Residual Properties of Finite Graphs

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Stralka's example

Stralka (1980) showed that there is a compact, totally disconnected partial order that is not a Priestley space.

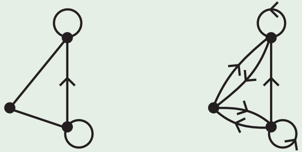
We will generalise this result.

Graphs

Definition

A digraph $\mathbf{G} = \langle G; \sim \rangle$ is a set of vertices G with a binary relation $\sim \subseteq G \times G$ corresponding to the set of edges.

Example



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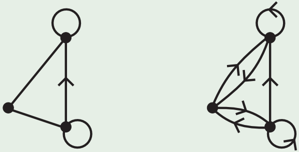

A graph is a symmetric digraph.

Quasi-equations

Definition

A quasi-equation (or quasi-identity) in the language of digraphs is a universally quantified (first order) sentence of the form:

$$\left(\mathop{\&}_{i \in I} \alpha_i \right) \longrightarrow \beta$$

where I is finite and possibly empty, and α_i, β are expressions of the form $x \sim y$ or $x = y$.

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If $\mathbf{G}$ is a digraph, we write $\mathrm{Th}_{\mathrm{qe}}(\mathbf{G})$ to denote the set of quasi-equations true in $\mathbf{G}$.

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Other classes axiomatisable by quasi-equations

- The class of pre-orders (also called quasi-orders).
- The class of equivalence relations.
- The class of k-colourable graphs. (This class is not finitely axiomatisable, but this fact requires some proof.)

Residual classes

Theorem (Malcev)

Let **G** *be a finite digraph. Then*

$$\mathcal{R}(\mathbb{S}(\mathbf{G})) = \mathbb{ISP}(\mathbf{G}) = \mathrm{Mod}(\mathrm{Th}_{\mathrm{qe}}(\mathbf{G})),$$

where $\mathcal{R}(\mathbb{S}(\mathbf{G}))$ *is the class of digraphs that are residually subgraphs of* **G**.

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H is residually a subgraph of **G** if

- for each $x, y \in \mathbf{H}$ with $x \neq y$, there exists a homomorphism $\phi : \mathbf{H} \to \mathbf{G}$ with $\phi(x) \neq \phi(y)$; and
- for each $x, y \in \mathbf{H}$ with $x \nsim y$, there exists a homomorphism $\phi : \mathbf{H} \to \mathbf{G}$ with $\phi(x) \nsim \phi(y)$

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- for each $x, y \in \mathbf{H}$ with $x \nrightarrow y$, there exists a homomorphism $\phi : \mathbf{H} \to \mathbf{G}$ with $\phi(x) \nrightarrow \phi(y)$

$\mathcal{R}(\mathbb{S}(\mathbf{G}))$ is also called the residual class of **G**.

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Let Σ be a set of digraph quasi-equations. Then $\mathrm{Mod}_{\mathrm{Bt}}(\Sigma)$ is the class of Boolean topological models of Σ, i.e. the class of digraphs **G** with a topology $\mathcal{T}$ such that:

- **G** satisfies Σ
- $\mathcal{T}$ is compact and totally disconnected
- the edge relation $\sim$ is topologically closed in $\mathbf{G} \times \mathbf{G}$.

Qe-axiomatisability

Let $\mathbf{G}$ be a finite digraph with the discrete topology. Then

$$\mathcal{R}_{\mathrm{CT}}(\mathbb{S}(\mathbf{G})) = \mathbb{ISP}(\mathbf{G}) \subseteq \mathrm{Mod}_{\mathrm{Bt}}(\mathrm{Th}_{\mathrm{qe}}(\mathbf{G})),$$

where $\mathcal{R}_{\mathrm{CT}}(\mathbb{S}(\mathbf{G}))$ is the topological residual class of $\mathbf{G}$, that is, the class of compact topological digraphs that are topologically residually subgraphs of $\mathbf{G}$.

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This is equivalent to

$$\mathcal{R}_{\mathrm{CT}}(\mathbb{S}(\mathbf{G})) = \mathrm{Mod}_{\mathrm{Bt}}(\mathrm{Th}_{\mathrm{qe}}(\Sigma))$$

for some set of quasi-equations Σ.

Stralka's example: a non-qe-axiomatisable $\mathcal{R}_{\mathrm{CT}}(\mathbb{S}(\mathbf{G}))$

Consider the graph **2**:

Note that $\mathcal{R}(\mathbb{S}(\mathbf{2})) = \mathrm{Mod}(\mathrm{Th}_{\mathrm{qe}}(\mathbf{2}))$ is the class of partial orders, and $\mathcal{R}_{\mathrm{CT}}(\mathbb{S}(\mathbf{2}))$ is the class of Priestley spaces.

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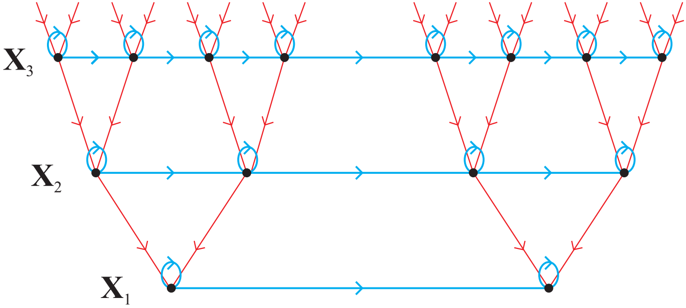
Let **X** be the inverse limit of the following system:



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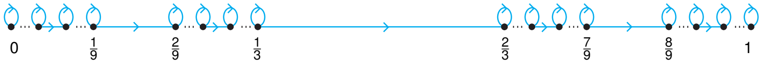

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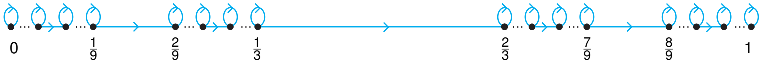

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Stralka gave a direct construction of **X**, but by constructing it as an inverse system we see how to generalise it.

Extending Stralka's example

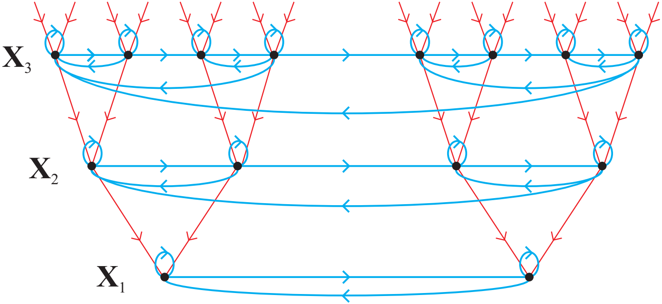
We extend Stralka's example to characterise qe-axiomatisability for classes of reflexive anti-symmetric digraphs.

Theorem

Let $\mathbf{G}$ *be a finite reflexive anti-symmetric digraph. Then* $\mathcal{R}_{\mathrm{CT}}(\mathbb{S}(\mathbf{G}))$ *is qe-axiomatisable if and only if* $\mathbf{2}$ *is not a subgraph of* $\mathbf{G}$*.*

Proof

The inverse system used in the proof:



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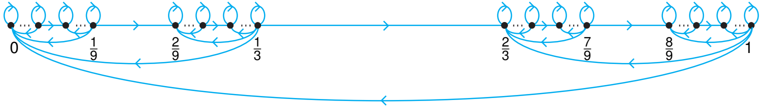

Classifying the residual classes generated by finite simple graphs

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1. $\mathbf{G}$ *consists of isolated vertices and 2-element paths;*
2. $\mathbf{G}$ *contains a 3-element path and is a disjoint union of complete bipartite graphs;*
3. *the following equivalent conditions hold:*
   - $\mathbf{G}$ *is not a disjoint union of complete bipartite graphs;*
   - $\mathcal{R}(\mathbb{S}(\mathbf{G}))$ *contains*  *;*
   - $\mathcal{R}(\mathbb{S}(\mathbf{G}))$ *contains all 2-colourable graphs;*
   - $\mathcal{R}(\mathbb{S}(\mathbf{G}))$ *is not finitely qe-axiomatisable.*

Qe-axiomatisability for simple graphs

Theorem

If **G** *is a finite simple graph,* $\mathcal{R}_{\mathrm{CT}}(\mathbb{S}(\mathbf{G}))$ *is qe-axiomatisable if and only if* **G** *consists of isolated vertices and 2-element paths.*

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By applying Erdös' probabilistic method, Feder and Vardi (1998) (and Hodkinson and Venema (2005)) construct an inverse system of finite graphs which shows that if **G** is a finite simple graph that is not a disjoint union of complete bipartite graphs, then $\mathcal{R}_{\mathrm{CT}}(\mathbb{S}(\mathbf{G}))$ is not qe-axiomatisable.

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The inverse limit is a structure $\mathbf{X}$ such that:

- $\mathbf{X} \in \mathrm{Mod}_{\mathrm{Bt}}(\mathrm{Th}_{\mathrm{qe}}(\mathbf{G}))$ (since $\mathbf{X}$ is 2-colourable)
- $\mathbf{X} \notin \mathcal{R}_{\mathrm{CT}}(\mathbb{S}(\mathbf{G}))$

Therefore $\mathcal{R}_{\mathrm{CT}}(\mathbb{S}(\mathbf{G}))$ is not qe-axiomatisable.

Some open problems

- Let **G** be a finite anti-reflexive, anti-symmetric digraph. When is $\mathcal{R}_{\mathrm{CT}}(\mathbb{S}(\mathbf{G}))$ qe-axiomatisable?

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- Let **G** be a finite anti-reflexive, anti-symmetric digraph. When is $\mathcal{R}_{\mathrm{CT}}(\mathbb{S}(\mathbf{G}))$ qe-axiomatisable?
- Let **G** be a finite digraph. Is it true that if $\mathcal{R}(\mathbb{S}(\mathbf{G}))$ is not finitely axiomatisable by quasi-equations, then $\mathcal{R}_{\mathrm{CT}}(\mathbb{S}(\mathbf{G}))$ is not qe-axiomatisable?

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