On the axiomatic system of $SBL_{\neg}$-algebras

Filip Švrček

Department of Algebra and Geometry
Faculty of Sciences
Palacký University of Olomouc
Czech Republic
svrcekf@inf.upol.cz

SSAOS 2008, Třešť

Outline

Definition ($SBL_\neg$-algebra, 1st part)

By an $SBL_\neg$-algebra we mean an algebra $A = (A; \vee, \wedge, \odot, \to, \neg, 0, 1)$ of the type $\langle 2, 2, 2, 2, 1, 0, 0 \rangle$, where

- $(A; \vee, \wedge, \odot, \to, 0, 1)$ is an SBL-algebra:

(BL1) $x \odot y = y \odot x$,
(BL2) $1 \odot x = x$,
(BL3) $x \odot (x \to y) = y \odot (y \to x)$,
(BL4) $(x \odot y) \to z = x \to (y \to z)$,
(BL5) $0 \to x = 1$,
(BL6) $((x \to y) \to z) \to (((y \to x) \to z) \to z) = 1$,
(SBL) $(x \odot y) \to 0 = (x \to 0) \vee (y \to 0)$,

$$x \vee y \overset{def}{=} ((x \to y) \to y) \wedge ((y \to x) \to x), \quad \text{(J)}$$

$$x \wedge y \overset{def}{=} x \odot (x \to y). \quad \text{(M)}$$

Definition ($SBL_\neg$-algebra, 1st part)

By an $SBL_\neg$-algebra we mean an algebra $A = (A; \vee, \wedge, \odot, \to, \neg, 0, 1)$ of the type $\langle 2, 2, 2, 2, 1, 0, 0 \rangle$, where

- $(A; \vee, \wedge, \odot, \to, 0, 1)$ is an *SBL*-algebra:

(BL1) $x \odot y = y \odot x$,
(BL2) $1 \odot x = x$,
(BL3) $x \odot (x \to y) = y \odot (y \to x)$,
(BL4) $(x \odot y) \to z = x \to (y \to z)$,
(BL5) $0 \to x = 1$,
(BL6) $((x \to y) \to z) \to (((y \to x) \to z) \to z) = 1$,
(SBL) $(x \odot y) \to 0 = (x \to 0) \vee (y \to 0)$,

$$x \vee y \overset{def}{=} ((x \to y) \to y) \wedge ((y \to x) \to x), \tag{J}$$

$$x \wedge y \overset{def}{=} x \odot (x \to y). \tag{M}$$

Definition ($SBL_\neg$-algebra, 1st part)

By an $SBL_\neg$-algebra we mean an algebra $A = (A; \vee, \wedge, \odot, \to, \neg, 0, 1)$ of the type $\langle 2, 2, 2, 2, 1, 0, 0\rangle$, where

- $(A; \vee, \wedge, \odot, \to, 0, 1)$ is an SBL-algebra:

(BL1) $x \odot y = y \odot x$,
(BL2) $1 \odot x = x$,
(BL3) $x \odot (x \to y) = y \odot (y \to x)$,
(BL4) $(x \odot y) \to z = x \to (y \to z)$,
(BL5) $0 \to x = 1$,
(BL6) $((x \to y) \to z) \to (((y \to x) \to z) \to z) = 1$,
(SBL) $(x \odot y) \to 0 = (x \to 0) \vee (y \to 0)$,

$$x \vee y \overset{def}{=} ((x \to y) \to y) \wedge ((y \to x) \to x), \qquad \text{(J)}$$

$$x \wedge y \overset{def}{=} x \odot (x \to y). \qquad \text{(M)}$$

Definition ($SBL_\neg$-algebra, 1st part)

By an $SBL_\neg$-algebra we mean an algebra $A = (A; \vee, \wedge, \odot, \rightarrow, \neg, 0, 1)$ of the type $\langle 2, 2, 2, 2, 1, 0, 0\rangle$, where

- $(A; \vee, \wedge, \odot, \rightarrow, 0, 1)$ is an SBL-algebra:

(BL1) $x \odot y = y \odot x$,
(BL2) $1 \odot x = x$,
(BL3) $x \odot (x \rightarrow y) = y \odot (y \rightarrow x)$,
(BL4) $(x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z)$,
(BL5) $0 \rightarrow x = 1$,
(BL6) $((x \rightarrow y) \rightarrow z) \rightarrow (((y \rightarrow x) \rightarrow z) \rightarrow z) = 1$,
(SBL) $(x \odot y) \rightarrow 0 = (x \rightarrow 0) \vee (y \rightarrow 0)$,

$$x \vee y \overset{def}{=} ((x \rightarrow y) \rightarrow y) \wedge ((y \rightarrow x) \rightarrow x), \qquad \text{(J)}$$

$$x \wedge y \overset{def}{=} x \odot (x \rightarrow y). \qquad \text{(M)}$$

Definition ($SBL_\neg$-algebra, 1st part)

By an $SBL_\neg$-algebra we mean an algebra $A = (A; \vee, \wedge, \odot, \rightarrow, \neg, 0, 1)$ of the type $\langle 2, 2, 2, 2, 1, 0, 0 \rangle$, where

- $(A; \vee, \wedge, \odot, \rightarrow, 0, 1)$ is an SBL-algebra:

(BL1) $x \odot y = y \odot x$,
(BL2) $1 \odot x = x$,
(BL3) $x \odot (x \rightarrow y) = y \odot (y \rightarrow x)$,
(BL4) $(x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z)$,
(BL5) $0 \rightarrow x = 1$,
(BL6) $((x \rightarrow y) \rightarrow z) \rightarrow (((y \rightarrow x) \rightarrow z) \rightarrow z) = 1$,
(SBL) $(x \odot y) \rightarrow 0 = (x \rightarrow 0) \vee (y \rightarrow 0)$,

$$x \vee y \overset{def}{=} ((x \rightarrow y) \rightarrow y) \wedge ((y \rightarrow x) \rightarrow x), \tag{J}$$

$$x \wedge y \overset{def}{=} x \odot (x \rightarrow y). \tag{M}$$

Definition ($SBL_\neg$-algebra, 2nd part)

- $\neg$ is a unary operation on A satisfying:

$$
\begin{array}{ll}
(\mathrm{SBL}_\neg 1) & \neg\neg x = x \,, \\
(\mathrm{SBL}_\neg 2) & {\sim} x \leq \neg x \,, \\
(\mathrm{SBL}_\neg 3) & \nu(x \to y) = \nu(\neg y \to \neg x) \,, \\
(\mathrm{SBL}_\neg 4) & \nu(x) \vee {\sim}\nu(x) = 1 \,, \\
(\mathrm{SBL}_\neg 5) & \nu(x \vee y) \leq \nu(x) \vee \nu(y) \,, \\
(\mathrm{SBL}_\neg 6) & \nu(x) \odot \nu(x \to y) \leq \nu(y) \,,
\end{array}
$$

$$
{\sim} x \stackrel{def}{=} x \to 0 \,,
$$

$$
\nu(x) \stackrel{def}{=} {\sim}\neg x \,.
$$

Definition ($SBL_\neg$-algebra, 2nd part)

- $\neg$ is a unary operation on A satisfying:

$$
\begin{array}{ll}
(\mathrm{SBL}_\neg 1) & \neg\neg x = x \,, \\
(\mathrm{SBL}_\neg 2) & \sim x \leq \neg x \,, \\
(\mathrm{SBL}_\neg 3) & \nu(x \to y) = \nu(\neg y \to \neg x) \,, \\
(\mathrm{SBL}_\neg 4) & \nu(x) \vee \sim\nu(x) = 1 \,, \\
(\mathrm{SBL}_\neg 5) & \nu(x \vee y) \leq \nu(x) \vee \nu(y) \,, \\
(\mathrm{SBL}_\neg 6) & \nu(x) \odot \nu(x \to y) \leq \nu(y) \,,
\end{array}
$$

$$\sim x \stackrel{def}{=} x \to 0 \,,$$

$$\nu(x) \stackrel{def}{=} \sim\neg x \,.$$

Definition ($SBL_\neg$-algebra, 2nd part)

- $\neg$ is a unary operation on A satisfying:

$$
\begin{array}{ll}
(\mathrm{SBL}_\neg 1) & \neg\neg x = x\ , \\
(\mathrm{SBL}_\neg 2) & \sim x \leq \neg x\ , \\
(\mathrm{SBL}_\neg 3) & \nu(x \to y) = \nu(\neg y \to \neg x)\ , \\
(\mathrm{SBL}_\neg 4) & \nu(x) \vee \sim\nu(x) = 1\ , \\
(\mathrm{SBL}_\neg 5) & \nu(x \vee y) \leq \nu(x) \vee \nu(y)\ , \\
(\mathrm{SBL}_\neg 6) & \nu(x) \odot \nu(x \to y) \leq \nu(y)\ ,
\end{array}
$$

$$\sim x \stackrel{def}{=} x \to 0\ ,$$

$$\nu(x) \stackrel{def}{=} \sim\neg x\ .$$

$SBL_\neg$-algebras = semantics for the $SBL_\neg$ logic:

Connectives: $\&$ (conj.), $\to$ (impl.), $\neg$ (strong neg.)
Truth constants: $\overline{0}$
Further connectives:

- $\varphi \wedge \psi$ is $\varphi \,\&(\varphi \to \psi)$;
- $\varphi \vee \psi$ is $((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi)$;
- $\varphi \equiv \psi$ is $(\varphi \to \psi) \,\&(\psi \to \varphi)$;
- $\sim\varphi$ is $\varphi \to \overline{0}$;
- $\Delta\varphi$ is $\sim\neg\varphi$.

Deductive rules: modus ponens, necessitation for Δ

$SBL_\neg$-algebras = semantics for the $SBL_\neg$ logic:

Connectives: $\&$ (conj.), $\rightarrow$ (impl.), $\neg$ (strong neg.)
Truth constants: $\overline{0}$
Further connectives:

- $\varphi \wedge \psi$ is $\varphi \,\&(\varphi \rightarrow \psi)$;
- $\varphi \vee \psi$ is $((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi)$;
- $\varphi \equiv \psi$ is $(\varphi \rightarrow \psi) \,\&(\psi \rightarrow \varphi)$;
- $\sim\varphi$ is $\varphi \rightarrow \overline{0}$;
- $\Delta\varphi$ is $\sim\neg\varphi$.

Deductive rules: modus ponens, necessitation for Δ

$SBL_\neg$-algebras = semantics for the $SBL_\neg$ logic:

Connectives: $\&$ (conj.), $\rightarrow$ (impl.), $\neg$ (strong neg.)
Truth constants: $\overline{0}$
Further connectives:

- $\varphi \wedge \psi$ is $\varphi \,\&(\varphi \rightarrow \psi)$;
- $\varphi \vee \psi$ is $((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi)$;
- $\varphi \equiv \psi$ is $(\varphi \rightarrow \psi)\,\&(\psi \rightarrow \varphi)$;
- $\sim\varphi$ is $\varphi \rightarrow \overline{0}$;
- $\Delta\varphi$ is $\sim\neg\varphi$.

Deductive rules: modus ponens, necessitation for Δ

SBL$_\neg$-algebras = semantics for the *SBL*$_\neg$ logic:

Connectives: $\&$ (conj.), $\rightarrow$ (impl.), $\neg$ (strong neg.)
Truth constants: $\overline{0}$
Further connectives:

- $\varphi \wedge \psi$ is $\varphi \,\&(\varphi \rightarrow \psi)$;
- $\varphi \vee \psi$ is $((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi)$;
- $\varphi \equiv \psi$ is $(\varphi \rightarrow \psi) \,\&(\psi \rightarrow \varphi)$;
- $\sim\varphi$ is $\varphi \rightarrow \overline{0}$;
- $\Delta\varphi$ is $\sim\neg\varphi$.

Deductive rules: modus ponens, necessitation for Δ

SBL$_\neg$-algebras = semantics for the *SBL*$_\neg$ logic:

Connectives: $\&$ (conj.), $\rightarrow$ (impl.), $\neg$ (strong neg.)
Truth constants: $\overline{0}$
Further connectives:

- $\varphi \wedge \psi$ is $\varphi \,\&(\varphi \rightarrow \psi)$;
- $\varphi \vee \psi$ is $((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi)$;
- $\varphi \equiv \psi$ is $(\varphi \rightarrow \psi)\,\&(\psi \rightarrow \varphi)$;
- $\sim\varphi$ is $\varphi \rightarrow \overline{0}$;
- $\Delta\varphi$ is $\sim\neg\varphi$.

Deductive rules: modus ponens, necessitation for Δ

Axioms:

(B1) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$,
(B2) $(\varphi \,\&\psi) \to \varphi$,
(B3) $(\varphi \,\&\psi) \to (\psi \,\&\varphi)$,
(B4) $(\varphi \,\&(\varphi \to \psi)) \to (\psi \,\&(\psi \to \varphi))$,
(B5) $(\varphi \to (\psi \to \chi)) \equiv ((\varphi \,\&\psi) \to \chi)$,
(B6) $((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$,
(B7) $\overline{0} \to \varphi$,
(STR) $(\varphi \,\&\psi \to \overline{0}) \to (\varphi \to \overline{0}) \vee (\psi \to \overline{0})$,
($1_\neg$) $\sim\sim\varphi \equiv \varphi$,
($2_\neg$) $\sim\varphi \to \neg\varphi$,
($3_\neg$) $\Delta(\varphi \to \psi) \to \Delta(\neg\psi \to \neg\varphi)$,
($4_\neg$) $\Delta\varphi \vee \sim\Delta\varphi$,
($5_\neg$) $\Delta(\varphi \vee \psi) \to (\Delta\varphi \vee \Delta\psi)$,
($6_\neg$) $\Delta(\varphi \to \psi) \to (\Delta\varphi \to \Delta\psi)$.

Theorem (Ciucci's result)

Let $A = (A; \vee, \wedge, \odot, \rightarrow, \neg, 0, 1)$ *be a structure such that*

1. $(A; \vee, \wedge, \odot, \rightarrow, 0, 1)$ *is an SBL-algebra,*
2. $\neg$ *is a unary operation on* A *satisfying for each* $x, y \in A$

$$(SBL_{\neg}1) \quad \neg\neg\, x = x \;,$$

$$(SBL_{\neg}3) \quad \nu(x \rightarrow y) = \nu(\neg y \rightarrow \neg x) \;,$$

$$(SBL_{\neg}6) \quad \nu(x) \odot \nu(x \rightarrow y) \leq \nu(y)$$

where

$$\sim x \overset{def}{=} x \rightarrow 0 \;,$$

$$\nu(x) \overset{def}{=} \sim\neg(x) \;.$$

Then, A is an $SBL_{\neg}$*-algebra and vice versa.*

Problem: Are the axioms from the foregoing theorem characterizing $SBL_{\neg}$-algebras mutually independent or not?

Showed: The axioms characterizing SBL-algebras, i.e. identities (BL1) – (BL6) and (SBL), are mutually independent.

Problem: Are the axioms from the foregoing theorem characterizing $SBL_{\neg}$-algebras mutually independent or not?

Showed: The axioms characterizing *SBL*-algebras, i.e. identities (BL1) – (BL6) and (SBL), are mutually independent.

Outline

Theorem

Let $A = (A; \vee, \wedge, \odot, \rightarrow, \neg, 0, 1)$ be an SBL-algebra satisfying moreover identities (SBL$_\neg$1) and (SBL$_\neg$3). Then the following hold in A:

(1) $x \rightarrow \sim (\sim y \rightarrow \neg(x \rightarrow y)) = \sim x$.

Substitution in (1)

If we put $x := \neg y, y := \neg x$ in (1), we get the identity

(2) $\neg y \rightarrow \sim (\nu(x) \rightarrow \neg(\neg y \rightarrow \neg x)) = \nu(y)$,

valid in every *SBL*-algebra satisfying also (SBL$_\neg$1) and (SBL$_\neg$3).

Theorem

Let $A = (A; \vee, \wedge, \odot, \to, \neg, 0, 1)$ *be an SBL-algebra satisfying moreover identities (SBL*$_\neg$*1) and (SBL*$_\neg$*3). Then the following hold in A:*

(1) $x \to \sim (\sim y \to \neg(x \to y)) = \sim x$.

Substitution in (1)

If we put $x := \neg y, y := \neg x$ in (1), we get the identity

(2) $\neg y \to \sim (\nu(x) \to \neg(\neg y \to \neg x)) = \nu(y)$,

valid in every *SBL*-algebra satisfying also (SBL$_\neg$1) and (SBL$_\neg$3).

Theorem

Let $A = (A; \vee, \wedge, \odot, \rightarrow, \neg, 0, 1)$ *be a structure such that*

1. $(A; \vee, \wedge, \odot, \rightarrow, 0, 1)$ *is an SBL-algebra,*
2. $\neg$ *is a unary operation on* A *satisfying for each* $x, y \in A$

$$(SBL_{\neg}1) \quad \neg\neg x = x \ ,$$

$$(SBL_{\neg}3) \quad \nu(x \rightarrow y) = \nu(\neg y \rightarrow \neg x) \ ,$$

where

$$\sim x \stackrel{def}{=} x \rightarrow 0 \ ,$$

$$\nu(x) \stackrel{def}{=} \sim\neg x \ .$$

Then, A *is an* $SBL_{\neg}$*-algebra and vice versa.*

Sketch of proof:

Does $(SBL_{\neg}6)$ follow from 1 and 2?

- $(SBL_{\neg}6) \quad \nu(x) \odot \nu(x \to y) \leq \nu(y)$

is equivalent to

- $(\nu(x) \odot \nu(x \to y)) \vee \nu(y) = \nu(y).$

$$\vdots$$

- $(\neg y \to ((\nu(x) \to (\nu(x \to y) \to 0)) \to (((\nu(y) \to (\nu(x) \odot \nu(x \to y))) \to (\nu(x) \odot \nu(x \to y))) \to 0))) \odot ((\nu(y) \to (\nu(x) \odot \nu(\neg y \to \neg x))) \to (\nu(x) \odot \nu(\neg y \to \neg x))) = \nu(y)$

Sketch of proof:

Does ($SBL_\neg 6$) follow from 1 and 2?

- ($SBL_\neg 6$) $\quad \nu(x) \odot \nu(x \to y) \leq \nu(y)$

is equivalent to

- $(\nu(x) \odot \nu(x \to y)) \vee \nu(y) = \nu(y)$.

$\vdots$

- $(\neg y \to ((\nu(x) \to (\nu(x \to y) \to 0)) \to (((\nu(y) \to (\nu(x) \odot \nu(x \to y))) \to (\nu(x) \odot \nu(x \to y))) \to 0))) \odot ((\nu(y) \to (\nu(x) \odot \nu(\neg y \to \neg x))) \to (\nu(x) \odot \nu(\neg y \to \neg x))) = \nu(y)$

Sketch of proof:

Does $(SBL_{\neg}6)$ follow from 1 and 2?

- $(SBL_{\neg}6)$ $\quad \nu(x) \odot \nu(x \to y) \leq \nu(y)$

is equivalent to

- $(\nu(x) \odot \nu(x \to y)) \vee \nu(y) = \nu(y).$

$$\vdots$$

- $(\neg y \to ((\nu(x) \to (\nu(x \to y) \to 0)) \to (((\nu(y) \to (\nu(x) \odot \nu(x \to y))) \to (\nu(x) \odot \nu(x \to y))) \to 0))) \odot ((\nu(y) \to (\nu(x) \odot \nu(\neg y \to \neg x))) \to (\nu(x) \odot \nu(\neg y \to \neg x))) = \nu(y)$

Sketch of proof:

Does ($SBL_\neg 6$) follow from 1 and 2?

- ($SBL_\neg 6$) $\quad \nu(x) \odot \nu(x \to y) \leq \nu(y)$

is equivalent to

- $(\nu(x) \odot \nu(x \to y)) \vee \nu(y) = \nu(y).$

$$\vdots$$

- $(\neg y \to ((\nu(x) \to (\nu(x \to y) \to 0)) \to (((\nu(y) \to (\nu(x) \odot \nu(x \to y))) \to (\nu(x) \odot \nu(x \to y))) \to 0))) \odot ((\nu(y) \to (\nu(x) \odot \nu(\neg y \to \neg x))) \to (\nu(x) \odot \nu(\neg y \to \neg x))) = \nu(y)$

Sketch of proof:

Does $(SBL_{\neg}6)$ follow from 1 and 2?

- $(SBL_{\neg}6)$ $\nu(x) \odot \nu(x \to y) \leq \nu(y)$

is equivalent to

- $(\nu(x) \odot \nu(x \to y)) \vee \nu(y) = \nu(y)$.

$\vdots$

- $(\neg y \to ((\nu(x) \to (\nu(x \to y) \to 0)) \to (((\nu(y) \to (\nu(x) \odot \nu(x \to y))) \to (\nu(x) \odot \nu(x \to y))) \to 0))) \odot ((\nu(y) \to (\nu(x) \odot \nu(\neg y \to \neg x))) \to (\nu(x) \odot \nu(\neg y \to \neg x))) = \nu(y)$

Sketch of proof:

Does $(SBL_{\neg}6)$ follow from 1 and 2?

- $(SBL_{\neg}6)$ $\quad \nu(x) \odot \nu(x \to y) \leq \nu(y)$

is equivalent to

- $(\nu(x) \odot \nu(x \to y)) \vee \nu(y) = \nu(y).$

$\vdots$

- $(\neg y \to ((\nu(x) \to (\nu(x \to y) \to 0)) \to (((\nu(y) \to (\nu(x) \odot \nu(x \to y))) \to (\nu(x) \odot \nu(x \to y))) \to 0))) \odot ((\nu(y) \to (\nu(x) \odot \nu(\neg y \to \neg x))) \to (\nu(x) \odot \nu(\neg y \to \neg x))) = \nu(y)$

After putting

- $M(x,y) := \neg y \to ((\nu(x) \to (\nu(x \to y) \to 0)) \to (((\nu(y) \to (\nu(x) \odot \nu(x \to y))) \to (\nu(x) \odot \nu(x \to y))) \to 0)),$
- $N(x,y) := (\nu(y) \to (\nu(x) \odot \nu(\neg y \to \neg x))) \to (\nu(x) \odot \nu(\neg y \to \neg x))$

we can write $(SBL_{\neg}6)$ as

- $M(x,y) \odot N(x,y) = \nu(y).$

$$\vdots$$

- $M(x,y) = 1,$
- $N(x,y) = \neg y \to\sim (\nu(x) \to \neg(\neg y \to \neg x)).$

Together we get: $(SBL_{\neg}6)$ is equivalent to

- $\neg y \to\sim (\nu(x) \to \neg(\neg y \to \neg x)) = \nu(y).$

After putting

- $M(x,y) := \neg y \to ((\nu(x) \to (\nu(x \to y) \to 0)) \to (((\nu(y) \to (\nu(x) \odot \nu(x \to y))) \to (\nu(x) \odot \nu(x \to y))) \to 0)),$
- $N(x,y) := (\nu(y) \to (\nu(x) \odot \nu(\neg y \to \neg x))) \to (\nu(x) \odot \nu(\neg y \to \neg x))$

we can write ($SBL_\neg 6$) as

- $M(x,y) \odot N(x,y) = \nu(y).$

$\vdots$

- $M(x,y) = 1,$
- $N(x,y) = \neg y \to \sim (\nu(x) \to \neg(\neg y \to \neg x)).$

Together we get: ($SBL_\neg 6$) is equivalent to

- $\neg y \to \sim (\nu(x) \to \neg(\neg y \to \neg x)) = \nu(y).$

After putting

- $M(x,y) := \neg y \rightarrow ((\nu(x) \rightarrow (\nu(x \rightarrow y) \rightarrow 0)) \rightarrow (((\nu(y) \rightarrow (\nu(x) \odot \nu(x \rightarrow y))) \rightarrow (\nu(x) \odot \nu(x \rightarrow y))) \rightarrow 0)),$
- $N(x,y) := (\nu(y) \rightarrow (\nu(x) \odot \nu(\neg y \rightarrow \neg x))) \rightarrow (\nu(x) \odot \nu(\neg y \rightarrow \neg x))$

we can write $(SBL_{\neg}6)$ as

- $M(x,y) \odot N(x,y) = \nu(y).$

$\vdots$

- $M(x,y) = 1,$
- $N(x,y) = \neg y \rightarrow\sim (\nu(x) \rightarrow \neg(\neg y \rightarrow \neg x)).$

Together we get: $(SBL_{\neg}6)$ is equivalent to

- $\neg y \rightarrow\sim (\nu(x) \rightarrow \neg(\neg y \rightarrow \neg x)) = \nu(y).$

After putting

- $M(x,y) := \neg y \to ((\nu(x) \to (\nu(x \to y) \to 0)) \to (((\nu(y) \to (\nu(x) \odot \nu(x \to y))) \to (\nu(x) \odot \nu(x \to y))) \to 0)),$
- $N(x,y) := (\nu(y) \to (\nu(x) \odot \nu(\neg y \to \neg x))) \to (\nu(x) \odot \nu(\neg y \to \neg x))$

we can write $(SBL_{\neg}6)$ as

- $M(x,y) \odot N(x,y) = \nu(y).$

$\vdots$

- $M(x,y) = 1,$
- $N(x,y) = \neg y \to \sim (\nu(x) \to \neg(\neg y \to \neg x)).$

Together we get: $(SBL_{\neg}6)$ is equivalent to

- $\neg y \to \sim (\nu(x) \to \neg(\neg y \to \neg x)) = \nu(y).$

After putting

- $M(x,y) := \neg y \to ((\nu(x) \to (\nu(x \to y) \to 0)) \to (((\nu(y) \to (\nu(x) \odot \nu(x \to y))) \to (\nu(x) \odot \nu(x \to y))) \to 0)),$
- $N(x,y) := (\nu(y) \to (\nu(x) \odot \nu(\neg y \to \neg x))) \to (\nu(x) \odot \nu(\neg y \to \neg x))$

we can write ($SBL_{\neg}6$) as

- $M(x,y) \odot N(x,y) = \nu(y).$

$\vdots$

- $M(x,y) = 1,$
- $N(x,y) = \neg y \to\sim (\nu(x) \to \neg(\neg y \to \neg x)).$

Together we get: ($SBL_{\neg}6$) is equivalent to

- $\neg y \to\sim (\nu(x) \to \neg(\neg y \to \neg x)) = \nu(y).$

Theorem

The axioms characterizing $SBL_{\neg}$-algebras according to the foregoing theorem are mutually independent.

Sketch of proof:

$$
\begin{array}{ll}
\text{(BL1)} & x \odot y = y \odot x \,, \\
\text{(BL2)} & 1 \odot x = x \,, \\
\text{(BL3)} & x \odot (x \to y) = y \odot (y \to x) \,, \\
\text{(BL4)} & (x \odot y) \to z = x \to (y \to z) \,, \\
\text{(BL5)} & 0 \to x = 1 \,, \\
\text{(BL6)} & ((x \to y) \to z) \to (((y \to x) \to z) \to z) = 1 \,, \\
\text{(SBL)} & (x \odot y) \to 0 = (x \to 0) \vee (y \to 0) \,, \\
(\text{SBL}_{\neg}1) & \neg\neg x = x \,, \\
(\text{SBL}_{\neg}3) & \nu(x \to y) = \nu(\neg y \to \neg x) \,;
\end{array}
$$

Theorem

The axioms characterizing $SBL_\neg$-algebras according to the foregoing theorem are mutually independent.

Sketch of proof:

$$
\begin{array}{rl}
(BL1) & x \odot y = y \odot x\ , \\
(BL2) & 1 \odot x = x\ , \\
(BL3) & x \odot (x \to y) = y \odot (y \to x)\ , \\
(BL4) & (x \odot y) \to z = x \to (y \to z)\ , \\
(BL5) & 0 \to x = 1\ , \\
(BL6) & ((x \to y) \to z) \to (((y \to x) \to z) \to z) = 1\ , \\
(SBL) & (x \odot y) \to 0 = (x \to 0) \vee (y \to 0)\ , \\
(SBL_\neg 1) & \neg\neg x = x\ , \\
(SBL_\neg 3) & \nu(x \to y) = \nu(\neg y \to \neg x)\ ;
\end{array}
$$

Theorem

The axioms characterizing $SBL_\neg$-algebras according to the foregoing theorem are mutually independent.

Sketch of proof:

$$
\begin{array}{rl}
\text{(BL1)} & x \odot y = y \odot x \ , \\
\text{(BL2)} & 1 \odot x = x \ , \\
\text{(BL3)} & x \odot (x \to y) = y \odot (y \to x) \ , \\
\text{(BL4)} & (x \odot y) \to z = x \to (y \to z) \ , \\
\text{(BL5)} & 0 \to x = 1 \ , \\
\text{(BL6)} & ((x \to y) \to z) \to (((y \to x) \to z) \to z) = 1 \ , \\
\text{(SBL)} & (x \odot y) \to 0 = (x \to 0) \vee (y \to 0) \ , \\
(\mathrm{SBL}_\neg 1) & \neg\neg x = x \ , \\
(\mathrm{SBL}_\neg 3) & \nu(x \to y) = \nu(\neg y \to \neg x) \ ;
\end{array}
$$

(BL1) $x \odot y = y \odot x$

Two-element set $A = \{0, 1\}$:

$\odot$	0	1
0	0	1
1	0	1

$\rightarrow$	0	1
0	1	1
1	1	1

$\wedge$	0	1
0	1	1
1	1	1

$\vee$	0	1
0	1	1
1	1	1

$\neg$	0	1
	0	1

$\sim$	0	1
	1	1

ν	0	1
	1	1

$0 \odot 1 = 1 \neq 0 = 1 \odot 0$

(BL2) $1 \odot x = x$

Two-element set $A = \{0, 1\}$:

$\odot$	0	1
0	0	1
1	1	1

$\rightarrow$	0	1
0	1	1
1	1	1

$\wedge$	0	1
0	1	1
1	1	1

$\vee$	0	1
0	1	1
1	1	1

$\neg$	0	1
	0	1

$\sim$	0	1
	1	1

ν	0	1
	1	1

$1 \odot 0 = 1 \neq 0$

(BL3) $x \odot (x \to y) = y \odot (y \to x)$

Two-element set $A = \{0, 1\}$:

$\odot$	0	1
0	0	0
1	0	1

$\to$	0	1
0	1	1
1	1	1

$\wedge$	0	1
0	0	0
1	1	1

$\vee$	0	1
0	1	1
1	1	1

$\neg$	0	1
	0	1

$\sim$	0	1
	1	1

ν	0	1
	1	1

$1 \odot (1 \to 0) = 1 \neq 0 = 0 \odot (0 \to 1)$

(BL4) $(x \odot y) \to z = x \to (y \to z)$

Three-element chain $B = \{0, x, 1\}$:

$\odot$	0	x	1
0	0	0	0
x	0	1	x
1	0	x	1

$\to$	0	x	1
0	1	1	1
x	0	1	1
1	0	x	1

$\wedge$	0	x	1
0	0	0	0
x	0	x	x
1	0	x	1

$\vee$	0	x	1
0	0	x	1
x	x	x	1
1	1	1	1

$\neg$	0	x	1
	1	x	0

$\sim$	0	x	1
	1	0	0

ν	0	x	1
	0	0	1

$(x \odot x) \to x = x \neq 1 = x \to (x \to x)$

(BL5) $0 \to x = 1$

Three-element set $B = \{0, x, 1\}$:

$\odot$	0	x	1
0	x	x	0
x	x	x	x
1	0	x	1

$\to$	0	x	1
0	1	0	1
x	1	1	1
1	0	x	1

$\wedge$	0	x	1
0	0	x	0
x	x	x	x
1	0	x	1

$\vee$	0	x	1
0	0	0	1
x	0	x	1
1	1	1	1

$\neg$	0	x	1
	0	1	x

$\sim$	0	x	1
	1	1	0

ν	0	x	1
	1	0	1

$0 \to x = x \neq 1$

(BL6) $((x \to y) \to z) \to (((y \to x) \to z) \to z) = 1$

Three-element set $B = \{0, x, 1\}$:

$\odot$	0	x	1
0	0	0	0
x	0	1	x
1	0	x	1

$\to$	0	x	1
0	1	1	1
x	0	x	1
1	0	x	1

$\wedge$	0	x	1
0	0	0	0
x	0	1	x
1	0	x	1

$\vee$	0	x	1
0	0	x	1
x	x	1	x
1	1	x	1

$\neg$	0	x	1
	1	x	0

$\sim$	0	x	1
	1	0	0

ν	0	x	1
	0	0	1

$((0 \to 0) \to x) \to (((0 \to 0) \to x) \to x) = x \to x = x \neq 1$

(SBL) $(x \odot y) \to 0 = (x \to 0) \vee (y \to 0)$

Three-element chain $B = \{0, x, 1\}$:

$\odot$	0	x	1
0	0	0	0
x	0	0	x
1	0	x	1

$\to$	0	x	1
0	1	1	1
x	x	1	1
1	0	x	1

$\wedge$	0	x	1
0	0	0	0
x	0	x	x
1	0	x	1

$\vee$	0	x	1
0	0	x	1
x	x	x	1
1	1	1	1

$\neg$	0	x	1
	1	x	0

$\sim$	0	x	1
	1	x	0

ν	0	x	1
	0	x	1

$(x \odot x) \to 0 = 1 \neq x = (x \to 0) \vee (x \to 0)$

(SBL$_\neg$1) $\neg\neg x = x$

Two-element chain $A = \{0, 1\}$:

$\odot$	0	1
0	0	0
1	0	1

$\to$	0	1
0	1	1
1	0	1

$\wedge$	0	1
0	0	0
1	0	1

$\vee$	0	1
0	0	1
1	1	1

$\neg$	0	1
	1	1

$\sim$	0	1
	1	0

ν	0	1
	0	0

$\neg\neg\, 0 = 1 \neq 0$

(SBL$_\neg$3) $\nu(x \to y) = \nu(\neg y \to \neg x)$

Two-element chain $A = \{0, 1\}$:

$\odot$	0	1
0	0	0
1	0	1

$\to$	0	1
0	1	1
1	0	1

$\wedge$	0	1
0	0	0
1	0	1

$\vee$	0	1
0	0	1
1	1	1

$\neg$	0	1
	0	1

$\sim$	0	1
	1	0

ν	0	1
	1	0

$\nu(1 \to 0) = \nu(0) = 1 \neq 0 = \nu(1) = \nu(0 \to 1) = \nu(\neg 0 \to \neg 1)$

Outline

Ciucci, D.
On the axioms of residuated structures: Independence, dependences and rough approximations.
Fund. Informaticae, 69: 367–402, 2006.

Esteva, F., Godo, L., Hájek, P., Navara, M.
Residuated fuzzy logic with an involutive negation.
Arch. Mat. Logic, 39: 103–124, 2000.

Hájek, P.
Metamathematics of Fuzzy Logic.
Kluwer, 1998.