The arity gap and generalizations of Świerczkowski's lemma

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Functions of several variables

Let A and B be arbitrary non-empty sets.

- A *function of several variables* from A to B is a map $f\colon A^n \to B$ for some positive integer n, called the *arity* of f.
- In the case that $A = B$, a map $f\colon A^n \to A$ is called an *operation* on A.
- The operation $(x_1, \ldots, x_n) \mapsto x_i$ is called the i-th n-ary *projection.*
- Operations on $\{0, 1\}$ are called *Boolean functions.*

Essential variables

Let $f\colon A^n \to B$.

The i-th variable is *essential* in f, if there exist elements $a_1, \ldots, a_n, b \in A$ such that

$$f(a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_n) \neq f(a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n).$$

Otherwise, the i-th variable is *inessential* in f.

The number of essential variables in f is called the *essential arity* of f, denoted $\operatorname{ess} f$.

Variable identification minors

Let $f\colon A^n \to B$.

For $i \neq j$, define the function $f_{i\leftarrow j}\colon A^n \to B$ by

$$f_{i\leftarrow j}(x_1, \dots, x_n) = f(x_1, \dots, x_{i-1}, x_j, x_{i+1}, \dots, x_n),$$

and call it a *variable identification minor* of f, obtained by identifying the i-th variable with the j-th variable.

Arity gap

The *arity gap* of f is defined as the smallest possible decrease in the number of essential variables of f when its essential variables are identified, i.e.,

$$\operatorname{gap} f = \min_{i \neq j} (\operatorname{ess} f - \operatorname{ess} f_{i \leftarrow j}),$$

where i and j range over the set of indices of essential variables of f.

Arity gap – examples

Example

Let $f\colon \{0,1\}^4 \to \{0,1\}$ be the Boolean function

$$x_1x_4 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4.$$

The variable identification minors of f are

$$\begin{aligned}
f_{1\leftarrow 2} &= x_2x_4 + x_2x_4 + x_2x_3x_4 + x_2x_3x_4 = 0,\\
f_{1\leftarrow 3} &= x_3x_4 + x_2x_3x_4 + x_3x_4 + x_2x_3x_4 = 0,\\
f_{1\leftarrow 4} &= x_4 + x_2x_4 + x_3x_4 + x_2x_3x_4,\\
f_{2\leftarrow 3} &= x_1x_4 + x_1x_3x_4 + x_1x_3x_4 + x_3x_4 = x_1x_4 + x_3x_4,\\
f_{2\leftarrow 4} &= x_1x_4 + x_1x_4 + x_1x_3x_4 + x_3x_4 = x_1x_3x_4 + x_3x_4,\\
f_{3\leftarrow 4} &= x_1x_4 + x_1x_2x_4 + x_1x_4 + x_2x_4 = x_1x_2x_4 + x_2x_4.
\end{aligned}$$

Thus, $\operatorname{gap} f = 1$.

Arity gap – examples

Example

Let $f\colon \{0,1\}^n \to \{0,1\}$ be the Boolean function

$$f(x_1, \ldots, x_n) = x_1 + \cdots + x_n.$$

It is easy to verify that $\operatorname{gap} f = 2$.

Example

Let $|A| = k$, and consider the function $f\colon A^k \to B$ given by

$$f(x_1, \ldots, x_k) = \begin{cases} b, & \text{if } x_i \neq x_j \text{ for all } i \neq j, \\ c, & \text{otherwise,} \end{cases}$$

where b and c are distinct elements of B. We have that $\operatorname{gap} f = k$.

Arity gap – examples

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Arity gap – background

The study of arity gap goes back to the 1963 paper by Salomaa, in which it was shown that the arity gap of every Boolean function is at most 2.

This upper bound was extended by Willard to a more general setting as follows:

Theorem

Suppose that $f \colon A^n \to B$ *depends on all of its variables. If* $n > |A|$*, then* $\operatorname{gap} f \leq 2$*.*

A. SALOMAA, On essential variables of functions, especially in the algebra of logic, *Ann. Acad. Sci. Fenn. Ser. A I. Math.* **339** (1963) 3–11.

R. WILLARD, Essential arities of term operations in finite algebras, *Discrete Math.* **149** (1996) 239–259.

Arity gap – background

For each positive integer n, define the function $\mathrm{oddsupp}\colon A^n \to \mathcal{P}(A)$ by

$$\mathrm{oddsupp}(a_1, \ldots, a_n) = \bigl\{a_i : |\{1, \ldots, n\} : a_j = a_i| \text{ is odd}\bigr\}.$$

A function $f\colon A^n \to B$ is *determined by* oddsupp if there is a nonconstant function $f^*\colon \mathcal{P}(A) \to B$ such that $f = f^* \circ \mathrm{oddsupp}$. The following is a consequence of theorems of Willard and Berman & Kisielewicz:

Corollary

Suppose that $f\colon A^n \to B$, $n > \max(|A|, 3)$, depends on all of its variables. If $\mathrm{gap}\, f = 2$, *then f is determined by* oddsupp.

J. BERMAN, A. KISIELEWICZ, On the number of operations in a clone, *Proc. Amer. Math. Soc.* **122** (1994) 359–369.

Arity gap – results

The previous theorems leave unsettled the arity gap of functions with small essential arity. In order to deal with the case that $\operatorname{ess} f \leq |A|$, we need to introduce some terminology.

For $n \geq 2$, the *diagonal* of A^n is defined as

$$A^n_{=} = \{\mathbf{a} \in A^n : a_i = a_j \text{ for some } i \neq j\}.$$

We define $A^1_{=} = A$. Note that if $n > |A|$, then $A^n_{=} = A^n$.

Let $f \colon A^n \to B$. Any function $g \colon A^n \to B$ satisfying $f|_{A^n_{=}} = g|_{A^n_{=}}$ is called a *support* of f.

Arity gap – results

The *quasi-arity* of $f\colon A^n \to B$, denoted $\operatorname{qa} f$, is defined as the minimum of the essential arities of the supports of f. If $\operatorname{qa} f = m$, we say that f is *quasi-m-ary*.

Note that if $n > |A|$, then quasi-arity simply means essential arity.

We say that f is *strongly determined by* oddsupp if $f|_{A^n_=} = f^* \circ \operatorname{oddsupp}^*$, where $f^*\colon \mathcal{P}'(A) \to B$ is a nonconstant function and $\operatorname{oddsupp}^*\colon A^n_= \to \mathcal{P}'(A)$ is defined as before, but here $\mathcal{P}'(A)$ denotes the set of odd or even—depending on the parity of n—subsets of A of cardinality at most $n - 2$.

Arity gap – results

Theorem

Suppose that $f\colon A^n \to B$, $n \geq 2$, $n \neq 3$, *depends on all of its variables.*

1. *For* $0 \leq m \leq n-3$, $\operatorname{gap} f = n - m$ *if and only if* $\operatorname{qa} f = m$.
2. $\operatorname{gap} f = 2$ *if and only if* $\operatorname{qa} f = n-2$ *or* $\operatorname{qa} f = n$ *and* f *is strongly determined by* oddsupp.
3. *Otherwise* $\operatorname{gap} f = 1$.

Arity gap – results

Theorem

Suppose that $f: \{0,1\}^n \to B$ *depends on all of its variables. Then* $\operatorname{gap} f = 2$ *if and only if* f *satisfies one of the following conditions:*

1. $n = 2$ *and* f *is a nonconstant function satisfying* $f(0,0) = f(1,1)$*;*
2. $f = g \circ h$*, where* $g: \{0,1\} \to B$ *is injective and* $h: \{0,1\}^n \to \{0,1\}$ *is a Boolean function equivalent to one of the following:*
   - $x_1 + x_2 + \cdots + x_m$ *for some* $m \geq 2$*,*
   - $x_1 x_2 + x_1$*,*
   - $x_1 x_2 + x_1 x_3 + x_2 x_3$*,*
   - $x_1 x_2 + x_1 x_3 + x_2 x_3 + x_1 + x_2$*.*

Otherwise $\operatorname{gap} f = 1$*.*

Świerczkowski's lemma

An operation $f\colon A^n \to A$ is called a *quasi-projection,* if there exists an n-ary projection $(x_1, \dots, x_n) \mapsto x_i$ on A that is a support of f.

Theorem (Świerczkowski's lemma)

Let $f\colon A^n \to A$ and $n \geq 4$. Then f is a quasi-projection if and only if every variable identification minor of f is a projection.

Generalizations of Świerczkowski's lemma

A function $f\colon A^n \to B$ is *quasi-constant*, if there exists a constant function that is a support of f, i.e., the restriction $f|_{A^n_=}$ is a constant function.

Theorem

Let $f\colon A^n \to B$.

1. *For $n \geq 2$, all variable identification minors of f are constant functions if and only if f is quasi-constant.*
2. *For $n = 2$ or $n \geq 4$, all variable identification minors of f are essentially unary if and only if f is quasi-unary.*

Furthermore, in parts 1 and 2, provided that $n \geq 4$, the variable identification minors of f are equivalent to the unique essentially at most unary support of f.

Consider the first item in our classification of functions by their arity gap.

Theorem

Suppose that $f\colon A^n \to B$ depends on all of its variables. For $0 \leq m \leq n-3$, we have that $\operatorname{gap} f = n - m$ if and only if $\operatorname{qa} f = m$.

The cases $m = 0$ and $m = 1$ correspond to parts 1 and 2 of the Theorem on the previous slide, so this can be viewed as yet another generalization of Świerczkowski's lemma.