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How subalgebra lattices of direct powers determine an algebra? in particular Clones of entropic algebras with unit

Dragan Mašulović and Reinhard Pöschel

University Novi Sad and Technische Universität Dresden

Summer School on General Algebra and Ordered Sets Třešť, September 2008



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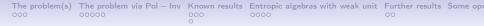
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Preliminary remark

This talk was inspired by papers of K. Kearnes and A. Szendrei

- Clones of finite groups. Algebra Universalis 54 (2005), 23-52.
- *Groups with identical subgroup lattices in all powers.* J. Group Theory, 7 (2004), 385–402.
- Clones of 2-step nilpotent groups. Algebra Universalis

First results presented by R. Pöschel at AAA74 (Tampere 2007) and AAA76 (Linz 2008) Now generalization



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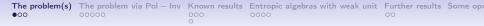
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Clo(A) clone of term operations = $\langle F \rangle_{O_A}$ clone generated by F in the clone Op(A) of all finitary operations on A

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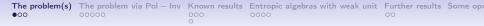
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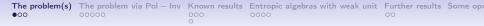
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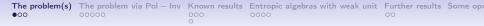
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$\operatorname{Sub}(\mathcal{A}^n) \cong \operatorname{Sub}(\mathcal{B}^n) \implies ???$

If \mathcal{A},\mathcal{B} have the same underlying set: $\mathsf{Sub}(\mathcal{A}^n) = \mathsf{Sub}(\mathcal{B}^n) \implies ???$

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e.g.,
$$\mathcal{A}, \mathcal{B} \in \mathcal{K}$$

Sub $(\mathcal{A}^n) \cong$ Sub $(\mathcal{B}^n) \xrightarrow{?}$ Clo $(\mathcal{A}) \cong$ Clo (\mathcal{B}) or $\mathcal{A} \cong \mathcal{B}$

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And if so, does n depend on the cardinality of $\mathcal{A}?$

unique modulo invariants : $\iff \ [orall n: {\sf Sub}({\mathcal A}^n) = {\sf Sub}({\mathcal B}^n)] \ \stackrel{!}{=}$



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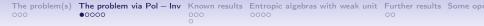
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The Galois connection Pol - Inv

induced by the relation function f preserves relation ϱ : $f \triangleright \varrho$ $F \subseteq Op(A)$ (set of all finitary operations $f : A^n \to A$) $Q \subseteq Rel(A)$ (set of all finitary relations $\varrho \subseteq A^m$) Inv $F := \{\varrho \in R_A \mid \forall f \in F : f \triangleright \varrho\}$ invariant relations

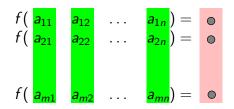


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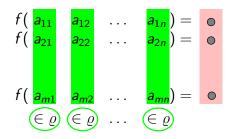
Inv $F := \{ \varrho \in R_A \mid \forall f \in F : f \triangleright \varrho \}$ Pol $Q := \{ f \in Op(A) \mid \forall \varrho \in Q : f \triangleright \varrho \}$ invariant relations polymorphisms

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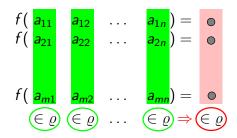
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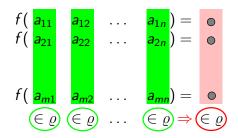
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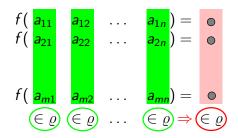
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Special case: commuting operations Operations $f : A^n \rightarrow A$ and $g : A^m \rightarrow A$ commute if

Then

 $f, g \text{ commute } \iff f \triangleright g^{\bullet} \iff g \triangleright f^{\bullet}$ where $f^{\bullet} := \{(a_1, \dots, a_n, b) \in A^{n+1} \mid f(a_1, \dots, a_n) = b\}$ is the TECHNISCHE graph of f

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Example
$$(f = g, n = m = 2)$$

A binary operation $+: A^2 \to A$ commutes with itself $(+ \triangleright +^{\bullet})$ iff

$$\forall a, b, c, d \in A : (a+b)+(c+d) = (a+c)+(b+d).$$

In particular, every commutative operation commutes with itself.



Theorem (Characterization of Galois closed elements) $\mathcal{A} = \langle A, F \rangle$ finite algebra.

- $Clo(A) = \langle F \rangle = Pol Inv F$ (clone generated by F)¹,
- m-Loc(F) = Pol Inv^(m) F
 If Clo(A) = Pol Inv^(m) F:
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- [Q] = Inv Pol Q (relational clone generated by Q).
- $\mathcal{A} = \langle \mathsf{A}, \mathsf{F} \rangle$ (arbitrary) algebra
 - Loc Clo(A) = Loc(F) = Pol Inv F (locally closed clone generated by F)



¹Lev Arkadevic Kalužnin, Лев Аркадевич Калужнин «♂» «≡» «≡»

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¹Lev Arkadevic Kalužnin, Лев Аркадевич Калужнин « 🔿 » « 🖹 » « 🗎 »

Theorem (Characterization of Galois closed elements) $\mathcal{A} = \langle A, F \rangle$ finite algebra.

- $Clo(A) = \langle F \rangle = Pol Inv F$ (clone generated by F)¹,
- *m*-Loc(*F*) = Pol Inv^(m) *F* If Clo(*A*) = Pol Inv^(m) *F*:
 Clo(*A*) determined by its *m*-ary invariants,
 m-locally closed clone, clone with *m*-interpolation property)
- [Q] = Inv Pol Q (relational clone generated by Q).

 $\mathcal{A} = \langle \mathsf{A}, \mathsf{F}
angle$ (arbitrary) algebra

Loc Clo(A) = Loc(F) = Pol Inv F (locally closed clone generated by F)



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Immediate consequence: An "answer" to our problem

Proposition. Let \mathcal{K} be the class of all algebras with a locally closed clone of term operations, i.e. $\operatorname{Clo}(\mathcal{A}) = \operatorname{Pol}\operatorname{Inv} F$ (in particular, \mathcal{K} contains all finite algebras). Then the clone of every $\mathcal{A} \in \mathcal{K}$ is unique in \mathcal{K} (as well as in the class $\mathcal{K}_{\mathcal{A}}$ of all algebras \mathcal{B} with $\operatorname{Clo}(\mathcal{A}) \subseteq \operatorname{Clo}(\mathcal{B})$) modulo invariants.

Proof. Let $\mathcal{A} = \langle \mathcal{A}, \mathcal{F} \rangle$, $\mathcal{B} = \langle \mathcal{A}, \mathcal{G} \rangle$ and $\mathsf{Inv} \mathcal{F} = \mathsf{Inv} \mathcal{G}$. Then

$$Clo(\mathcal{A}) \stackrel{\mathcal{A} \subseteq \mathcal{N}}{=} Loc Clo(\mathcal{A}) = Pol Inv F = Pol Inv G$$
$$= Loc Clo(\mathcal{B}) \stackrel{\mathcal{B} \subseteq \mathcal{N}}{=} Clo(\mathcal{B})$$
$$\stackrel{\mathcal{B} \subseteq \mathcal{N}}{=} Clo(\mathcal{A}) \stackrel{\mathcal{B} \subseteq \mathcal{N}}{=} Clo(\mathcal{A}) \stackrel{\mathcal{B} \subseteq \mathcal{N}}{\longrightarrow} Clo(\mathcal{A})$$

Immediate consequence: An "answer" to our problem locally closed \implies clone unique modulo invariants (in \mathcal{K}) $Clo(\mathcal{A}) = Pol Inv F \implies Clo(\mathcal{A})$ unique modulo invariants (in \mathcal{K})

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Immediate consequence: An "answer" to our problem *m*-locally closed \implies clone unique modulo *m*-ary invariants (in \mathcal{K})

 $Clo(\mathcal{A}) = Pol Inv^{(m)} F \stackrel{? \neq !}{\Longrightarrow} Clo(\mathcal{A})$ unique modulo *m*-ary invariants (in \mathcal{K}

Proposition. Let \mathcal{K}_m be the class of all algebras with an m-locally closed clone of term operations, i.e. $\operatorname{Clo}(\mathcal{A}) = \operatorname{Pol} \operatorname{Inv}^{(m)} \mathcal{F}$ (in particular, \mathcal{K} contains all m-locally closed finite algebras). Then the clone of every $\mathcal{A} \in \mathcal{K}_m$ is unique in \mathcal{K} (as well as in the class $\mathcal{K}_{\mathcal{A}}$ of all algebras \mathcal{B} with $\operatorname{Clo}(\mathcal{A}) \subseteq \operatorname{Clo}(\mathcal{B})$) modulo m-ary invariants.

Proof. Let $\mathcal{A} = \langle \mathcal{A}, \mathcal{F} \rangle$, $\mathcal{B} = \langle \mathcal{A}, \mathcal{G} \rangle$ and $\mathsf{Inv} \mathcal{F} = \mathsf{Inv} \mathcal{G}$. Then

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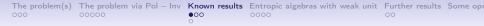
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Near unanimity term operation

Theorem

Let $\mathcal{A} = \langle A, F \rangle$ be a finite algebra such that there is a (d+1)-ary near unanimity term operation. Then $\operatorname{Clo} \mathcal{A} = \langle F \rangle = \operatorname{Pol} \operatorname{Inv}^{(d)} F$ (in particular, $\operatorname{Clo}(\mathcal{A})$ is unique in \mathcal{K}_d modulo d-ary invariants).

near unanimity operation $f: A^n \to A$:

$$f(x, y, \dots, y) = y$$
$$f(y, x, \dots, y) = y$$
$$\vdots$$
$$f(y, y, \dots, x) = y$$



Generalization: *k*-edge term operation

Kearnes/Szendrei (personal communication May 2007)

Theorem Let $\mathcal{A} = \langle A, F \rangle$ be a finite algebra such that (1) \mathcal{A} has a k-edge term for some $k \in \mathbb{N}$, (2) \mathcal{A} generates a residually small variety. Then $\operatorname{Clo}(\mathcal{A}) = \operatorname{Pol} \operatorname{Inv}^{(d)} F$ for some d (depending only on the cardinality $|\mathcal{A}|$).



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k-edge term

Generalization of both, near-unanimity term and Mal'cev term A k-edge term is a (k + 1)-ary term satisfying the identities:

$$e(x, x, y, y, y, ..., y, y) = y$$

$$e(x, y, x, y, y, ..., y, y) = y$$

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...

$$e(y, y, y, y, y, ..., y, x) = y$$

(introduced by J. Berman, P. Idziak, P. Markovic, R. McKenzie, M. Valeriote, R. Willard: *Tractability and learnability arising from algebras with few subpowers*, Proceedings of the 22nd Annual IEEE Symposium on Logic in Computer Science, 2007)

Is every group determined (up to isomorphism) by the subgroup lattices of its finite powers?

$[\forall n \in \mathbb{N} \operatorname{Sub}(G^n) \cong \operatorname{Sub}(H^n)] \implies G \cong H$?

But for abelian groups we have (R. Baer, 1939): Sub $(G^3) \cong$ Sub $(H^3) \implies G \cong H$

moreover, ${
m Sub}(G^3)={
m Sub}(H^3)\implies G=H$ (diploma thesis H.A. Pham '07)

More general (Kearnes/Szendrei): $G = \langle A, \cdot \rangle$, $H = \langle A, \odot \rangle$ finite groups with abelian Sylow subgroups: $Sub(G^3) = Sub(H^3) \implies Clo(G) = Clo(H)$ (term equivalent)

cyclic Sylow subgroups: Sub $(G^2) \cong$ Sub $(H^2) \implies G, H$ weakly isomorphic



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An algebra $\mathcal{A} = \langle A, F \rangle$ is called *entropic* if every two operations $f, g \in F$ commute (i.e. $\forall f, g \in F : f \triangleright g^{\bullet}$)

 $e \in A \text{ weakly neutral for an operation } f : A^n \to A : \iff n \ge 2 \text{ and } \forall i \in \{1, \dots, n\} \forall x \in A : f(e, \dots, e, x, e, \dots, e) = x (x \text{ at } i\text{-th place})$

 $\mathcal{A} = \langle A, F \rangle$ entropic with weakly neutral element : $\iff \mathcal{A}$ entropic and $\exists e \in A$: e is weakly neutral for every $f \in F$ (consequently all operations $f \in F$ have arity at least 2)





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Entropic algebras and monoids

Lemma

Let $\mathcal{A} = \langle A, F, e \rangle$ be an entropic algebra with weakly neutral element e. Take any $f \in F$ (! arity of f is at least 2) and define

$$x \cdot y := f(x, y, e, \ldots, e),$$

Then $\mathcal{M} = \langle A, \cdot, e \rangle$ is a commutative monoid (called monoid associated to \mathcal{A}).

Theorem

Let $\mathcal{A} = \langle A, F, e \rangle$ be an entropic algebra with weakly neutral element e. Then \mathcal{A} is term-equivalent to any associated monoid $\mathcal{M} = \langle A, \cdot, e \rangle$: $Clo(\mathcal{A}) = Clo(\mathcal{M})$.

idea of the proof: For *m*-ary $g \in F$ show $g(x_1, x_2, \dots, x_m) = x_1 \cdot x_2 \cdot \dots \cdot x_m.$





Entropic algebras and monoids

Lemma

Let $\mathcal{A} = \langle A, F, e \rangle$ be an entropic algebra with weakly neutral element e. Take any $f \in F$ (! arity of f is at least 2) and define

$$x \cdot y := f(x, y, e, \ldots, e),$$

Then $\mathcal{M} = \langle A, \cdot, e \rangle$ is a commutative monoid (called monoid associated to \mathcal{A}).

Theorem

Let $\mathcal{A} = \langle A, F, e \rangle$ be an entropic algebra with weakly neutral element e. Then \mathcal{A} is term-equivalent to any associated monoid $\mathcal{M} = \langle A, \cdot, e \rangle$: $\mathsf{Clo}(\mathcal{A}) = \mathsf{Clo}(\mathcal{M})$.

idea of the proof: For *m*-ary $g \in F$ show $g(x_1, x_2, \ldots, x_m) = x_1 \cdot x_2 \cdot \ldots \cdot x_m.$





Crucial Lemma for the proof

Lemma

Let $f : A^n \to A$ and $g : A^m \to A$ be operations on A with $m \le n$ such that f and g commute and have a common weakly neutral element e (thus $2 \le m \le n$). Then

 $g(x_1,\ldots,x_m)=f(x_1,\ldots,x_m,e,\ldots,e)$

(in particular f = g for m = n).





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Entropic algebras are unique modulo ternary invariants

Theorem

- The clone of each entropic algebra A = ⟨A, F, e⟩ with weakly neutral element is determined by its ternary invariants: Clo(A) = Pol Inv⁽³⁾ F.
- (2) Let \mathcal{E} be the class of all entropic algebras with a weakly neutral element. Then the clone of each $\mathcal{A} \in \mathcal{E}$ is unique in \mathcal{E} modulo ternary invariants:

 $\mathcal{A}, \mathcal{B} \in \mathcal{E}, \ \mathsf{Sub}(\mathcal{A}^3) = \mathsf{Sub}(\mathcal{B}^3) \implies \mathsf{Clo}(\mathcal{A}) = \mathsf{Clo}(\mathcal{B})$

(3) Let \mathcal{E}_n be the class of all entropic algebras $\mathcal{A} = \langle A, f, e \rangle$ with a weakly neutral element and one n-ary fundamental operation. Then each $\mathcal{A} \in \mathcal{E}_n$ is unique in \mathcal{E}_n modulo ternary invariants:

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Algebras $\mathcal{A} = \langle A, f^{\mathcal{A}} \rangle$ (Generalization of semilattices)

Proposition

Let ${\cal K}$ be a class of algebras ${\cal A}=\langle A,f\rangle$ with one n-ary operation satisfying

(1) \mathcal{A} is entropic, i.e. f commutes with itself ($f \triangleright f^{\bullet}$),

(2) f is idempotent, i.e., f(x,...,x) = x

(3) f is cyclic commutative, i.e.,

 $f(x_1, x_2, \ldots, x_n) = f(x_2, \ldots, x_n, x_1)$

Then the algebras $\mathcal{A} \in \mathcal{K}$ are unique in \mathcal{K} modulo (n+1)-ary invariants, i.e.

 $\forall \mathcal{A}, \mathcal{B} \in \mathcal{K} : \mathsf{Sub}(\mathcal{A}^{n+1}) = \mathsf{Sub}(\mathcal{B}^{n+1}) \implies \mathcal{A} = \mathcal{B}.$

Examples: $A = \langle A, \Lambda \rangle$ semilattice (Λ associative, commutative idempotent)

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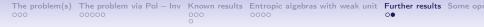
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Crucial Lemma for the proof

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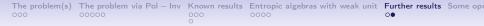
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Then f = g

$((1) implies g^{\bullet} \in \mathsf{Inv}^{(n+1)}{f})$

Proof. $f(x_{1},...,x_{n}) \stackrel{(2)}{=} g(f(x_{1},...,x_{n}),...,f(x_{1},...,x_{n}))$ $\stackrel{(3)}{=} g(f(x_{1},x_{2},...,x_{n}),f(x_{2},x_{3},...,x_{1}),...,f(x_{n},x_{1},...,x_{n-1}))$ $\stackrel{(1)}{=} f(g(x_{1},x_{2},...,x_{n}),g(x_{2},x_{3},...,x_{1}),...,g(x_{n},x_{1},...,x_{n-1}))$ $\stackrel{(3)}{=} f(g(x_{1},...,x_{n}),...,g(x_{1},...,x_{n})) \stackrel{(2)}{=} g(x_{1},...,x_{n})$



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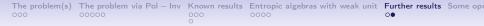
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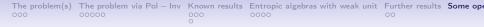
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 Which groups are determined by their ternary invariants? (Clo(G) = Pol Inv⁽³⁾ G ?)

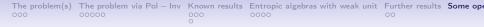
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Then the clone of every algebra in $\mathcal K$ is unique modulo invariants.

 Find further algebraic properties (P) for operations f : Aⁿ → A such that the algebras ⟨A, f⟩ (or their clone) are unique modulo (n-ary) invariants in the class *K* := {⟨A, f⟩ | f satisfies (P)}.



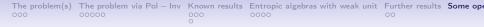


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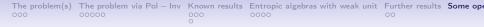


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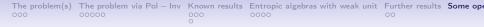
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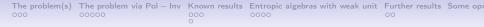


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- F contains all projections $(e_i^n(x_1,...,x_n) = x_i)$
- F is closed under composition i.e. if f, g₁,..., g_n ∈ F (f n-ary, g_i m-ary), then

 $f[g_1,\ldots,g_n]\in F$

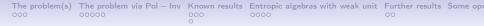
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e.g., for a group $G = \langle A, \cdot, -^1 \rangle$, the clone Clo(G) of term functions (= clone generated by the operation $x \cdot y$ of multiplication and taking inverse x^{-1}) consists of all functions definable by a semigroup word:

$$f(x_1,\ldots,x_n) = x_{i_1}^{s_1}\cdot\ldots\cdot x_{i_t}^{s_t}$$

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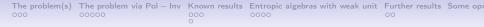
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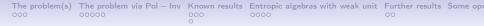
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e.g., for a group $G = \langle A, \cdot, {}^{-1} \rangle$, the clone Clo(G) of term functions (= clone generated by the operation $x \cdot y$ of multiplication and taking inverse x^{-1}) consists of all functions definable by a semigroup word:

$$f(x_1,\ldots,x_n) = x_{i_1}^{s_1}\cdot\ldots\cdot x_{i_t}^{s_t}$$

(where $x_{i_j} \in \{x_1, \ldots, x_n\}$ and $s_j \in \mathbb{N}, \in \mathbb{Z}$)



A set F of finitary functions $f : A^n \to A$ (on a base set A) is called *clone*, if

- F contains all projections $(e_i^n(x_1,...,x_n) = x_i)$
- F is closed under composition i.e. if $f, g_1, \ldots, g_n \in F$ (f n-ary, g_i m-ary), then

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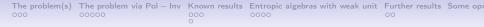
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