On the Maximal Regular Subsemigroups of Some Transformation Semigroups

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Let $X_n = \{1, \ldots, n\}$ be an n - element set ($n \in \boldsymbol{N}^* := \boldsymbol{N} \setminus \{0\}$). The **full transformation semigroup** T_n is the set of all mappings, written on the right, of X_n into itself with the composition of mappings as multiplication. A subsemigroup of T_n is called a **transformation semigroup**.

Each ideal of T_n has the form

$$K(n,r) = \{\alpha \in T_n : |im\ \alpha| \le r\} \text{ for } 1 \le r \le n.$$

Thus $K(n,n) = T_n$ and $K(n,n-1) = Sing_n$, the semigroup of all singular selfmaps (non-permutation transformations) on X_n.

It is well known that the Green's equivalences on T_n are:

$$\alpha\mathcal{L}\beta \iff im\alpha = im\beta$$

$$\alpha\mathcal{R}\beta \iff ker\alpha = ker\beta$$

$$\alpha\mathcal{J}\beta \iff |im\alpha| = |im\beta|$$

Thus, the semigroup $K(n,r)$ is the union of $\mathcal{J}$-classes $J_1, J_2, \ldots, J_r$, where

$$J_k := \{\alpha \in T_n : |im\ \alpha| = k\} \text{ for } k = 1, 2, \ldots, r.$$

Let us denote by Λ_r the collection of all subsets of X_n of cardinality r. Since two transformations are $\mathcal{L}$-related iff they have the same image, a typical $\mathcal{L}$-class is completely determined by the image set A of transformations in it. Therefore a typical $\mathcal{L}$-class in J_r may be denoted by

$$L_A := \{\alpha \in K(n,r) : im\ \alpha = A\}$$

for some set A in Λ_r.

We say that π is an equivalence relation on X_n with weight r if $|X_n/\pi| = r$. Let Ω_r be the collection of all equivalence relations on X_n with weight r. Since two transformations are $\mathcal{R}$-related iff they have the same kernel, a typical $\mathcal{R}$-class is completely determined by the kernel of any transformations in it. Then a typical $\mathcal{R}$-class in J_r may be denoted by

$$R_\pi := \{\alpha \in K(n, r) : ker\ \alpha = \pi\},$$

for some equivalence relation π in Ω_r.

Since two transformations are $\mathcal{H}$-related iff they have the same image and the same kernel, a typical $\mathcal{H}$-class is completely determined by the image set A and by the kernel π of any transformations in it. Then a typical $\mathcal{H}$-class in J_r may be denoted by

$$H_{(\pi,A)} := R_\pi \cap L_A$$

for some equivalence relation π in Ω_r and some set A in Λ_r.

The $\mathcal{J}$ - class (respectively $\mathcal{L}$ - class, $\mathcal{R}$ - class, $\mathcal{H}$ - class) containing the element $\alpha \in T_n$ will be denoted by J_α (respectively L_α, R_α, H_α).

Definition 1 *Let* $A \subseteq X_n$ *and let* π *be an equivalence relation on* X_n. *If* $| \bar{x} \cap A | = 1$ *for all* $\bar{x} \in X_n/\pi$, *then* A *is called a* **cross-section** *of* π, *denoted by* $A \# \pi$.

Let U be a subset of the semigroup T_n. We denote by $E(U)$ the set of all idempotents in the set U.

Proposition 1 *Let* α *and* β *be two elements of the class* $J_r \subset T_n$ $(1 \leq r \leq n-1)$. *Then the following statements are equivalent:*

1) $\alpha\beta \in J_r$. 2) $L_\alpha \cap R_\beta \subset E(J_r)$.

3) $\alpha\beta \in R_\alpha \cap L_\beta$. 4) $L_\alpha R_\beta = J_r$.

Theorem 1 (Taijie You) *Let $n \geq 3$. Then a subsemigroup of T_n is maximal regular iff it belongs to one of the following types:*

$K(n, n-2) \cup S_n$, *where S_n is the symmetric group.*

$K(n, n-1) \cup G$, *where G is a maximal subgroup of S_n.*

Theorem 2 (Taijie You) *Let $n \geq 3$ and $2 \leq r \leq n-1$. Then a subsemigroup of $K(n,r)$ is maximal regular iff it belongs to one of the following types:*

$S_A := K(n, r-1) \cup (J_r \setminus L_A)$, *where $A \in \Lambda_r$.*

$S_\pi := K(n, r-1) \cup (J_r \setminus R_\pi)$, *where $\pi \in \Omega_r$.*

Definition 2 *Let* $2 \leq r \leq n-1$. *A pair* (Λ, Ω) *is called a* **coupler** *of* (Λ_r, Ω_r) *if the following four conditions are satisfied:*

1. $\emptyset \neq \Lambda \subseteq \Lambda_r$ *and* $\emptyset \neq \Omega \subseteq \Omega_r$.

2. *There is no* $A \in \Lambda$ *and no* $\pi \in \Omega$ *with* $A \# \pi$.

3. *For all* $A \in \Lambda_r \backslash \Lambda$ *there is* $\pi \in \Omega$ *with* $A \# \pi$.

4. *For all* $\pi \in \Omega_r \backslash \Omega$ *there is* $A \in \Lambda$ *with* $A \# \pi$.

Definition 3 *A transformation* $\alpha \in T_n$ *is called* **isotone** *if* $x \leq y \implies x\alpha \leq y\alpha$ *for all* $x, y \in X_n$*,* **antitone** *if* $x \leq y \implies x\alpha \geq y\alpha$ *for all* $x, y \in X_n$ *and* **monotone** *if it is isotone or antitone.*

Definition 4 *The set* $O_n := \{\alpha \in Sing_n : \alpha$ *is isotone* $\}$ *forms a semigroup under the operation of composition, called* **the semigroup of all isotone transformations***.*

Definition 5 For $1 \leq r \leq n-1$ we put:
1) $K^i(n,r) := O_n \cap K(n,r)$.
2) $J^i_r := O_n \cap J_r$.

Clearly, $K^i(n,r)$ forms an ideal of the semigroup O_n and $K^i(n,n-1) = O_n$.

Let Ω_r^i be the collection of all equivalence relations π on X_n with $|X_n/\pi| = r$ such that all classes $\bar{x} \in X_n/\pi$ are convex subsets of X_n, in the sense that $x, y \in \bar{x}$ and $x \leq z \leq y$ together imply that $z \in \bar{x}$.

Let us denote by Ω_r^c the set of all equivalence relations $\pi \in \Omega_r^i$ for which exists $x \leq n - r + 1$ such that $\{x, x+1, x+2, \ldots, x+r-1\} \# \pi$.

The $\mathcal{R}$ -, $\mathcal{L}$ - and $\mathcal{H}$ - classes of the class J_r^i have the following form:

$$R_\pi^i := \{\alpha \in J_r^i : \text{ ker } \alpha = \pi\}, \text{ where } \pi \in \Omega_r^i,$$

$$L_A^i := \{\alpha \in J_r^i : \text{ im } \alpha = A\}, \text{ where } A \in \Lambda_r,$$

$$H_{(\pi,A)}^i := R_\pi^i \cap L_A^i.$$

We put $E_r := E(J^i_r)$. It is well known that

Proposition 2 *Let* $1 \leq r \leq n-1$. *Then*

$$J^i_r \subseteq \langle E_r \rangle.$$

Theorem 3 (Dimitrova and Koppitz) *Let* $n \geq 3$ *and* $2 \leq r \leq n-1$. *Then a subsemigroup of* $K^i(n,r)$ *is maximal iff it belongs to one of the following types*

$S_A := K^i(n,r-1) \cup (J^i_r \setminus L^i_A)$, *where* $A \in \Lambda_r$*;*

$S_\pi := K^i(n,r-1) \cup (J^i_r \setminus R^i_\pi)$, *where* $\pi \in \Omega^i_r \setminus \Omega^c_r$*;*

$S_{(A,\pi)} := K^i(n,r-1) \cup \bigcup\{L^i_A : A \in \Lambda\} \cup \bigcup\{R^i_\pi : \pi \in \Omega\}$ *for some coupler* (Λ, Ω) *of* (Λ_r, Ω^i_r).

Proposition 3 *Let* $2 \leq r \leq n-1$ *and* $\alpha, \beta \in J_r^i$. *Then* $\alpha\beta\alpha = \alpha$ *iff* $\ker \alpha\beta = \ker \alpha$ *and* $im\beta\alpha = im\alpha$.

Proposition 4 *Let* S *be a regular subsemigroup of* O_n *with the property that* S *intersects each* $\mathcal{L}$ *- class and each* $\mathcal{R}$ *- class of* O_n. *Then* $S = O_n$.

Theorem 4 *Let* $n \geq 3$ *and* $2 \leq r \leq n-1$. *Then a subsemigroup of* $K^i(n,r)$ *is a maximal regular subsemigroup of* $K^i(n,r)$ *iff it belongs to one of the following types:*

$S_A \; := \; K^i(n, r-1) \cup (J_r^i \setminus L_A^i)$, *where* $A \in \Lambda_r$.

$S_\pi \; := \; K^i(n, r-1) \cup (J_r^i \setminus R_\pi^i)$, *where* $\pi \in \Omega_r^i \setminus \Omega_r^c$.

From Theorem 3 and Theorem 4 we get the following

Corollary 1 *A regular subsemigroup of* $K^i(n,r)$ *is a maximal regular subsemigroup iff it is a maximal subsemigroup of* $K^i(n,r)$.

The set $M_n := \{\alpha \in Sing_n : \alpha$ is monotone $\}$ forms a semigroup under the operation of composition, called the **semigroup of all monotone transformations**.

Definition 6 *For* $1 \leq r \leq n-1$ *we put:*
1) $K^m(n,r) := M_n \cap K(n,r)$.
2) $J_r^m := M_n \cap J_r$.

Clearly, $K^m(n,r)$ forms an ideal of the semigroup M_n and $K^m(n,n-1) = M_n$.

An immediate but important property is that the product of two isotone transformations or two antitone transformations is an isotone transformation, and the product of an isotone transformation with an antitone transformation, or vice-versa, is an antitone transformation.

Let us denote by J_r^a the set of all antitone transformations in the class J_r. Therefore for the class J_r^m we have also $J_r^m = J_r^i \cup J_r^a$.

The $\mathcal{R}$ -, $\mathcal{L}$ - and $\mathcal{H}$ - classes of the class J_r^a have the following form:

$$R_\pi^a \ := \ \{\alpha \in J_r^a : \ \text{ker } \alpha = \pi\}, \text{ where } \pi \in \Omega_r^i,$$

$$L_A^a \ := \ \{\alpha \in J_r^a : \ \text{im } \alpha = A\}, \text{ where } A \in \Lambda_r,$$

$$H_{(\pi,A)}^a \ := \ R_\pi^a \cap L_A^a.$$

Then for the $\mathcal{R}$ -, $\mathcal{L}$ - and $\mathcal{H}$ - classes of the class J_r^m we have

$$R_\pi^m := R_\pi^i \cup R_\pi^a,$$

$$L_A^m := L_A^i \cup L_A^a,$$

$$H_{(\pi,A)}^m := H_{(\pi,A)}^i \cup H_{(\pi,A)}^a.$$

Therefore, the $\mathcal{H}$ - classes of the semigroup M_n contain exactly two transformations with the same image and the same kernel - one isotone and one antitone, i.e. $|H_{(\pi,A)}^m| = 2$.

Proposition 5 *For* $1 \leq r \leq n-1$*, the following hold:*
1) $K^m(n,r) = \langle E_r, \gamma\rangle$ *for some* $\gamma \in J_r^a$*.*
2) $K^m(n,r) = \langle J_r^a \rangle$*.*

Theorem 5 (Dimitrova and Koppitz) *Let* $n \geq 3$ *and* $2 \leq r \leq n-1$*. Then a subsemigroup* S *of* $K^m(n,r)$ *is maximal iff* $S = K^m(n,r-1) \cup J_r^i$ *or there exists a maximal subsemigroup* S^i *of* $K^i(n,r)$ *such that* $S = \bigcup\{H_\alpha^m : \alpha \in S^i\}$*.*

Proposition 6 *Let* $2 \leq r \leq n-1$ *and* $\alpha, \beta \in J_r^a$*. Then* $\alpha\beta\alpha = \alpha$ *iff* $\ker \alpha\beta = \ker \alpha$ *and* $im\beta\alpha = im\alpha$*.*

Proposition 7 *Let* $1 \leq r \leq n-1$*. Then the ideal* $K^m(n,r)$ *is regular.*

Theorem 6 *Let* $n \geq 3$ *and* $2 \leq r \leq n-1$. *Then a subsemigroup* S *of* $K^m(n,r)$ *is a maximal regular subsemigroup of* $K^m(n,r)$ *iff* $S = K^m(n,r-1) \cup J_r^i$ *or there exists a maximal regular subsemigroup* S^i *of* $K^i(n,r)$ *such that* $S = \cup\{H_\alpha^m : \alpha \in S^i\}$.

Theorem 5 and Theorem 6 provide the following corollary

Corollary 2 *A regular subsemigroup of* $K^m(n,r)$ *is a maximal regular subsemigroup iff it is a maximal subsemigroup of* $K^m(n,r)$.