Rank of divisors on tropical curves

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Overview of the talk

Divisors on Riemann surfaces

Divisors on tropical curves

Our results

Divisors on Riemann surfaces

Riemann surfaces

- A Riemann surface is a second countable connected Hausdorff space with a complex structure.
- genus g=0, Riemann sphere
- compact Riemann surfaces
  every compact Riemann surface is diffeomorphic to the g-holed torus
- a meromorphic function is a function which is analytic except for finetely many poles

Divisors on Riemann surfaces

- a divisor is a formal sum of finitely many points of the surface
- a divisor is effective if all its coefficients are non-negative
- a principal divisor is a divisor $D(f)$ associated with a meromorphic function f:

$$D(f) = \sum_{\text{zeroes } w} w - \sum_{\text{poles } w} w$$

  multiplicities are counted in the sum
- $L(D_0)$ is the class of meromorphic functions f such that $D(f) + D_0$ is effective
- $L(D_0)$ is a finite dimensional $\mathbb{C}$-vector space

Riemann-Roch theorem

$$\dim(L(D)) - \dim(L(K-D)) = \deg(D) + 1 - g$$

- g is the genus of the Riemann surface X
- K is the canonical divisor of X
- the degree of D, denoted by $\deg(D)$, is $\sum_{v \in \mathrm{supp}(D)} D(v)$

Reformulation of Riemann-Roch theorem

- $L(D_0)$ is the class of meromorphic functions f such that $D(f) + D_0$ is effective
- two divisors are equivalent if their difference is a principal divisor
- $|D|$ is the class of all effective divisors equivalent to D
- the rank of D is $r(D) = \dim(|D|) = \dim(L(D)) - 1$

$$r(D) - r(K - D) = \deg(D) + 1 - g$$

Divisors on tropical curves

Tropical curves and graphs

- a graph is a set of vertices and edges joining pairs of vertices

- a tropical curve can be viewed as a metric graph
  each edge is associated with a segment of a certain length
    which defines a topology on the curve/graph
  edges incident with degree-one vertices can have infinite lengths and
    their end-points are called unbounded ends

- a tropical curve with all segments of the same length and no infinite segments is understood as a graph

Riemann surfaces vs. tropical curves

Riemann surfaces	Tropical curves
Meromorphic functions	Rational functions
	(piecewise linear functions)
Divisors	Divisors
$D(f) = \sum \text{zeroes} - \sum \text{poles}$	$D(f)(v) = \sum \text{slopes at } v$
Genus	Cyclomatic number
Divisor equivalence	$D - D'$ is principal
Canonical divisor	$K(v) = \deg(v) - 2$

Riemann surfaces vs. graphs

Riemann surfaces	Graphs
Meromorphic functions	Integer potential functions
	In-/Out-going current for voltages
Divisors	Divisors
$D(f) = \sum \text{zeroes} - \sum \text{poles}$	$D : V(G) \to \mathbb{Z}$
Genus	Cyclomatic number
Divisor equivalence	$D - D'$ is principal
Canonical divisor	$K(v) = \deg(v) - 2$

Chip-firing game

- an equivalance of divisors on graphs definied in a different way

- vertices are assigned chips (negative numbers allowed)
- at each move, a selected vertex sends one chip to each neighbors
- the goal is to have only non-negative numbers of chips

- divisors describe the game configurations
- two divisors are equivalent if they can be reached

Analogue of Riemann-Roch theorem

Riemann-Roch theorem [Baker and Norine 2006]

$$r(D) - r(K - D) = \deg(D) - g + 1$$

- the rank $r(D)$ is the maximal non-negative integer K such that $D - E$ is equivalent to an effective divisor for every effective divisor E of degree K
  if $|D| = \emptyset$, then $r(D) = -1$
- Corollary: if there are at least g chips, the game can be won

Our results

Divisors on graphs and tropical curves

- a natural correspondance between graphs and tropical curves
- let G be a graph and Γ the corresponding tropical curve
- let D be a divisor on G and thus on Γ
- What is the relation between $r_G(D)$ and $r_\Gamma(D)$?
  no inequality apriori clear
  We prove that the ranks are the same.
- Corollary:
  If G' is a uniform subdivision of G, then $r_G(D) = r_{G'}(D)$.

Rank of divisors of tropical curves

- algorithmic corollary of our structural results
- the rank of a divisor on a graph can be computed
  What about divisors on tropical curves?
  The definition involves infinite objects.
- the rank of a divisor on a tropical curve can be computed

Thank you for your attention!