Identities in Tropical Matrix Semigroups and the Bicyclic Monoid

Laure Daviaud[1], Marianne Johnson[2] & **Mark Kambites**[2].

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[1]University of Warsaw (then), University of Warwick (now).
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[2]University of Manchester.

Tropical???

Definition

$$\mathbb{T} = \mathbb{R} \cup \{-\infty\}$$

Binary operations: $x \oplus y = \max(x, y)$ and $x \otimes y = x + y \quad (= \text{“}xy\text{”})$.

Properties

$\mathbb{T}$ *is an* **idempotent semifield***:*

- $(\mathbb{T}, \oplus)$ *is a commutative monoid with identity* $-\infty$*;*
- $-\infty$ *is a zero element for* $\otimes$*;*
- $(\mathbb{T} \setminus \{-\infty\}, \otimes)$ *is an abelian group with identity* 0*;*
- $\otimes$ *distributes over* $\oplus$*;*
- $x \oplus x = x$

In fact $x \oplus y$ *is either* x *or* y*.*

Definition

Tropical algebra or **max-plus algebra** is linear algebra where the base field is replaced by the tropical semiring.

Definition

Tropical geometry is (roughly!) algebraic geometry where the base field is replaced by the tropical semiring.

Applications

Tropical methods have applications in . . .

- Combinatorial Optimisation
- Discrete Event Systems
- Control Theory
- Formal Languages and Automata
- Phylogenetics
- Statistical Inference
- Geometric Group Theory
- Enumerative Algebraic Geometry
- Semigroup Theory

Tropical Polynomials

The tropical polynomial $x^2 \oplus x \oplus 1$ defines the function $x \mapsto \max(2x, x, 1)$.

The tropical polynomial $x^2 \oplus 1$ defines the function $x \mapsto \max(2x, 1)$.

These are the same function!

Definition

Two tropical polynomials are **equivalent** if they define the same function.

Definition

A term in a formal tropical polynomial is called **essential** if there is a value of the variable(s) for which only that term attains the maximum. A formal tropical polynomial is **essential** if every term is essential.

Computing Essential Polynomials

Proposition

Every tropical polynomial is equivalent to a unique essential polynomial. This is obtained by discarding all the non-essential terms.

- Each term of a tropical polynomial defines a (classical) linear function.
- To check if a term is essential is therefore a (classical continuous) linear programming problem.
- Given a tropical polynomial, we can compute the equivalent essential polynomial in polynomial time by checking if each term is essential and discarding those which are not.
- In particular, we can check in polynomial time whether two tropical polynomials are equivalent.
- All of this works with multiple variables.
- In fact with **one** variable and assuming a suitable model of computation we can do it in **linear** time (see Butkovic 2010).

Tropical Matrix Semigroups

Definition

$M_n(\mathbb{T})$ is the semigroup of $n \times n$ matrices over $\mathbb{T}$, under the natural matrix multiplication induced by $\oplus$ and $\otimes$.

- Studied implicitly for 50+ years with many interesting specific results (e.g. Gaubert, Cohen-Gaubert-Quadrat, d'Alessandro-Pasku).
- Since about 2008, systematic structural study using the tools of semigroup theory
  (Hollings, Izhakian, Johnson, Kambites, Taylor, Wilding).

Philosophy

The algebra of $M_n(\mathbb{T})$ *mirrors the geometry of* **tropical convex sets**.

Semigroup Identities

A **semigroup identity** is a pair of non-empty words, usually written $u = v$ over some alphabet Σ.

A semigroup S **satisfies** the identity $u = v$ if every morphism from the free semigroup Σ^+ to S sends u and v to the same place.

(In other words, if u and v evaluate to the same element for every substitution of elements in S for the letters in Σ.)

For example, a semigroup satisfies ...

- ... $AB = BA$ if and only if it is commutative;
- ... $A^2 = A$ if and only if it is idempotent;
- ... $AB = A$ if and only if it is a left-zero semigroup.

Tropical Matrix Identities

Theorem (d'Alessandro-Pasku 2003)

The semigroup $M_n(\mathbb{T})$ has **polynomial growth**. *(For any finite subset F, the number of distinct elements which can be written as products of k elements from F is bounded above by a polynomial in k.)*

Question (Izhakian-Margolis 2010)

Does $M_n(\mathbb{T})$ satisfy a semigroup identity?

- Yes, when $n = 1$ ($AB = BA$).
- Yes, when $n = 2$ (Izhakian-Margolis 2010, identity of length 40, reduced to 34 by Daviaud-Johnson 2017).
- Yes, when $n = 3$ (Shitov 2014, identity of length 2,714,856).
- Yes in general (very recent preprint Izhakian-Merlet 2018).

Construction of examples, but no general understanding.

Upper Triangular Tropical Matrices

Definition

- A tropical matrix is **upper triangular** if all entries below the main diagonal are $-\infty$.
- $UT_n(\mathbb{T})$ is the semigroup of all $n \times n$ upper triangular tropical matrices.

Question

Does $UT_n(\mathbb{T})$ satisfy a semigroup identity?

- Yes, when $n = 1$.
- Yes, when $n = 2$ (Izhakian-Margolis 2010, shortest has length 20).
- Yes in general (Izhakian 2013–16, Okniński 2015, Taylor 2016).

Results for $M_n(\mathbb{T})$ are based on those for $UT_n(\mathbb{T})$.
Constructions of examples, beginning to glimpse a general understanding.

The Bicyclic Monoid.

Definition

The **bicyclic monoid** $\mathbb{B}$ is the monoid with presentation.

$$\langle p, q \mid pq = 1 \rangle.$$

The bicyclic monoid is ...

- ...the monoid generated by the partial functions

$$p : \mathbb{N} \to \mathbb{N} \setminus \{0\}, \quad n \mapsto n+1$$

$$q : \mathbb{N} \setminus \{0\} \to \mathbb{N}, \quad n \mapsto n-1.$$

- ...the syntactic monoid of the language of Dyck words.
- ...the natural algebraic model of a counter or a one-sided shift.

It also ubiquitous in (infinite) semigroup theory.

Identities in the Bicyclic Monoid

Theorem (Adjan 1966)

The bicyclic monoid $\mathbb{B}$ satisfies the identity

$$ABBA\ AB\ ABBA\ =\ ABBA\ BA\ ABBA$$

and no shorter identity.

Theorem (Shleifer 1990, exhaustive computer search)

Up to obvious manipulations, there are exactly **two** *identities of this length which hold in $\mathbb{B}$. (The other is $ABBA\ AB\ BAAB\ =\ ABBA\ BA\ BAAB$).*

Theorem (consequence of Scheiblich 1971)

The bicyclic monoid satisfies the same identities as the **free monogenic inverse monoid***.*

Theorem (Shneerson 1989)

The bicyclic monoid does not have a finite basis of identities.

Identities in $UT_2(\mathbb{T})$

Theorem (Izhakian-Margolis 2010)

$UT_2(\mathbb{T})$ satisfies Adjan's identity $ABBA\ AB\ ABBA = ABBA\ BA\ ABBA$.

Proposition (Izhakian-Margolis 2010)

The bicyclic monoid $\mathbb{B}$ embeds in $UT_2(\mathbb{T})$.

Corollary

Every identity satisfied in $UT_2(\mathbb{T})$ is satisfied in $\mathbb{B}$.

Question (Izhakian-Margolis 2010)

Do $UT_2(\mathbb{T})$ and $\mathbb{B}$ satisfy exactly the same identities?

Theorem (Chen-Hu-Luo-Sapir 2016)

$UT_2(\mathbb{T})$ has no finite basis of identities.

The Technical Bit That Shows There Is Some Content

- Let $w = w_1 \dots w_k$ be a word over an alphabet Σ.
- For each $s \in \Sigma$ and $0 \leq i \leq |w|$, let $\lambda_s^w(i)$ be the number of occurrences of s in the first i letters of the word w.
- For each $t \in \Sigma$ define a formal tropical polynomial

$$f_t^w = \bigoplus_{w_i = t} \bigotimes_{s \in \Sigma} x_s^{\lambda_s^w(i-1)}$$

in the variables x_s for $s \in \Sigma$.

Theorem (Daviaud-Johnson-K. 2018)

An identity $w = v$ is satisfied in $UT_2(\mathbb{T})$ if and only if for each $t \in \Sigma$, the tropical polynomials f_t^w and f_t^v are equivalent.

Corollary

Identities in $UT_2(\mathbb{T})$ can (really!) be checked in polynomial time.

In the special case of a 2-letter identity, a projectivisation trick allows us to reduce to (twice as many) one-variable polynomials:

Theorem (Daviaud-Johnson-K. 2018)

Suppose w and v are words over a 2-letter alphabet Σ. Then the identity $w = v$ is satisfied in $UT_2(\mathbb{T})$ if and only if for each $t \in \Sigma$,

- $f_t^w(x, 1)$ *is equivalent to* $f_t^v(x, 1)$*; and*
- $f_t^w(x, -1)$ *is equivalent to* $f_t^v(x, -1)$.

Corollary

Assuming a suitable model of computation, 2-letter identities in $UT_2(\mathbb{T})$ can be checked in **linear** *time.*

Example: Shleifer's identity

- Let's check if $ABBAABBAAB = ABBABABAAB$ holds in $UT_2(\mathbb{T})$.
- Set $w = ABBA\ AB\ BAAB$ and $v = ABBA\ BA\ BAAB$.
- It suffices to check if f_t^w is equivalent to f_t^v for all $t \in \{A, B\}$ ...
- ...or if $f_t^w(x, b)$ is equivalent to $f_t^v(x, b)$ for $t \in \{A, B\}$, $b \in \{1, -1\}$.
- For example, from the definitions

$$f_A^w(x, 1) = 0 \oplus (x + 2) \oplus (2x + 2) \oplus (3x + 4) \oplus (4x + 4)$$

$$f_A^v(x, 1) = 0 \oplus (x + 2) \oplus (2x + 3) \oplus (3x + 4) \oplus (4x + 4)$$

- These differ only in the red terms, which are inessential (check!).
- So the polynomials are equivalent.
- After checking the other three possibilities, we conclude that Shleifer's identity holds in $UT_2(\mathbb{T})$.

Theorem (Daviaud-Johnson-K. 2018)

The monoid $UT_2(\mathbb{T})$ satisfies exactly the same identities as the bicyclic monoid $\mathbb{B}$ (and the free monogenic inverse monoid).

Proof.

Given values of the variables which falsify an identity in $UT_2(\mathbb{T})$, manipulate them to construct values which falsify the identity inside an embedded copy of $\mathbb{B}$. □

Corollary

Efficient algorithms to check identities in the bicyclic monoid (see also Pastijn 2006 for an alternative but related approach).

Remark

From a computational perspective, the "big picture" is passage from a discrete to a continuous setting, so that we can do continuous linear programming instead of integer programming.

Other Corollaries

Corollary

The subsemigroups of $UT_2(\mathbb{T})$ obtained by restricting the on-or-above diagonal entries to lie in $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Q} \cup \{-\infty\}$, $\mathbb{Z} \cup \{-\infty\}$ or $\mathbb{N} \cup \{-\infty\}$ all satisfy the same identities as $\mathbb{B}$.

Corollary

Various continuous versions of $\mathbb{B}$ satisfy the same identities as $\mathbb{B}$.

Corollary

More information on the relationship between $\mathbb{B}$ and the free monogenic inverse monoid.

The Details

L. Daviaud, M. Johnson & M. Kambites, *Identities in upper triangular tropical matrix semigroups and the bicyclic monoid*, Journal of Algebra Vol.501 (2018), pp.503–525.

The Future

- Digest Izhakian-Merlet.
- Efficient algorithms and usable theoretical descriptions for identities holding in $M_n(\mathbb{T})$ and $UT_n(\mathbb{T})$.
- Johnson-Tran (preprint 2018) have made a good start for $UT_n(\mathbb{T})$:
  - use lattice polytopes to describe identities in $UT_n(\mathbb{T})$;
  - for 2-letter identities in $\mathbb{B}$ (or equivalently $UT_2(\mathbb{T})$), an efficient enumeration algorithm and a shorter proof of Adjan's theorem;
  - similar polytope characterisation for higher n (but barriers to efficient computational application);
  - numerical data and consequent conjectures linking semigroup theory, geometry, probability and combinatorics.
- Applications to other interesting semigroups representable by tropical matrices, such as plactic monoids (Cain-Klein-Kubat-Malheiro-Okniński, preprint 2017).