# Span and Chainability in Non-metric Continua

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# Chainability

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DEFINITION A continuum $X$ is chainable if every open cover has an open cover refinement which is a chain.

# Span

DEFINITION A continuum $X$ has span zero if every subcontinuum $Z$ of $X \times X$, which projects onto the same set on both coordinates, has a nonempty intersection with the diagonal $\Delta_X = \{(x,x) \mid x \in X\}$ of $X$. Otherwise we say that $X$ has span non-zero.

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OUR RESULT If there is a non-metric counterexample, there is also a metric counterexample.

# Wallman's representation theorem

DEFINITION  A lattice is called disjunctive if it models the following sentence

$$\forall ab\ \exists c\ (a \not\leq b \rightarrow c \neq 0 \text{ and } c \leq a \text{ and } b \wedge c = 0).$$

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DEFINITION A lattice is called normal if it models the following sentence

$$\forall ab\, \exists cd\ (a \sqcap b = 0 \rightarrow a \wedge d = 0 \text{ and } b \wedge c = \mathbf{0} \text{ and } c \vee d = 1).$$

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Wallman's representation extends to lattice homomorphisms and provides a functor.

# Elementarity

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LÖWENHEIM-SKOLEM THEOREM Let $A$ be an infinite $\mathcal{L}$-structure and let $X \subset A$. Denote $\kappa = \max(|\mathcal{L}|, |X|)$. Then for every cardinal $\lambda$ such that $\kappa \leq \lambda \leq |A|$, there exists an elementary substructure $B$ of $A$ such that $X \subset B$ and $|B| = \lambda$.

# Elementarity in set theory

For a cardinal $\theta$, $H(\theta)$ denotes the set of all sets whose transitive closure has cardinality less then $\theta$.

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$$H(\theta) \models \text{ ZFC - P } .$$

If $\mathcal{M}$ is an elementary submodel of $H(\theta)$ such that $2^X \in \mathcal{M}$ then $L = \mathcal{M} \cap 2^X$ is an elementary sublattice of $2^X$. Similarly $K = \mathcal{M} \cap 2^{X \times X}$ is an elementary sublattice of $2^{X \times X}$.

# Applying elementarity

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THEOREM (van der Steeg 2003) $X$ is chainable if and only if $wL$ is chainable.

# Keisler-Shelah theorem

KEISLER-SHELAH THEOREM Let $\kappa$ be a cardinal, $\lambda = \min\{\mu \mid \kappa^\mu > \kappa\}$ and let $A$ and $B$ be two elementarily equivalent $\mathcal{L}$-structures with $\mathrm{card}(A), \mathrm{card}(B) < \lambda$. Then there exists an ultrafilter $\mathcal{U}$ over $\kappa$ such that $\prod_{\mathcal{U}} A$ and $\prod_{\mathcal{U}} B$ are isomorphic.

# Reflecting span zero

THEOREM (DB+KPH 2008) If $X$ is a continuum having span zero, then $wL$ has span zero as well.

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Proof

$$\begin{array}{ccc} K & \xrightarrow{e} & 2^{X\times X} \\ \Big\downarrow \Delta & & \Big\downarrow \Delta \\ \prod_{\mathcal{U}} K & \xrightarrow{h} & \prod_{\mathcal{U}} 2^{X\times X} \end{array} \qquad (1)$$

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\begin{array}{ccc}
wL \times wL \cong wK & \xleftarrow{w(e)} & X \times X \\
\Big\uparrow {\scriptstyle w(\Delta)} & & \Big\uparrow {\scriptstyle w(\Delta)} \\
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$$Z' = w(\Delta) \circ w(h)^{-1}[w(\textstyle\prod_{\mathcal{U}} Z)].$$

THANK YOU