Localification of variable-basis topological systems

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Outline

Topological systems

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2008 S. Solovyov introduces the category of variable-basis topological systems over an arbitrary variety of algebras generalizing the respective notion of S. Vickers.

!!! The latter category provides a single framework in which to treat both variable-basis lattice-valued topological spaces and the respective algebraic structures underlying their topologies.

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Current talk

- The above-mentioned framework is good on the topological side (spatialization of variable-basis topological systems is possible) and is bad on the algebraic one (the procedure of localification collapses).
- Stimulated by the deficiency we introduced a modified version of the category of variable-basis topological systems.
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Ω-algebras and Ω-homomorphisms

- Let $\Omega = (n_\lambda)_{\lambda \in \Lambda}$ be a class of cardinal numbers.

Definition 1

- An Ω-algebra is a pair $(A, (\omega_\lambda^A)_{\lambda \in \Lambda})$ (denoted by A), where A is a set and $(\omega_\lambda^A)_{\lambda \in \Lambda}$ is a family of maps $A^{n_\lambda} \xrightarrow{\omega_\lambda^A} A$.
- An Ω-homomorphism $(A, (\omega_\lambda^A)_{\lambda \in \Lambda}) \xrightarrow{f} (B, (\omega_\lambda^B)_{\lambda \in \Lambda})$ is a map $A \xrightarrow{f} B$ such that $f \circ \omega_\lambda^A = \omega_\lambda^B \circ f^{n_\lambda}$ for every $\lambda \in \Lambda$.

Definition 2

- **Alg**(Ω) is the category of Ω-algebras and Ω-homomorphisms.
- $|-|$ is the forgetful functor to the category **Set** (sets).

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Varieties of algebras

Definition 3

- Let $\mathcal{M}$ (resp. $\mathcal{E}$) be the class of Ω-homomorphisms with injective (resp. surjective) underlying maps.
- A variety of Ω-algebras is a full subcategory of $\mathbf{Alg}(\Omega)$ closed under the formation of products, $\mathcal{M}$-subobjects (subalgebras) and $\mathcal{E}$-quotients (homomorphic images).
- The objects (resp. morphisms) of a variety are called algebras (resp. homomorphisms).

Example 4

The categories **Frm**, **SFrm** and **SQuant** of frames, semiframes and semi-quantales (popular in lattice-valued topology) are varieties.

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Q-powersets

- From now one fix a variety **A** and an algebra Q.

Definition 5

Given a set X, Q^X is the Q-powerset of X.

- An arbitrary element of Q^X is denoted by p (with indices).
- Q^X is an algebra with operations lifted point-wise from Q by

$$(\omega_\lambda^{Q^X}(\langle p_i \rangle_{n_\lambda}))(x) = \omega_\lambda^Q(\langle p_i(x) \rangle_{n_\lambda}).$$

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Image and preimage operators

- Let $X \xrightarrow{f} Y$ be a map and let $A \xrightarrow{g} B$ be a homomorphism.
- There exist:
  - the standard image and preimage operators $\mathcal{P}(X) \xrightarrow{f^{\to}} \mathcal{P}(Y)$ and $\mathcal{P}(Y) \xrightarrow{f^{\leftarrow}} \mathcal{P}(X)$;
  - the Zadeh preimage operator $Q^Y \xrightarrow{f_Q^{\leftarrow}} Q^X$ defined by $f_Q^{\leftarrow}(p) = p \circ f$;
  - a map $A^X \xrightarrow{g_{\to}^X} B^X$ defined by $g_{\to}^X(p) = g \circ p$.

Lemma 6

For every map $X \xrightarrow{f} Y$ and every homomorphism $A \xrightarrow{g} B$, both $Q^Y \xrightarrow{f_Q^{\leftarrow}} Q^X$ and $A^X \xrightarrow{g_{\to}^X} B^X$ are homomorphisms.

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Fixed-basis topological spaces

Definition 7

- Given a set X, a subset τ of Q^X is a Q-topology on X provided that τ is a subalgebra of Q^X.
- A Q-topological space (also called a Q-space) is a pair (X, τ), where X is a set and τ is a Q-topology on X.
- A map $(X, \tau) \xrightarrow{f} (Y, \sigma)$ between Q-spaces is Q-continuous provided that $(f_Q^{\leftarrow})^{\rightarrow}(\sigma) \subseteq \tau$.

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- Q-**Top** is the category of Q-spaces and Q-continuous maps.
- $|-|$ is the forgetful functor to the category **Set**.

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Notations

- From now on introduce the following notations:
  - The dual of the category **A** is denoted by **LoA** (the "**Lo**" comes from "localic").
  - The objects (resp. morphisms) of **LoA** are called localic algebras (resp. homomorphisms).
  - The respective homomorphism of a localic homomorphism f is denoted by f^{op} and vice versa.
  - To distinguish between maps and homomorphisms denote them by "f, g" and "φ, ψ" respectively.

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Variable-basis preimage operator

Definition 9

Given a **Set** $\times$ **LoA**-morphism $(X, A) \xrightarrow{(f,\varphi)} (Y, B)$, there exists the Rodabaugh preimage operator $B^Y \xrightarrow{(f,\varphi)^{\leftarrow}} A^X$ defined by $(f, \varphi)^{\leftarrow}(p) = \varphi^{op} \circ p \circ f$.

Lemma 10

For every **Set** $\times$ **LoA**-*morphism* $(X, A) \xrightarrow{(f,\varphi)} (Y, B)$, *the diagram*

$$\begin{array}{ccc} B^Y & \xrightarrow{(\varphi^{op})^Y_{\rightarrow}} & A^Y \\ {\scriptstyle f_B^{\leftarrow}}\downarrow & (f,\varphi)^{\leftarrow} & \downarrow{\scriptstyle f_A^{\leftarrow}} \\ B^X & \xrightarrow{(\varphi^{op})^X_{\rightarrow}} & A^X \end{array}$$

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Variable-basis topological spaces

Definition 11

- Given a subcategory **C** of **LoA**, the category **C-Top** comprises the following data:
  - Objects: **C**-topological spaces or **C**-spaces (X, A, τ), where (X, A) is a **Set** $\times$ **C**-object and (X, τ) is an A-space.
  - Morphisms: **C**-continuous pairs $(X, A, \tau) \xrightarrow{(f,\varphi)} (Y, B, \sigma)$, where (f, φ) is a **Set** $\times$ **C**-morphism and $((f, \varphi)^{\leftarrow})^{\rightarrow}(\sigma) \subseteq \tau$.
- $|-|$ is the forgetful functor to the category **Set** $\times$ **C**.

- **C-Top** generalizes the respective category of S. E. Rodabaugh.
- This talk considers the case **C** = **LoA**.
- Call **LoA**-spaces by spaces and **LoA**-continuity by continuity.

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Satisfaction relation

Definition 12

Let X be a set and A be a frame. Then $X \xrightarrow{\models} A$ is a satisfaction relation on (X, A) if $\models$ is a binary relation from X to A satisfying the following join interchange law and meet interchange law:

- For any family $\{a_i\}_{i \in I}$ of elements of A,
$$x \models \bigvee_{i \in I} a_i \text{ iff } x \models a_i \text{ for at least one } i \in I.$$
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- A topological system is a triple $(X, A, \models)$, where (X, A) is a **Set** $\times$ **Loc**-object and $\models$ is a satisfaction relation on (X, A).
- Elements of X are points and elements of A are opens.
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$$(X, A, \models_1) \xrightarrow{f=(\mathrm{pt}\, f, (\Omega f)^{op})} (Y, B, \models_2),$$

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From variable-basis to fixed-basis

Definition 15

- For a **C**-object Q, Q-**TopSys** is the subcategory of **C-TopSys** of all **C**-systems $(X, Q, B, \models)$ with basis Q and all continuous f such that $\Sigma f = 1_Q$.
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Lemma 16

- *The subcategory Q-**TopSys** is full iff $\mathbf{C}(Q, Q) = \{1_Q\}$.*
- *If Q is an initial (terminal) object in **A**, then Q-**TopSys** is full.*

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$2 = \{\bot, \top\}$ is initial in **Frm**. The full subcategory 2-**TopSys** of **Loc**-**TopSys** is isomorphic to the category **TopSys** of S. Vickers.

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Given a set K, the subcategory K-**TopSys** of **LoSet**-**TopSys** is isomorphic to the category $\mathtt{Chu}(\mathbf{Set}, K)$ of Chu spaces over K. K-**TopSys** is full iff K is the empty set or a singleton.

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From spaces to systems

Lemma 19

There exists a full embedding **LoA-Top** $\xhookrightarrow{E_T}$ **LoA-TopSys** *with*

$$E_T((X, A, \tau) \xrightarrow{(f,\varphi)} (Y, B, \sigma)) =$$

$$(X, A, \tau, \models_1) \xrightarrow{(f,\varphi,((f,\varphi)^{\leftarrow})^{op})} (Y, B, \sigma, \models_2)$$

where $\models_i(z, p) = p(z)$.

Proof.

As an example show that $E_T(f, \varphi)$ is in **LoA-TopSys**:

$$\models_1(x, (f, \varphi)^{\leftarrow}(p)) = \models_1(x, \varphi^{op} \circ p \circ f) =$$

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From systems to spaces: spatialization

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There exists a functor **LoA-TopSys** $\xrightarrow{\text{Spat}}$ **LoA-Top** *defined by*

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As an example show that Spat(f) is in **LoA-Top**:

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E_T and Spat form an adjoint pair

Theorem 21

Spat *is a right-adjoint-left-inverse of* E_T.

Proof.

- Given a system $(X, A, B, \models)$,

$$E_T\,\mathrm{Spat}(X, A, B, \models) \xrightarrow{(1_X, 1_A, \Phi^{op})} (X, A, B, \models)$$

with $\Phi(b) = \models(-, b)$ provides an E_T-(co-universal) map.

- Straightforward computations show that Spat $E_T = 1_{\mathbf{LoA\text{-}Top}}$.

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LoA-Top *is isomorphic to a full (regular mono)-coreflective subcategory of* **LoA-TopSys**.

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$$E_L^Q(B \xrightarrow{\varphi} C) = (\mathrm{Pt}_Q(B), Q, B, \models_1) \xrightarrow{((\varphi^{op})^{\leftarrow}_Q, 1_Q, \varphi)} (\mathrm{Pt}_Q(C), Q, C, \models_2)$$

  where $\mathrm{Pt}_Q(B) = \mathbf{A}(B, Q)$ *and* $\models_i(p, d) = p(d)$.
- E_L^Q *is full iff* $\mathbf{A}(Q, Q) = \{1_Q\}$.
- *If* Q *is an initial (terminal) object in* $\mathbf{A}$, *then* E_L^Q *is full.*

From systems to localic algebras: localification

Lemma 24

There exists a functor **LoA-TopSys** $\xrightarrow{\text{Loc}}$ **LoA** *defined by*

$$\text{Loc}((X, A, B, \models_1) \xrightarrow{f} (Y, C, D, \models_2)) = B \xrightarrow{(\Omega f)^{op}} D.$$

Lemma 25

- Loc *is a left inverse of* E_L^Q.
- *In general* E_L^Q *has neither left nor right adjoint and therefore* Loc *is neither left nor right adjoint of* E_L^Q.

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E_L^Q has neither left nor right adjoint

Proof.

- If E_L^Q has a left adjoint, then it preserves limits. In particular, it preserves terminal objects. However, $\mathbf{1}$ is a terminal object in **Frm** and $E_L^2(\mathbf{1}) = (\mathrm{Pt}_2(\mathbf{1}), \mathbf{2}, \mathbf{1}, \models) = (\varnothing, \mathbf{2}, \mathbf{1}, \models)$ is not a terminal object in **Loc-TopSys**.
- If E_L^Q has a right adjoint, then it preserves colimits and, in particular, initial objects. However, $\mathbf{2}$ is an initial object in **Frm** and $E_L^2(\mathbf{2}) = (\mathrm{Pt}_2(\mathbf{2}), \mathbf{2}, \mathbf{2}, \models) = (\mathbf{1}, \mathbf{2}, \mathbf{2}, \models)$ is not an initial object in **Loc-TopSys**.

E_L^Q has neither left nor right adjoint

Proof.

- If E_L^Q has a left adjoint, then it preserves limits. In particular, it preserves terminal objects. However, 1 is a terminal object in **Frm** and $E_L^2(1) = (\mathrm{Pt}_2(1), 2, 1, \models) = (\varnothing, 2, 1, \models)$ is not a terminal object in **Loc-TopSys**.
- If E_L^Q has a right adjoint, then it preserves colimits and, in particular, initial objects. However, 2 is an initial object in **Frm** and $E_L^2(2) = (\mathrm{Pt}_2(2), 2, 2, \models) = (1, 2, 2, \models)$ is not an initial object in **Loc-TopSys**.

From localic algebras to systems again

Definition 26

Let $\mathbf{LoA}_i \times \mathbf{LoA}$ be the subcategory of $\mathbf{LoA} \times \mathbf{LoA}$ with the same objects and with (φ, ψ) in $\mathbf{LoA}_i \times \mathbf{LoA}$ iff φ is a $\mathbf{LoA}$-isomorphism.

Lemma 27

- *There exists an embedding* $\mathbf{LoA}_i \times \mathbf{LoA} \xhookrightarrow{E^i_L} \mathbf{LoA}\text{-}\mathbf{TopSys}$ *defined by*

$$E^i_L((A,B) \xrightarrow{(\varphi,\psi)} (C,D)) =$$

$$(\mathrm{Pt}_A(B), A, B, \models_1) \xrightarrow{((|\psi^{op}|, \varphi^{-1})^{\leftarrow}, \varphi, \psi)} (\mathrm{Pt}_C(D), C, D, \models_2)$$

where $\mathrm{Pt}_A(B) = \mathbf{A}(B, A)$ *and* $\models_i(p, e) = p(e)$.

- *In general* E^i_L *is non-full.*

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Modified variable-basis topological systems

Definition 28

- Given a subcategory **C** of **A**, the category **C-TopSys** comprises the following data:
  - Objects: **C**-topological systems or **C**-systems $(X, A, B, \models)$, where (X, A, B) is a $\mathbf{Set} \times \mathbf{C} \times \mathbf{C}^{op}$-object and $X \times B \xrightarrow{\models} A$ is a map (satisfaction relation) such that for every $x \in X$,
  $$B \xrightarrow{\models(x,-)} A \text{ is a homomorphism.}$$
  - Morphisms: **C**-continuous maps
  $$(X, A, B, \models_1) \xrightarrow{f=(\mathrm{pt}\, f, \Sigma f, (\Omega f)^{op})} (Y, C, D, \models_2),$$
  where f is a $\mathbf{Set}\times\mathbf{C}\times\mathbf{C}^{op}$-morphism and for every $x \in X$, $d \in D$,
  $$\models_2(\mathrm{pt}\, f(x), d) = \Sigma f(\models_1(x, \Omega f(d))).$$
- $| - |$ is the forgetful functor to the category $\mathbf{Set} \times \mathbf{C} \times \mathbf{C}^{op}$.

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Some remarks

- Given a subcategory **C** of **A**, the categories $\mathbf{C}^{op}$-**TopSys** and **C**-**TopSys** have (eventually) the same objects.
- For a **C**-object Q, Q-**TopSys** is (eventually) a subcategory of both $\mathbf{C}^{op}$-**TopSys** and **C**-**TopSys**.
- Let **D** be the subcategory of **C** with the same objects and with φ in **C** iff φ is an isomorphism. Then the categories $\mathbf{D}^{op}$-**TopSys** and **D**-**TopSys** are isomorphic.

- The following considers the category **A**-**TopSys**.
- Call **A**-systems by systems and **A**-continuity by continuity.

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From algebras to systems

Lemma 29

There exists a full embedding $\mathbf{A} \times \mathbf{LoA} \xhookrightarrow{E_L} \mathbf{A\text{-}TopSys}$ *with*

$$E_L((A,B) \xrightarrow{(\varphi,\psi)} (C,D)) =$$

$$(\mathtt{Pt}_A(B), A, B, \models_1) \xrightarrow{((|\psi^{op}|,\varphi^{op})^{\leftarrow},\varphi,\psi)} (\mathtt{Pt}_C(D), C, D, \models_2)$$

where $\mathtt{Pt}_A(B) = \mathbf{A}(B,A)$ *and* $\models_i(p,e) = p(e)$.

Proof.

As an example show that $E_L(\varphi,\psi)$ is in **A-TopSys**:

$$\models_2(((|\psi^{op}|,\varphi^{op})^{\leftarrow})(p),d) = \models_2(\varphi \circ p \circ |\psi^{op}|, d) =$$
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From systems to algebras: localification

Lemma 30

There exists a functor $\mathbf{A\text{-}TopSys} \xrightarrow{\text{Loc}} \mathbf{A} \times \mathbf{LoA}$ *defined by*

$$\text{Loc}((X, A, B, \models_1) \xrightarrow{f} (Y, C, D, \models_2)) = (A, B) \xrightarrow{(\Sigma f, (\Omega f)^{op})} (C, D).$$

E_L and Loc form an adjoint pair

Theorem 31

Loc *is a left-adjoint-left-inverse of* E_L.

Proof.

- Given a system $(X, A, B, \models)$,

$$(X, A, B, \models) \xrightarrow{(f, 1_A, 1_B)} E_L \operatorname{Loc}(X, A, B, \models)$$

with $f(x) = \models(x, -)$ provides an E_L-universal map.
- Straightforward computations show that $\operatorname{Loc} E_L = 1_{\mathbf{A} \times \mathbf{LoA}}$.

Corollary 32

$\mathbf{A} \times \mathbf{LoA}$ *is isomorphic to a full reflective subcategory of* **A-TopSys** *which (in general) is neither mono- nor epi-reflective.*

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From spaces to systems

Definition 33

Let $\mathbf{LoA}\text{-}\mathbf{Top}_i$ be the subcategory of $\mathbf{LoA}\text{-}\mathbf{Top}$ with the same objects and with (f, φ) in $\mathbf{LoA}\text{-}\mathbf{Top}_i$ iff φ is a localic isomorphism.

Lemma 34

There exists an embedding $\mathbf{LoA}\text{-}\mathbf{Top}_i \xhookrightarrow{E^i_T} \mathbf{A}\text{-}\mathbf{TopSys}$ *with*

$$E^i_T((X, A, \tau) \xrightarrow{(f,\varphi)} (Y, B, \sigma)) =$$

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where $\models_j(z, p) = p(z)$. *In general* E^i_T *is non-full.*

From spaces to systems

Definition 33

Let **LoA-Top**$_i$ be the subcategory of **LoA-Top** with the same objects and with (f, φ) in **LoA-Top**$_i$ iff φ is a localic isomorphism.

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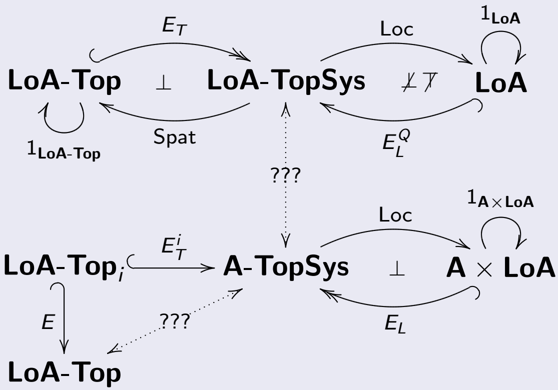
Spatialization & Localification

Basic functorial relationships



A-TopSys versus **LoA-TopSys** and **LoA-Top**

Problem 35

How are the categories **A-TopSys** *and* **LoA-TopSys** *related?*

Problem 36

Are there any non-trivial functorial relationships between **A-TopSys** *and* **LoA-Top***?*

A-TopSys versus **LoA-TopSys** and **LoA-Top**

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Algebraic properties of **A-TopSys**

Lemma 37

The concrete category (**LoA-TopSys**, $|-|$) *has the following properties:*

- $|-|$ *creates isomorphisms;*
- $|-|$ *is adjoint;*
- **LoA-TopSys** *is (Epi, Mono-Source)-factorizable;*

and therefore it is essentially algebraic.

Problem 38

- *Is the concrete category* (**A-TopSys**, $|-|$) *essentially algebraic?*
- *What about algebraicity?*

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Thank you for your attention!