Distributivity and complementarity of quasiorder lattices of monounary algebras

Danica Jakubíková-Studenovská

P.J.Šafárik University, Košice, Slovakia

Třešť, Czech republic
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the lattice $\operatorname{Quord}\mathcal{A}$ **of all quasiorders of an algebra** $\mathcal{A}$

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  - how endomorphisms of quasiorders behave
  - when $\mathrm{End}\ q \subseteq \mathrm{End}\ q'$ for quasiorders q, q' on a set A ($\mathrm{End}\ q$ is the set of all mappings preserving q)
  - description of the quasiorder lattice of the algebra $(A, \mathrm{End}\ q)$

AIM

Find necessary and sufficient conditions for a monounary algebra (A, f), under which the lattice $\mathrm{Quord}\,(A, f)$ is

- **modular** or
- **distributive** or
- **complementary**, respectively.

Distributivity - connected monounary algebra

Theorem

(J. Berman, 1972) If $n \in N$, then θ is a congruence relation of an n-element cycle (C, f) if and only if there is $d \in N$ such that d divides n and for each $x \in C$,

$$[x]_\theta = \left\{x, f^d(x), \dots, f^{(\frac{n}{d}-1)\,d}(x)\right\}.$$

- denoted θ_d
- θ_d is the smallest congruence containing the pair $(x, f^d(x))$

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Lemma

Let (A, f) be an n-element cycle, $n \in N$. Then $\mathrm{Quord}\,(A, f) = \mathrm{Con}\,(A, f) = \{\theta_d : d/n\}$.

Corollary

Let (A, f) *be an* n*-element cycle,* $n \in N$*. Then the lattice* $\mathrm{Quord}\,(A, f)$ *is distributive.*

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Corollary

Let (A, f) be an n-element cycle, $n \in N$. Then the lattice $\mathrm{Quord}\,(A, f)$ is distributive.

- Let (A, f) be a monounary algebra. Then $\mathrm{Quord}\,(A, f) = \mathrm{Con}\,(A, f)$ if and only if (A, f) is a cycle.

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- Let (A, f) be a monounary algebra. Then $\mathrm{Quord}\,(A, f) = \mathrm{Con}\,(A, f)$ if and only if (A, f) is a cycle.

Lemma

Let (A, f) *be a connected monounary algebra containing an* n*-element cycle* $C, |A| = n + 1 \geq 3$*. Then* $\mathrm{Quord}\,(A, f) \cong \mathrm{Quord}(C, f) \times M_2 = \mathrm{Con}\,(C, f) \times M_2$ *and* $\mathrm{Quord}\,(A, f)$ *is distributive.*

Distributivity - connected monounary algebra

- Assume that (A, f) is a connected monounary algebra, $|A| \geq 2$ and that none of the following conditions is valid:
  - (A, f) a cycle
  - (A, f) contains an n-element cycle $|A| = n + 1$

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Then $\mathrm{Quord}\,(A, f)$ contains a pentagon, thus it fails to be modular.

Distributivity - general case

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Theorem

Let (A, f) be a monounary algebra. The following conditions are equivalent:

(i) *The lattice* $\mathrm{Quord}\,(A, f)$ *is modular.*

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Let (A, f) be a monounary algebra. The following conditions are equivalent:

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(ii) *The lattice* $\mathrm{Quord}\,(A, f)$ *is distributive.*

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Theorem

Let (A, f) be a monounary algebra. The following conditions are equivalent:

(i) *The lattice* $\mathrm{Quord}\,(A, f)$ *is modular.*

(ii) *The lattice* $\mathrm{Quord}\,(A, f)$ *is distributive.*

(iii) *Either* $|A| \leq 2$ *or* (A, f) *is connected and there exists a cycle* C *of* (A, f) *such that* $|A| \leq |C| + 1$*.*

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- n is square-free,
- for each $a \in A$, the element $f(a)$ is cyclic.

Complementarity - equivalence relations

- For $\alpha \in \mathrm{Quord}\,(A, f)$, define $\bar{\alpha}$:

$$(b, a) \in \bar{\alpha} \Longleftrightarrow (a, b) \in \alpha.$$

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- For $a \in A$ denote by $C(a)$ the cycle, containing $f(a)$.
- Relation R: If B, D are cycles of (A, f), then $B\ R\ D$, if there are $k \in N$, cycles $B = C_0, C_1, \ldots, C_k = D$, elements $c_0 \in C_0, c_1 \in C_1, \ldots, c_k \in C_k$ such that for each $i \in \{0, 1, \ldots, k-1\}$, $(c_i, c_{i+1}) \in \alpha \cup \bar{\alpha}$.

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- For $a, b \in A$, set

$$a\ r\ b \Longleftrightarrow C(a)\ R\ C(b).$$

The relation r is an equivalence on A.

Complementarity - auxiliary results

Assumption. The equivalence r on A has **only one** class. Let $\alpha \in \mathrm{Quord}\,(A, f)$.

- A': all noncyclic elements x of A such that $(x, f^n(x)) \notin \alpha$ and $(f^n(x), x) \notin \alpha$.

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- ρ on A': $(a, b) \in \rho$ if $a, b \in A'$, $f(a) = f(b)$ and there are $k \in N$ and $a = u_0, u_1, \ldots, u_k = b$ elements of A' such that $(\forall i \in \{0, \ldots, k-1\})(f(a) = f(u_i), (u_i, u_{i+1}) \in \alpha \cup \bar{\alpha})$.

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    1. $(\forall x \in D \setminus P(D))(\exists y \in P(D))((x, y) \in \alpha, (y, x) \in \alpha)$;

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    1. $(\forall x \in D \setminus P(D))(\exists y \in P(D))((x, y) \in \alpha, (y, x) \in \alpha)$;
    2. $(\forall x, y \in P(D))((x, y) \in \alpha \Rightarrow (y, x) \notin \alpha)$.

Complementarity - construction

- *Step (a).* Let x, y belong to the same cycle C, $y = f^k(x)$, $\alpha \restriction C = \theta_d$, d/n and let $e = \frac{n}{d}$. We set $(x, y) \in \beta$ if and only if e/k.

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- *Step (b).* Let $x \in C_1$, $y \in C_2$, where C_1 and C_2 are distinct cycles. We put $(x, y) \in \beta$ if and only if there are $a \in C_1$ and $b \in C_2$ with $(b, a) \in \alpha, (a, b) \notin \alpha$.

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- *Step (c).* Suppose that $x, y \in P(D)$ for some $D \in A'/\rho$. Then $(x, y) \in \beta$ if and only if and $(y, x) \in \alpha$.

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- *Step (c).* Suppose that $x, y \in P(D)$ for some $D \in A'/\rho$. Then $(x, y) \in \beta$ if and only if and $(y, x) \in \alpha$.
- *Step (d1).* Suppose that x belongs to a cycle C, y is noncyclic, $C(y) = C$. Further let $\alpha \restriction C = \theta_d$, d/n, $e = \frac{n}{d}$. If $y \notin A'$, then $(x, y) \in \beta$ if and only if $(f^n(y), y) \notin \alpha, (y, f^n(y)) \in \alpha, x = f^k(y), e/k$.

Complementarity - construction

- *Step (d'1).* Suppose that y belongs to a cycle C, x is noncyclic, $C(x) = C$. Further let $\alpha \restriction C = \theta_d$, d/n, $e = \frac{n}{d}$. If $x \notin A'$, then $(x, y) \in \beta$ if and only if $(f^n(x), x) \in \alpha, (x, f^n(x)) \notin \alpha, y = f^k(x), e/k.$
- *Step (d2).* Suppose that x belongs to a cycle C, y is noncyclic, $C(y) = C$. Further let $\alpha \restriction C = \theta_d$, d/n, $e = \frac{n}{d}$. If $y \in A'$, then $(x, y) \in \beta$ if and only if there is $D \in A'/\rho$ such that $y \in P(D), x = f^k(y), e/k$ and $(y, p(D)) \in \alpha$.
- *Step (d'2).* Suppose that y belongs to a cycle C, x is noncyclic, $C(x) = C$. Further let $\alpha \restriction C = \theta_d$, d/n, $e = \frac{n}{d}$. If $x \in A'$, then $(x, y) \in \beta$ if and only if there is $D \in A'/\rho$ such that $x \in P(D), y = f^k(x), e/k$ and $(x, p(D)) \in \alpha$.
- *Step (e).* Suppose that x, y satisfy none of the assumptions of the previous steps. Then $(x, y) \in \beta$ if and only if $(x, f^n(x)) \in \beta$, $(f^n(x), f^n(y)) \in \beta$, $(f^n(y), y) \in \beta$.

Complementarity - main result

$A/r = \{A_j : j \in J\}$

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Theorem

Let $\alpha \in \mathrm{Quord}\,(A, f)$, $j \in J$. *Then there exists a complement* β_j *of* $\alpha_j = \alpha \restriction A_j$ *in the lattice* $\mathrm{Quord}\,(A_j, f)$.

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Theorem

If $\alpha \in \mathrm{Quord}\,(A, f)$ *and* $|A/r| = 1$, *then the conditions*

- *each connected component of* (A, f) *contains a cycle,*
- *there is* $n \in N$ *such that each cycle of* (A, f) *has* n *elements,*
- n *is square-free,*
- *for each* $a \in A$, *the element* $f(a)$ *is cyclic*

are necessary and sufficient for the existence of a complement of α *in the lattice* $\mathrm{Quord}\,(A, f)$.

Complementarity - general case?

HYPOTHESIS

Theorem

Let (A, f) be a monounary algebra. The lattice $\mathrm{Quord}\,(A, f)$ is complementary if and only if

- *each connected component of (A, f) contains a cycle,*
- *there is $n \in N$ such that each cycle of (A, f) has n elements,*
- *n is square-free,*
- *for each $a \in A$, the element $f(a)$ is cyclic.*