 = g$.

Assume now that U_δ and g_δ are chosen for all δ less then some γ. If γ is limit, we set $U_\gamma = \bigcup_{\delta<\gamma} U_\delta$.

Let $\gamma = \lambda + 1$. If $U_\lambda = X$, we set $\kappa = \lambda$ and stop the procedure. Otherwise we apply Lemma 5.4 to $K = X \setminus U_\lambda$ and obtain an open set $U \subset X$ intersecting K along with a $\Sigma_{\alpha+1}(\mathrm{Hs}(U \cap K))$-measurable function g on $U \cap K$ satisfying $\sigma(g, f) < \varepsilon$ on $U \cap K$. We set $U_\gamma = U_\lambda \cup U$ and $g_\lambda = g$. This finishes the construction.

Let $g \colon X \to F$ be defined as $g = g_\gamma$ on $U_{\gamma+1} \setminus U_\gamma$, $\gamma < \kappa$. By [12, Proposition 2.2], g is a $\Sigma_{\alpha+1}(\mathrm{Hs}(X))$-measurable function.

By the procedure above we can approximate uniformly f by $\Sigma_{\alpha+1}(\mathrm{Hs}(X))$-measurable functions which yields that f itself is $\Sigma_{\alpha+1}(\mathrm{Hs}(X))$-measurable (see Proposition 4.2(b)). But f is a Baire function. Thus Theorem 5.2 and Corollary 5.5 in [20] imply $f \in \mathcal{C}_\alpha(X)$. This finishes the proof. □

Proof of Corollary 2.5. (S) Let X, F and f be as in the premise. We proceed by induction. If $\alpha = 1$, by [7, Theorem 2.7], f can be extended to mapping h from $\mathfrak{A}_2(X, F)$. By Theorem 2.4, $h \in \mathcal{C}_1(X, F)$, which in turn by virtue of [7, Corollary 3.5] gives $h \in \mathfrak{A}_1(X, F)$. Thus h is the sought extension.

Assume now that the assertion holds for each $\beta < \alpha$ for some $\alpha \in (1, \omega_1)$. Let $f \colon \operatorname{ext} X \to F$ be a bounded mapping in $\mathcal{C}_\alpha(X, F)$ and let $L = \overline{\mathrm{aco}} f(\operatorname{ext} X)$. By Proposition 1.4, there exists a sequence (f_n) in $\mathcal{C}_{\alpha_n}(\operatorname{ext} X, L)$ converging pointwise to f, where $\alpha_n < \alpha$, $n \in \mathbb{N}$. Let $h_n \in \mathfrak{A}_{\alpha_n}(X, F)$ be the extension of f_n, $n \in \mathbb{N}$. Since h_n is strongly affine, $h_n(X) \subset L$ by Lemma 1.1(c). (Indeed, let

$$B = \{x \in X;\, h_n(x) \in L\}.$$

Then B is a Baire subset containing $\operatorname{ext} X$ and thus carries all maximal measures. For a fixed $x \in X$, let $\mu \in \mathcal{M}_x(X)$ be maximal. Then by Lemma 1.1(b),

$$h_n(x) = \mu(h_n) = \int_B h_n \,\mathrm{d}\mu \in L.)$$

Let $h \in \mathfrak{A}_{1+\alpha}(X, F)$ be the extension of f provided by [7, Theorem 2.7(S)]. Set

$$B = \{x \in X;\, h_n(x) \to h(x)\}$$

Then B is a Baire set containing $\operatorname{ext} X$. For a fixed $x \in X$, let $\mu \in \mathcal{M}_x(X)$ be maximal. Using Lemma 1.1(e) we infer

$$h(x) = \mu(h) = \int_B h \,\mathrm{d}\mu = \lim_{n\to\infty} \int_B h_n \,\mathrm{d}\mu = h_n(x).$$

Thus $h_n \to h$ on X, which means that the extension h is in $\mathfrak{A}_\alpha(X, F)$.

The cases (R) and (C) are now complete analogues of the case (S), one just uses [7, Theorem 2.7(R),(C)] along with [7, Lemma 3.12(d) and Lemma 3.14(c)].

□

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