# An Example of Commutative Basic Algebra which is not an MV-algebra

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# MV-algebras

The concept of MV-algebra as an algebraic axiomatization of Łukasiewicz many-valued propositional logic was introduced by C.C.Chang.

**Definition**

An MV-algebra is an algebra $\mathbf{A} = (A, \oplus, \neg, 0)$ of type $(2, 1, 0)$ satisfying the identities:

(M1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$

(M2) $x \oplus y = y \oplus x$

(M3) $x \oplus 0 = x$

(M4) $\neg\neg x = x$

(M5) $x \oplus \neg 0 = \neg 0$

(M6) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x.$

# MV-algebras

As known, if $\mathbf{A} = (A, \oplus, \neg, 0)$ as MV-algebra then $(A, \vee, \wedge, 1, 0)$, where

$$x \vee y := \neg(\neg x \oplus y) \oplus y$$
$$x \wedge y := \neg(\neg x \vee \neg y)$$
$$1 := \neg 0$$

is a bounded distributive lattice. Induced order is given by:

$$x \leq y \text{ iff } \neg x \oplus y = 1.$$

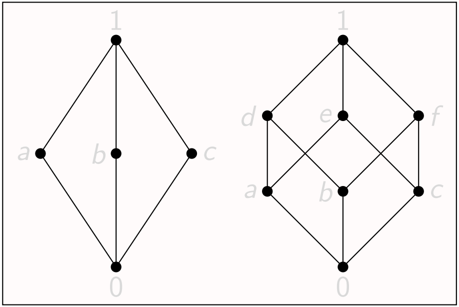
Moreover, the mapping $^{a} : [a, 1] \rightarrow [a, 1]$ defined by $x^{a} = \neg x \oplus a$ is an antitone involution on $[a, 1]$ (i.e. $x \leq y$ iff $y^{a} \leq x^{a}$ and $(x^{a})^{a} = x$).

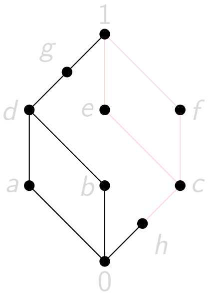
# Lattices with Section Antitine Involutions

## Definition

A lattice with section antitone involutions is a system $\mathbf{L} = (L, \vee, \wedge, ({}^{a})_{a \in L}, 0, 1)$ where $(L, \vee, \wedge, 0, 1)$ is a bounded lattice such that every principal order filter $[a, 1]$ (called a section) possesses an antitone involution $x \mapsto x^{a}$.

## Example



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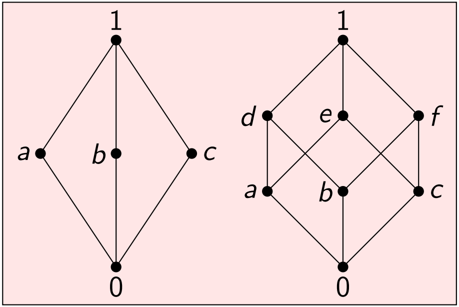


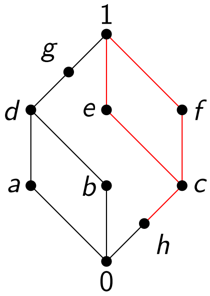

# Lattice with Section Antitone Involutions

The family $({}^{a})_{a \in L}$ of section antitone involutions being partial unary operations on $L$ can be equivalently replaced by a single binary operation $\rightarrow$ defined by

$$x \rightarrow y := (x \vee y)^{y}.$$

This easy "trick" allows one to treate lattices with section antitone involutions as total algebras $(L, \vee, \wedge, \rightarrow, 0, 1)$ or even $(L, \rightarrow, 0, 1)$ that form a variety (see e.g. [CHK1] or [CE]).
In [CHK1] MV-algebras were characterized as those lattices with section antitone involutions satysfying the so-called exchange identity (EI):

$$x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z). \qquad (EI)$$

# The Correspondence Theorem

**Theorem**

**(i)** *Let* $\mathbf{L} = (A, \vee, \wedge, ({}^a)_{a \in A}, 0, 1)$ *be a lattice with section antitone involutions. Then the assigned algebra* $\mathcal{A}(\mathbf{L}) = (L, \oplus, \neg, 0)$*, where*

$$x \oplus y := (x^0 \vee y)^y \quad \text{and} \quad \neg x := x^0$$

*satisfies the identites*

(A1) $x \oplus 0 = x$

(A2) $\neg\neg x = x$

(A3) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$

(A4) $\neg(\neg(\neg(x \oplus y) \oplus y) \oplus z) \oplus (x \oplus z) = 1.$

# The Correspondence Theorem

**(ii)** Conversely, given an algebra $\mathbf{A} = (A, \oplus, \neg, 0)$ satisfying the identites (A1)-(A4), then for every $a \in A$, the mapping

$$x \mapsto x^a := \neg x \oplus a$$

is an antitone involution on the section $[a, 1]$, and the structure $\mathcal{L}(\mathbf{A}) = (A, \vee, \wedge, ({}^a)_{a \in A}, 0, 1)$ is a lattice with section antitone involutions.

**(iii)** The corespondence is one-to-one, i.e. $\mathcal{L}(\mathcal{A}(\mathbf{L})) = \mathbf{L}$ and $\mathcal{A}(\mathcal{L}(\mathbf{A})) = \mathbf{A}$.

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# Basic Algebras

Algebras satisfying the identities (A1)-(A4) are called **basic algebras**. Hence, basic algebras form the variety of type $\langle 2, 1, 0\rangle$. It has been proved [CHK1] that this variety is arithmetical.

We call a basic algebra $\mathbf{A} = (A, \oplus, \neg, 0)$ **commutative** if $\oplus$ is commutative.

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**b)** Every MV-algebra is a commutative basic algebra.

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# Commutative basic algebras-theorems

**Theorem**

*Lattices induced by commutative basic algebras are distributive.*

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*Subdirectly irreducible commutative basic algebras are chains.*

# Commutative basic algebras

Given a basic algebra $\mathbf{A} = (\mathbf{A}, \oplus, \neg, \mathbf{0})$ we use the following derived operations:

$$a \odot b := \neg(\neg a \oplus \neg b)$$

$$a \to b := \neg a \oplus b$$

**Theorem**

*Commutative basic algebras are residuated structures, i.e. the* **adjointness condition**

$$x \odot y \leq z \quad \textit{iff} \quad x \leq y \to z$$

*holds.*

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# Are commutative basic algebras an MV-algebras?

- Are all commutative basic algebras an MV-algebras?
- Are complete commutative basic algebras an MV-algebras?

In [BoHa1] we have proved:
If there is a complete commutative basic algebra which is not an MV-algebra then there is a commutative basic algebra (which is not an MV-algebra) on the interval of reals.
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# Important properties of commutative basic algebras.

**Claim** A basic algebra is commutative if and only if satisfies the contraposition law ($x \to y = \neg y \to \neg x$).

**Remark** We can describe the operation "$\to$" as follows:

$$x \to y = \begin{cases} x^y & \text{if} \quad y \leq x \\ 1 & \text{otherwise.} \end{cases}$$

in any linearly ordered commutative basic algebra.

**Theorem (Fixpoint theorem)**

*If $\mathbf{A} = (A, \oplus, \neg, 0)$ is a complete commutative basic algebra and $x \in A$ then there is a unique $x^* \in [x, 1]$ such that $x^* = x^* \to x$.*

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## Theorem

*Any commutative basic algebra* $\mathbf{A} = ([0,a], \oplus, \neg, 0)$ *is isomorphic to a commutative basic algebra* $\mathbf{A}' = ([0,a], \oplus', \neg', 0)$ *such that* $\neg' x = a - x$ *for any* $x \in [0,a]$.

**Sketch of proof.** Let $\alpha_* : [0_*, a] \longrightarrow [a/2, a]$ be any order isomorphism. Define the mapping

$$\alpha(x) := \begin{cases} \alpha_*(x) & \text{if} \quad x \in [0_*, a] \\ 12 - \alpha_*(\neg x) & \text{otherwise.} \end{cases}$$

Putting

$$x \to' y := \alpha(\alpha^{-1}(x) \to \alpha^{-1}(y)),$$

one can easily prove that for $x \oplus' y := \neg x \to' y$, $\mathbf{A}' = ([0,a], \oplus', \neg', 0)$ is a commutative basic algebra for which $\mathbf{A}' \cong \mathbf{A}$, the isomorphism of which is given by $\alpha$. ∎

We construct a commutative basic algebra on the interval $[0, 12]$ of reals. Without lost of generality, we may suppose that the operation $\neg$ is defined by $\neg x := 12 - x$.

## Lemma

*Let* $\mathbf{A} = ([0, 12], \oplus, \neg, 0)$ *be a commutative basic algebra. Then the function* $f : [0, 12]^2 \longrightarrow [0, 12]$, *where* $f(x, y) := x \rightarrow y$ *is continuous* (*in a usual sense*) *and for all* $y \in [0, 12]$ *and all* $x, x_1, x_2 \in [y, 12]$ *we have*

(i) $f(x, y) = f(\neg y, \neg x)$

(ii) *if* $x \geq y$ *then* $f(f(x, y), y) = x$

(iii) *if* $x_1 \leq x_2$ *then* $f(x_1, y) \geq f(x_2, y)$.

As well-known, the implication $\rightarrow_{MV}$ on $[0, 12]$ considered as an MV-algebra is given by stipulation $x \rightarrow_{MV} y := 12 - x + y$. Consider another function $f(x, y)$ of the form:

$$f(x, y) := 12 - x + y + d(x, y),$$

where $d(x, y)$ measures the "difference" of $f(x, y)$ and "$x \rightarrow_{MV} y$". The idea of constructing a commutative basic algebra which is not MV-algebra is based on finding the non-zero function $d(x, y)$. Hence, given $f(x, y)$ as before, we derive the properties of $d(x, y)$.

### Lemma

*For all* $x, y \in [0, 12]$ *with* $x \geq y$ *we have* $d(x, y) = d(\neg y, \neg x) = d(f(x, y), y)$.

Now, consider the following sets:

$$g = \{\langle x_*, x\rangle | \, x \in [0, 12]\}, h = \{\langle 12 - x, 12 - x_*\rangle | x \in [0, 12]\},$$

$$k = \{\langle x, \neg x\rangle | x \in [6, 12]\}.$$

The continuity of $f$ yields that all $g$, $h$, $k$ are the continuous curves. Moreover, they divide the area $\{\langle x, y\rangle \in [0, 12]^2 | x \geq y\}$ into six parts.

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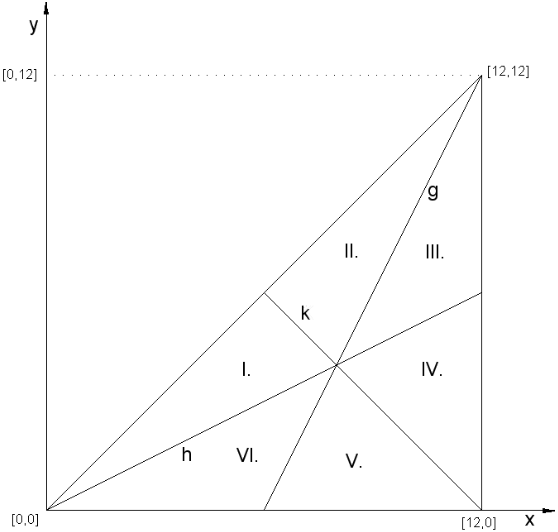
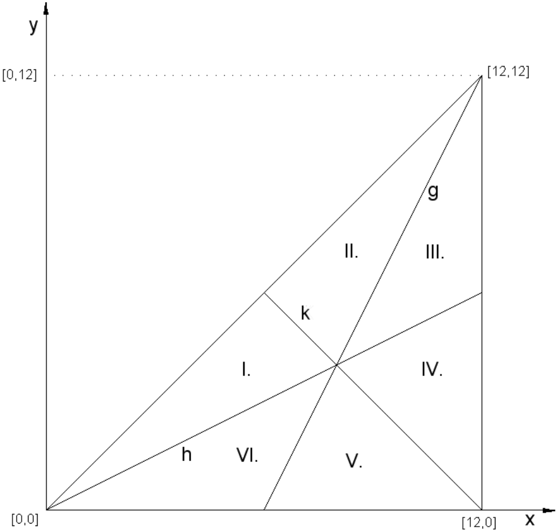

y
[0,12]
[12,12]
g
II.
III.
k
I.
IV.
h
VI.
V.
[0,0]
[12,0]
x

The following table describes the membership of points into the above areas.

| $\langle x, y \rangle$ | $\langle \neg y, \neg x \rangle$ | $\langle f(x, y), y \rangle$ |
|---|---|---|
| I. | II. | IV. |
| II. | I. | III. |
| III. | VI. | II. |
| IV. | V. | I. |
| V. | IV. | VI. |
| VI. | III. | V. |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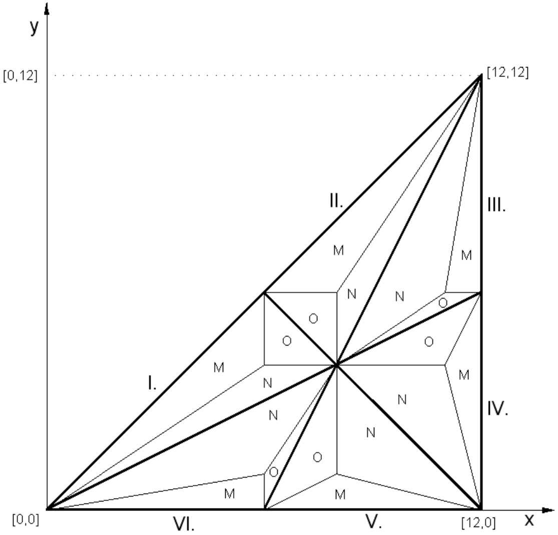
x

Now we present a commutative basic algebra $\mathbf{A} = ([0, 12], \oplus, \neg, 0)$ by constructing the function $f(x, y)$. Let the curves $g$, $h$, $k$ be defined as follows:

- $g$ is an abscissa from $[6, 0]$ to $[12, 12]$
- $h$ is an abscissa from $[0, 0]$ to $[6, 12]$
- $k$ is an abscissa from $[6, 6]$ to $[12, 0]$.

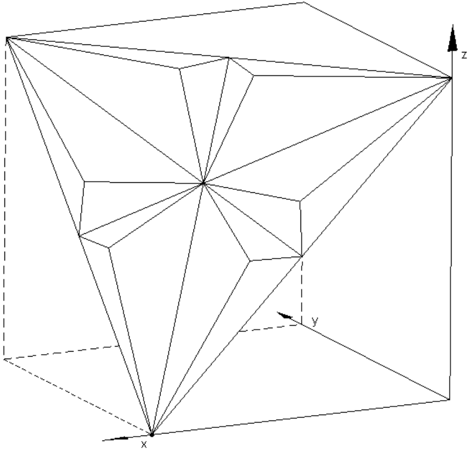
Thus, the curves $g$, $h$, $k$ coincide on the standard MV-algebra on $[0, 12]$. On the the area I. (see Figure 2.1.) we define the function $d(x, y)$ as a "pyramid" and with vertex in $[6, 4]$ with height 1.8. So, the area I. is splitting into the three subareas $M$, $N$, $O$.

$$d(x, y) := \begin{cases} 0.9x - 0.9y & \text{if} \quad x \in M \\ 1.8y - 0.9x & \text{if} \quad x \in N \\ 10.8 - 0.9x - 0.9y & \text{if} \quad x \in O. \end{cases}$$


y
x
[0,12]
[12,12]
[0,0]
[12,0]
I.
II.
III.
IV.
V.
VI.
M
N
O

| | M | N | O |
|---|---|---|---|
| I. | [0;0],[6;6],[6;4] | [0;0],[6;4],[8;4] | [6;4],[8;4],[6;6] |
| II. | [6;6],[8;6],[12;12] | [8;4],[8;6],[12;12] | [8;4],[8;6],[6;6] |
| III. | [12;6],[12;12],[11.8;6] | [8;4],[12;12],[11.8;6] | [8;4],[12;6],[11.8;6] |
| IV. | [12;0],[11.8;4],[12;6] | [12;0],[8;4],[11.8;4] | [8;4],[12;6],[11.8;4] |
| V. | [6;0],[12;0],[8;0.2] | [12;0],[8;4],[8;0.2] | [6;0],[8;4],[8;0.2] |
| VI. | [0;0],[6;0],[6;0.2] | [0;0],[8;4],[6;0.2] | [6;0],[8;4],[6;0.2] |

| $f(x,y)$ | M | N | O |
|---|---|---|---|
| I. | $12 - 0.1x + 0.1y$ | $12 - 1.9x + 2.8y$ | $22.8 - 1.9x + 0.1y$ |
| II. | $12 - 0.1x + 0.1y$ | $22.8 - 2.8x + 1.9y$ | $1.2 - 0.1x + 1.9y$ |
| III. | $120 - 10x + y$ | $\frac{12\cdot 1.9}{2.8} - \frac{1}{2.8}x + \frac{1.9}{2.8}y$ | $12 - 10x + 19y$ |
| IV. | $120 - 10x + y$ | $\frac{12}{1.9} - \frac{1}{1.9}x + \frac{2.8}{1.9}y$ | $12 - \frac{1}{1.9}x + \frac{1}{19}y$ |
| V. | $12 - x + 10y$ | $\frac{12\cdot 2.8}{1.9} - \frac{2.8}{1.9}x + \frac{1}{1.9}y$ | $\frac{12}{1.9} - \frac{1}{19}x + \frac{1}{1.9}y$ |
| VI. | $12 - x + 10y$ | $12 - \frac{1.9}{2.8}x + \frac{1}{2.8}y$ | $120 - 19x + 10y$ |


z
y
x

Since $f(x, y)$ is piecewise linear, so these are also $f(f(x, y), y)$ and $f(\neg y, \neg x)$. It can be checked that $f(f(x, y), y) = x$ and $f(\neg y, \neg x) = f(x, y)$ on all of the boudaries of areas I.-VI., hence $f(x, y)$ fulfils these identities everywhere. Thus, if we denote

$$x \oplus y = \neg x \to y = f(\neg x, y)$$

then $\mathbf{A} = ([0, 12], \oplus, \neg, 0)$ is a commutative basic algebra.

Finally, one can compute that $10 \to (8 \to 4) = 10 \to 8 = 10$ whereas $8 \to (10 \to 4) \doteq 8 \to 6.95 \doteq 11.89$. Thus in $\mathbf{A}$ the exchange identity does not hold and $\mathbf{A}$ is not an MV-algebra.

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