Constructions of small A-loops

Přemysl Jedlička[1], Michael Kinyon[2], Petr Vojtěchovský[2]

[1]Department of Mathematics
Faculty of Engineering (former Technical Faculty)
Czech University of Life Sciences (former Czech University of Agriculture), Prague

[2]Department of Mathematics
University of Denver

SSAOS 2008
Třešť, 6 September 2008

Quasigroups

Definition

Let $(G, \cdot)$ be a groupoid. The mapping $L_x : a \mapsto xa$ is called the *left translation* and the mapping $R_x : a \mapsto ax$ the right translation.

Definition (Combinatorial)

A groupoid $(Q, \cdot)$ is called a *quasigroup* if the mappings L_x and R_x are bijections for each $x \in Q$.

Definition (Universal algebraic)

The algebra $(Q, \cdot, /, \backslash)$ is called a *quasigroup* if it satisfies the following identities:

$$x\backslash(x \cdot y) = y \qquad (x \cdot y)/y = x$$

$$x \cdot (x\backslash y) = y \qquad (x/y) \cdot y = x$$

Quasigroups

Definition

Let $(G, \cdot)$ be a groupoid. The mapping $L_x : a \mapsto xa$ is called the *left translation* and the mapping $R_x : a \mapsto ax$ the right translation.

Definition (Combinatorial)

A groupoid $(Q, \cdot)$ is called a *quasigroup* if the mappings L_x and R_x are bijections for each $x \in Q$.

Definition (Universal algebraic)

The algebra $(Q, \cdot, /, \backslash)$ is called a *quasigroup* if it satisfies the following identities:

$$x\backslash(x \cdot y) = y \qquad (x \cdot y)/y = x$$

$$x \cdot (x\backslash y) = y \qquad (x/y) \cdot y = x$$

Loops

Definition

A quasigroup Q is called a *loop* if it contains the identity element.

Example (A minimal nonassociative loop)

	1	2	3	4	5
1	1	2	3	4	5
2	2	1	5	3	4
3	3	4	1	5	2
4	4	5	2	1	3
5	5	3	4	2	1

Definition

Let Q be a loop. An element $a \in Q$ belongs to the *center* of Q if $ax = xa$, $a \cdot xy = ax \cdot y$, $x \cdot ay = xa \cdot y$, and $x \cdot ya = xy \cdot a$, for all $x, y \in Q$.

Loops

Definition

A quasigroup Q is called a *loop* if it contains the identity element.

Example (A minimal nonassociative loop)

	1	2	3	4	5
1	1	2	3	4	5
2	2	1	5	3	4
3	3	4	1	5	2
4	4	5	2	1	3
5	5	3	4	2	1

Definition

Let Q be a loop. An element $a \in Q$ belongs to the *center* of Q if $ax = xa$, $a \cdot xy = ax \cdot y$, $x \cdot ay = xa \cdot y$, and $x \cdot ya = xy \cdot a$, for all $x, y \in Q$.

Multiplication Groups

Definitions

Let Q be a loop.

- The group generated by L_x and R_x, for all $x \in Q$, is called *the multiplication group* of Q and it is denoted by $\mathrm{Mlt}(Q)$.
- The subgroup of $\mathrm{Mlt}(Q)$ stabilizing the neutral element of Q is called *the inner mapping group* of Q and it is denoted by $\mathrm{Inn}(Q)$.

Fact

An inner mapping of a loop needs not to be an automorphism.

Definition

A loop Q is called an *A-loop* if $\mathrm{Inn}(Q) \leq \mathrm{Aut}(Q)$.

Multiplication Groups

Definitions

Let Q be a loop.

- The group generated by L_x and R_x, for all $x \in Q$, is called *the multiplication group* of Q and it is denoted by $\mathrm{Mlt}(Q)$.
- The subgroup of $\mathrm{Mlt}(Q)$ stabilizing the neutral element of Q is called *the inner mapping group* of Q and it is denoted by $\mathrm{Inn}(Q)$.

Fact

An inner mapping of a loop needs not to be an automorphism.

Definition

A loop Q is called an *A-loop* if $\mathrm{Inn}(Q) \leq \mathrm{Aut}(Q)$.

Multiplication Groups

Definitions

Let Q be a loop.

- The group generated by L_x and R_x, for all $x \in Q$, is called *the multiplication group* of Q and it is denoted by $\mathrm{Mlt}(Q)$.
- The subgroup of $\mathrm{Mlt}(Q)$ stabilizing the neutral element of Q is called *the inner mapping group* of Q and it is denoted by $\mathrm{Inn}(Q)$.

Fact

An inner mapping of a loop needs not to be an automorphism.

Definition

A loop Q is called an *A-loop* if $\mathrm{Inn}(Q) \leq \mathrm{Aut}(Q)$.

Multiplication Groups

Definitions

Let Q be a loop.

- The group generated by L_x and R_x, for all $x \in Q$, is called *the multiplication group* of Q and it is denoted by $\mathrm{Mlt}(Q)$.
- The subgroup of $\mathrm{Mlt}(Q)$ stabilizing the neutral element of Q is called *the inner mapping group* of Q and it is denoted by $\mathrm{Inn}(Q)$.

Fact

An inner mapping of a loop needs not to be an automorphism.

Definition

A loop Q is called an *A-loop* if $\mathrm{Inn}(Q) \le \mathrm{Aut}(Q)$.

Multiplication Groups

Definitions

Let Q be a loop.

- The group generated by L_x and R_x, for all $x \in Q$, is called *the multiplication group* of Q and it is denoted by $\mathrm{Mlt}(Q)$.
- The subgroup of $\mathrm{Mlt}(Q)$ stabilizing the neutral element of Q is called *the inner mapping group* of Q and it is denoted by $\mathrm{Inn}(Q)$.

Fact

An inner mapping of a loop needs not to be an automorphism.

Definition

A loop Q is called an *A-loop* if $\mathrm{Inn}(Q) \leq \mathrm{Aut}(Q)$.

Variety of A-loops

Fact

Let Q be a loop. The inner mapping group of Q is generated by the mappings

$$L_{xy}^{-1}L_xL_y, \qquad R_{xy}^{-1}R_xR_y \quad \text{and} \quad L_x^{-1}R_x,$$

where $x, y \in Q$.

Corollary

A loop is an A-loop if it satisfies the following three identities:

$$(xy)\backslash(x(y \cdot uv)) = ((xy)\backslash(x \cdot yu)) \cdot ((xy)\backslash(x \cdot yv)),$$
$$((uv \cdot x)y)/(xy) = ((ux \cdot y)/(xy)) \cdot ((vx \cdot y)/(xy)),$$
$$x\backslash(uv \cdot x) = (x\backslash(ux)) \cdot (x\backslash(vx)).$$

Variety of A-loops

Fact

Let Q be a loop. The inner mapping group of Q is generated by the mappings

$$L_{xy}^{-1}L_xL_y, \qquad R_{xy}^{-1}R_xR_y \quad \text{and} \quad L_x^{-1}R_x,$$

where $x, y \in Q$.

Corollary

A loop is an A-loop if it satisfies the following three identities:

$$(xy)\backslash(x(y \cdot uv)) = ((xy)\backslash(x \cdot yu)) \cdot ((xy)\backslash(x \cdot yv)),$$
$$((uv \cdot x)y)/(xy) = ((ux \cdot y)/(xy)) \cdot ((vx \cdot y)/(xy)),$$
$$x\backslash(uv \cdot x) = (x\backslash(ux)) \cdot (x\backslash(vx)).$$

Smallest Moufang Loop

Construction by O. Chein:

1	2	3	4	5	6	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$	$\bar{6}$
2	1	4	3	6	5	$\bar{2}$	$\bar{1}$	$\bar{6}$	$\bar{5}$	$\bar{4}$	$\bar{3}$
3	6	5	2	1	4	$\bar{3}$	$\bar{4}$	$\bar{5}$	$\bar{6}$	$\bar{1}$	$\bar{2}$
4	5	6	1	2	3	$\bar{4}$	$\bar{3}$	$\bar{2}$	$\bar{1}$	$\bar{5}$	$\bar{6}$
5	4	1	6	3	2	$\bar{5}$	$\bar{6}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$
6	3	2	5	4	1	$\bar{6}$	$\bar{5}$	$\bar{4}$	$\bar{3}$	$\bar{2}$	$\bar{1}$
$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$	$\bar{6}$	1	2	3	4	5	6
$\bar{2}$	$\bar{1}$	$\bar{6}$	$\bar{5}$	$\bar{4}$	$\bar{3}$	2	1	4	3	6	5
$\bar{3}$	$\bar{4}$	$\bar{5}$	$\bar{6}$	$\bar{1}$	$\bar{2}$	3	6	5	2	1	4
$\bar{4}$	$\bar{3}$	$\bar{2}$	$\bar{1}$	$\bar{5}$	$\bar{6}$	4	5	6	1	2	3
$\bar{5}$	$\bar{6}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	5	4	1	6	3	2
$\bar{6}$	$\bar{5}$	$\bar{4}$	$\bar{3}$	$\bar{2}$	$\bar{1}$	6	3	2	5	4	1

Smallest Conjugacy Closed Loop

Construction by A. Drápal:
We take a group $G(+)$, an automorphism $f \in \mathrm{Aut}(G)$ and $t \in G$ satisfying $f^2(x) = t^{-1}xt$ and $f(t) \neq t$. We construct

$$x * y = \begin{array}{c|c} x+y & \overline{f(x)+y} \\ \hline \overline{x+y} & f(x)+y+t \end{array}$$

Example

1	2	3	$\bar{1}$	$\bar{2}$	$\bar{3}$
2	3	1	$\bar{3}$	$\bar{1}$	$\bar{2}$
3	1	2	$\bar{2}$	$\bar{3}$	$\bar{1}$
$\bar{1}$	$\bar{2}$	$\bar{3}$	2	3	1
$\bar{2}$	$\bar{3}$	$\bar{1}$	1	2	3
$\bar{3}$	$\bar{1}$	$\bar{2}$	3	1	2

Smallest A-loop

Construction by R. H. Bruck & L. J. Paige:
We take a group G and a nontrivial automorphism $f \in \mathrm{Aut}(G)$. We construct

$$x * y = \begin{array}{c|c} x+y & \overline{x+y} \\ \hline f(x+y) & f^{-1}(x+y) \end{array}$$

Example

1	2	3	$\bar{1}$	$\bar{2}$	$\bar{3}$
2	3	1	$\bar{2}$	$\bar{3}$	$\bar{1}$
3	1	2	$\bar{3}$	$\bar{1}$	$\bar{2}$
$\bar{1}$	$\bar{3}$	$\bar{2}$	1	3	2
$\bar{3}$	$\bar{2}$	$\bar{1}$	3	2	1
$\bar{2}$	$\bar{1}$	$\bar{3}$	2	1	3

Commutative A-loops of Order 8

1	2	3	4	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$
2	3	4	1	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{1}$
3	4	1	2	$\bar{3}$	$\bar{4}$	$\bar{1}$	$\bar{2}$
4	1	2	3	$\bar{4}$	$\bar{1}$	$\bar{2}$	$\bar{3}$
$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	1	4	3	2
$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{1}$	4	3	2	1
$\bar{3}$	$\bar{4}$	$\bar{1}$	$\bar{2}$	3	2	1	4
$\bar{4}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	2	1	4	3

1	2	3	4	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$
2	1	4	3	$\bar{2}$	$\bar{1}$	$\bar{4}$	$\bar{3}$
3	4	1	2	$\bar{3}$	$\bar{4}$	$\bar{1}$	$\bar{2}$
4	3	2	1	$\bar{4}$	$\bar{3}$	$\bar{2}$	$\bar{1}$
$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	1	3	4	2
$\bar{2}$	$\bar{1}$	$\bar{4}$	$\bar{3}$	3	1	2	4
$\bar{3}$	$\bar{4}$	$\bar{1}$	$\bar{2}$	4	2	1	3
$\bar{4}$	$\bar{3}$	$\bar{2}$	$\bar{1}$	2	4	3	1

1	2	3	4	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$
2	1	4	3	$\bar{2}$	$\bar{1}$	$\bar{4}$	$\bar{3}$
3	4	1	2	$\bar{3}$	$\bar{4}$	$\bar{1}$	$\bar{2}$
4	3	2	1	$\bar{4}$	$\bar{3}$	$\bar{2}$	$\bar{1}$
$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	1	2	4	3
$\bar{2}$	$\bar{1}$	$\bar{4}$	$\bar{3}$	2	1	3	4
$\bar{3}$	$\bar{4}$	$\bar{1}$	$\bar{2}$	4	3	1	2
$\bar{4}$	$\bar{3}$	$\bar{2}$	$\bar{1}$	3	4	2	1

1	2	3	4	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$
2	1	4	3	$\bar{2}$	$\bar{1}$	$\bar{4}$	$\bar{3}$
3	4	1	2	$\bar{3}$	$\bar{4}$	$\bar{1}$	$\bar{2}$
4	3	2	1	$\bar{4}$	$\bar{3}$	$\bar{2}$	$\bar{1}$
$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	2	1	3	4
$\bar{2}$	$\bar{1}$	$\bar{4}$	$\bar{3}$	1	2	4	3
$\bar{3}$	$\bar{4}$	$\bar{1}$	$\bar{2}$	3	4	2	1
$\bar{4}$	$\bar{3}$	$\bar{2}$	$\bar{1}$	4	3	1	2

Construction of the Smallest Commutative A-loops

Proposition (P.J.,M.K.,P.V.)

Let $(G,+)$ be an abelian group, f an automorphism of G and t a fixed point of f. We define an operation $*$ on $Q = G \cup \bar{G}$ as follows:

$$x * y = x + y, \qquad \bar{x} * y = \overline{x + y},$$
$$x * \bar{y} = \overline{x + y}, \qquad \bar{x} * \bar{y} = f(x + y) + t.$$

Then Q is a loop and

- *Q is associative if and only if f is trivial;*
- *if f is not trivial then $Z(Q) = \{x \in G;\ f(x) = x\}$;*
- *Q is an A-loop if and only if $f(2x) = 2x$, for all $x \in G$.*

Construction of the Smallest Commutative A-loops

Proposition (P.J.,M.K.,P.V.)

Let $(G, +)$ be an abelian group, f an automorphism of G and t a fixed point of f. We define an operation $*$ on $Q = G \cup \bar{G}$ as follows:

$$x * y = x + y, \qquad \bar{x} * y = \overline{x + y},$$
$$x * \bar{y} = \overline{x + y}, \qquad \bar{x} * \bar{y} = f(x + y) + t.$$

Then Q is a loop and

- *Q is associative if and only if f is trivial;*
- *if f is not trivial then $Z(Q) = \{x \in G;\ f(x) = x\}$;*
- *Q is an A-loop if and only if $f(2x) = 2x$, for all $x \in G$.*

Construction of the Smallest Commutative A-loops

Proposition (P.J.,M.K.,P.V.)

Let $(G,+)$ be an abelian group, f an automorphism of G and t a fixed point of f. We define an operation $*$ on $Q = G \cup \bar{G}$ as follows:

$$x * y = x + y, \qquad \bar{x} * y = \overline{x + y},$$
$$x * \bar{y} = \overline{x + y}, \qquad \bar{x} * \bar{y} = f(x + y) + t.$$

Then Q is a loop and

- *Q is associative if and only if f is trivial;*
- *if f is not trivial then $Z(Q) = \{x \in G;\ f(x) = x\}$;*
- *Q is an A-loop if and only if $f(2x) = 2x$, for all $x \in G$.*

Construction of the Smallest Commutative A-loops

Proposition (P.J.,M.K.,P.V.)

Let $(G, +)$ *be an abelian group,* f *an automorphism of* G *and* t *a fixed point of* f*. We define an operation* $*$ *on* $Q = G \cup \bar{G}$ *as follows:*

$$x * y = x + y, \qquad \bar{x} * y = \overline{x + y},$$
$$x * \bar{y} = \overline{x + y}, \qquad \bar{x} * \bar{y} = f(x + y) + t.$$

Then Q *is a loop and*

- Q *is associative if and only if* f *is trivial;*
- *if* f *is not trivial then* $Z(Q) = \{x \in G;\ f(x) = x\}$*;*
- Q *is an A-loop if and only if* $f(2x) = 2x$*, for all* $x \in G$*.*

Commutative A-loops of Order 8 — Constructions

Examples

The commutative A-loops of order 8 are

1. $G = \mathbb{Z}_4$, $f : x \mapsto 3x$ and $t = 0$ or 2;
2. $G = \mathbb{Z}_2^2$, f of order 2 and t neutral;
3. $G = \mathbb{Z}_2^2$, f of order 2 and t not neutral.
4. $G = \mathbb{Z}_2^2$, f of order 3 and t neutral;

Corollary

There exist commutative A-loops with trivial center for any size 2^k with $k > 2$.

Commutative A-loops of Order 8 — Constructions

Examples

The commutative A-loops of order 8 are

1. $G = \mathbb{Z}_4$, $f : x \mapsto 3x$ and $t = 0$ or 2;
2. $G = \mathbb{Z}_2^2$, f of order 2 and t neutral;
3. $G = \mathbb{Z}_2^2$, f of order 2 and t not neutral.
4. $G = \mathbb{Z}_2^2$, f of order 3 and t neutral;

Corollary

There exist commutative A-loops with trivial center for any size 2^k with $k > 2$.

Commutative A-loops of Order 8 — Constructions

Examples

The commutative A-loops of order 8 are

1. $G = \mathbb{Z}_4$, $f : x \mapsto 3x$ and $t = 0$ or 2;
2. $G = \mathbb{Z}_2^2$, f of order 2 and t neutral;
3. $G = \mathbb{Z}_2^2$, f of order 2 and t not neutral.
4. $G = \mathbb{Z}_2^2$, f of order 3 and t neutral;

Corollary

There exist commutative A-loops with trivial center for any size 2^k with $k > 2$.

Commutative A-loops of Order 8 — Constructions

Examples

The commutative A-loops of order 8 are

1. $G = \mathbb{Z}_4$, $f : x \mapsto 3x$ and $t = 0$ or 2;
2. $G = \mathbb{Z}_2^2$, f of order 2 and t neutral;
3. $G = \mathbb{Z}_2^2$, f of order 2 and t not neutral.
4. $G = \mathbb{Z}_2^2$, f of order 3 and t neutral;

Corollary

There exist commutative A-loops with trivial center for any size 2^k with $k > 2$.

Commutative A-loops of Order 8 — Constructions

Examples

The commutative A-loops of order 8 are

1. $G = \mathbb{Z}_4$, $f : x \mapsto 3x$ and $t = 0$ or 2;
2. $G = \mathbb{Z}_2^2$, f of order 2 and t neutral;
3. $G = \mathbb{Z}_2^2$, f of order 2 and t not neutral.
4. $G = \mathbb{Z}_2^2$, f of order 3 and t neutral;

Corollary

There exist commutative A-loops with trivial center for any size 2^k with $k > 2$.

Commutative A-loops of Order 8 — Constructions

Examples

The commutative A-loops of order 8 are

1. $G = \mathbb{Z}_4$, $f : x \mapsto 3x$ and $t = 0$ or 2;
2. $G = \mathbb{Z}_2^2$, f of order 2 and t neutral;
3. $G = \mathbb{Z}_2^2$, f of order 2 and t not neutral.
4. $G = \mathbb{Z}_2^2$, f of order 3 and t neutral;

Corollary

There exist commutative A-loops with trivial center for any size 2^k with $k > 2$.

Cocycles in Groups

Definition

Let G be a group and V an abelian group. A mapping $\theta : G^2 \to V$ is called a *group cocycle* if, for all g, h, k in G,

$$\theta(g,1) = \theta(1,g) = 0,$$

$$\theta(g,hk) + \theta(h,k) = \theta(g,h)^k + \theta(gh,k).$$

Theorem

Ler G be a group and V an abelian group. The set $G \times V$ with the operation

$$(g,u) \cdot (h,v) = (gh, \theta(g,h) + u^h + v)$$

is a group denoted by $E(\theta, G, V)$.
On the other hand, every group E, with a normal abelian subgroup V is isomorphic to $E(\theta, E/V, V)$, for some cocycle θ.

Cocycles in Groups

Definition

Let G be a group and V an abelian group. A mapping $\theta : G^2 \to V$ is called a *group cocycle* if, for all g, h, k in G,

$$\theta(g,1) = \theta(1,g) = 0,$$
$$\theta(g,hk) + \theta(h,k) = \theta(g,h)^k + \theta(gh,k).$$

Theorem

Ler G be a group and V an abelian group. The set $G \times V$ with the operation

$$(g,u)\cdot(h,v) = (gh, \theta(g,h) + u^h + v)$$

is a group denoted by $E(\theta, G, V)$.
On the other hand, every group E, with a normal abelian subgroup V is isomorphic to $E(\theta, E/V, V)$, for some cocycle θ.

Normal Subloops

Definition

A subloop S of a loop L is called *normal* if it is preserved by any inner mapping of L.

Proposition

The center of a loop is a normal subgroup.

Proof.

Any inner mapping fixes the center pointwise. □

Normal Subloops

Definition

A subloop S of a loop L is called *normal* if it is preserved by any inner mapping of L.

Proposition

The center of a loop is a normal subgroup.

Proof.

Any inner mapping fixes the center pointwise. □

Normal Subloops

Definition

A subloop S of a loop L is called *normal* if it is preserved by any inner mapping of L.

Proposition

The center of a loop is a normal subgroup.

Proof.

Any inner mapping fixes the center pointwise. □

Cocycle Extensions of A-loops

Theorem (R. H. Bruck & L. J. Paige, special version)

Let Z be an elementary abelian 2-group and K a commutative A-loop of exponent 2. Let $\theta : K \times K \to Z$ be a loop cocycle satisfying $\theta(x,y) = \theta(y,x)$, for every $x,y \in K$, $\theta(x,x) = 1$, for every $x \in K$, and

$$\theta(x,y)\theta(x',y)\theta(xx',y)\theta(x,x')\theta(xy,z)\theta(x'y,z)\theta(y,z)\theta((xx')y,z) = \theta(R(y,z)x,yz)\theta(R(y,z)x',yz)\theta(R(y,z)(xx'),yz)\theta(R(y,z)x,R(y,z)x,R(y,z)x')$$

for every $x,y,z,x' \in K$, where $R(y,z) = R_yR_zR_{yz}^{-1}$. Then $K \ltimes_\theta R$ is a commutative A-loop of exponent 2.
Conversely, every commutative A-loop of exponent two that is a central extension of Z by K can be represented in this manner.

Enumeration

Numbers of non-associative commutative A-loops which are 2-loops. The columns show, for each order, the number of all commutative A-loops, the number of commutative A-loops of exponent 2 with nontrivial center and the number of commutative A-loops with trivial center.

loop order	all loops	e.2 + n.c.	trivial center
8	4	1	1
16	46	10	2
32	?	355	6 + ?

Questions

Question

Does there exist any finite simple non-associative commutative A-loop? If does, it has to be a p-loop of order p^k, for some prime $p \neq 3$.

Question

Construct a finite commutative A-loop with trivial center and exponent different from 2.

Question

Prove that there is no non-associative commutative A-loop of order p^2, for any prime p.

Questions

Question

Does there exist any finite simple non-associative commutative A-loop? If does, it has to be a p-loop of order p^k, for some prime $p \neq 3$.

Question

Construct a finite commutative A-loop with trivial center and exponent different from 2.

Question

Prove that there is no non-associative commutative A-loop of order p^2, for any prime p.

Questions

Question

Does there exist any finite simple non-associative commutative A-loop? If does, it has to be a p-loop of order p^k, for some prime $p \neq 3$.

Question

Construct a finite commutative A-loop with trivial center and exponent different from 2.

Question

Prove that there is no non-associative commutative A-loop of order p^2, for any prime p.

Bibliography

R. H. Bruck, J. L. Paige:
Loops whose inner mappings are automorphisms
The Annals of Math., 2nd Series, **63**, no. 2, (1956), 308–323

P. Jedlička, M. Kinyon, P. Vojtěchovský:
Commutative A-loops: Construction and enumeration
preprint

P. Jedlička, M. Kinyon, P. Vojtěchovský:
The structure of commutative A-loops (preprint)

M. Kinyon, K. Kunen, J. D. Phillips:
Every diassociative A-loop is Moufang
Proc. Amer. Math. Soc. **130** (2002), 619–624

M. Kinyon, K. Kunen, J. D. Phillips:
Some notes on the structure of A-loops (preprint)