Logic, Algebra and Implication

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SSAOS 2018

Outline

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- Protoalgebraic logics and their subclasses are based on a general notion of equivalence.
- Implication has a crucial role in reasoning (entailment, consequence, preservation of truth,...)
- The goal of this course is to present an AAL theory based on implication, together with a wealth of examples of (non-)classical logics.

Outline

Basic syntactical notions – 1

Propositional language: a countable type $\mathcal{L}$, i.e. a function $ar\colon C_{\mathcal{L}} \to \mathrm{N}$, where $C_{\mathcal{L}}$ is a countable set of symbols called connectives, giving for each one its arity. Nullary connectives are also called truth-constants. We write $\langle c, n\rangle \in \mathcal{L}$ whenever $c \in C_{\mathcal{L}}$ and $ar(c) = n$.

Formulas: Let Var be a fixed infinite countable set of symbols called variables. The set $Fm_{\mathcal{L}}$ of formulas in $\mathcal{L}$ is the least set containing Var and closed under connectives of $\mathcal{L}$, i.e. for each $\langle c, n\rangle \in \mathcal{L}$ and every $\varphi_1, \dots, \varphi_n \in Fm_{\mathcal{L}}$, $c(\varphi_1, \dots, \varphi_n)$ is a formula.

Substitution: a mapping $\sigma\colon Fm_{\mathcal{L}} \to Fm_{\mathcal{L}}$, such that $\sigma(c(\varphi_1, \dots, \varphi_n)) = c(\sigma(\varphi_1), \dots, \sigma(\varphi_n))$ holds for each $\langle c, n\rangle \in \mathcal{L}$ and every $\varphi_1, \dots, \varphi_n \in Fm_{\mathcal{L}}$.

Consecution: a pair $\Gamma \rhd \varphi$, where $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$.

Basic syntactical notions – 2

A set L of consecutions can be seen as a relation between sets of formulas and formulas. We write '$\Gamma \vdash_{\mathrm{L}} \varphi$' instead of '$\Gamma \rhd \varphi \in \mathrm{L}$'.

Definition

A set L of consecutions in $\mathcal{L}$ is called a logic in $\mathcal{L}$ whenever

- If $\varphi \in \Gamma$, then $\Gamma \vdash_{\mathrm{L}} \varphi$. (Reflexivity)
- If $\Delta \vdash_{\mathrm{L}} \psi$ for each $\psi \in \Gamma$ and $\Gamma \vdash_{\mathrm{L}} \varphi$, then $\Delta \vdash_{\mathrm{L}} \varphi$. (Cut)
- If $\Gamma \vdash_{\mathrm{L}} \varphi$, then $\sigma[\Gamma] \vdash_{\mathrm{L}} \sigma(\varphi)$ for each substitution σ. (Structurality)

Observe that reflexivity and cut entail:

- If $\Gamma \vdash_{\mathrm{L}} \varphi$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash_{\mathrm{L}} \varphi$. (Monotonicity)

The least logic Min is described as:

$$\Gamma \vdash_{\mathrm{Min}} \varphi \qquad \text{iff} \qquad \varphi \in \Gamma.$$

Basic syntactical notions – 3

Theorem: a consequence of the empty set

(note that Min has no theorems).

Inconsistent logic Inc: the set all consecutions

(equivalently: a logic where all formulas are theorems).

Almost Inconsistent logic AInc: the maximum logic without theorems (note that $\Gamma, \varphi \vdash_{\mathrm{AInc}} \psi$).

Theory: a set of formulas T such that if $T \vdash_{\mathrm{L}} \varphi$ then $\varphi \in T$. By $\mathrm{Th}(\mathrm{L})$ we denote the set of all theories of L.

Note that

- $\mathrm{Th}(\mathrm{L})$ can be seen as a closure system. By $\mathrm{Th}_{\mathrm{L}}(\Gamma)$ we denote the theory generated in $\mathrm{Th}(\mathrm{L})$ by Γ (i.e., the intersection of all theories containing Γ).
- $\mathrm{Th}_{\mathrm{L}}(\Gamma) = \{\varphi \in Fm_{\mathcal{L}} \mid \Gamma \vdash_{\mathrm{L}} \varphi\}$.
- The set of all theorems is the least theory and it is generated by the empty set.

Basic syntactical notions – 4

Axiomatic system: a set $\mathcal{AS}$ of consecutions closed under substitutions. An element $\Gamma \rhd \varphi$ is an

- axiom if $\Gamma = \emptyset$,
- finitary deduction rule if Γ is a finite,
- infinitary deduction rule otherwise.

An axiomatic system is finitary if all its rules are finitary.

Proof: a proof of a formula φ from a set of formulas Γ in $\mathcal{AS}$ is a well-founded tree labeled by formulas such that

- its root is labeled by φ and leaves by axioms of $\mathcal{AS}$ or elements of Γ and
- if a node is labeled by ψ and $\Delta \neq \emptyset$ is the set of labels of its preceding nodes, then $\Delta \rhd \psi \in \mathcal{AS}$.

We write $\Gamma \vdash_{\mathcal{AS}} \varphi$ if there is a proof of φ from Γ in $\mathcal{AS}$.

Basic syntactical notions – 5

Lemma

Let $\mathcal{AS}$ be an axiomatic system. Then $\vdash_{\mathcal{AS}}$ is the least logic containing $\mathcal{AS}$.

Presentation: We say that $\mathcal{AS}$ is an axiomatic system for (or a presentation of) the logic L if $\mathrm{L} = \vdash_{\mathcal{AS}}$. A logic is said to be finitary if it has some finitary presentation.

Lemma

A logic L is finitary iff for each set of formulas $\Gamma \cup \{\varphi\}$ we have: if $\Gamma \vdash_{\mathrm{L}} \varphi$, then there is a finite $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vdash_{\mathrm{L}} \varphi$.

Note that $\mathrm{Inc}, \mathrm{AInc}, \mathrm{Min}$ are finitary because:

Inc	is axiomatized by	axioms $\{\varphi \mid \varphi \in Fm_{\mathcal{L}}\}$
AInc	is axiomatized by	unary rules $\{\varphi \rhd \psi \mid \varphi, \psi \in Fm_{\mathcal{L}}\}$
Min	is axiomatized by	by the empty set

More interesting examples

Finitary axiomatic system for BCI in $\mathcal{L}_{\to} = \{\to\}$

B $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$

C $(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$

I $\varphi \to \varphi$

MP $\varphi, \varphi \to \psi \rhd \psi$

Finitary axiomatic system for BCK in $\mathcal{L}_{\to} = \{\to\}$

B $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$

C $(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$

K $\varphi \to (\psi \to \varphi)$

MP $\varphi, \varphi \to \psi \rhd \psi$

Examples of proofs – 1

Let φ be an arbitrary formula. We show that $\vdash_{\mathrm{BCK}} \varphi \to \varphi$:

- a $\varphi \to ((\varphi \to (\psi \to \varphi)) \to \varphi)$ (K)
- b $[\varphi \to ((\varphi \to (\psi \to \varphi)) \to \varphi)] \to [(\varphi \to (\psi \to \varphi)) \to (\varphi \to \varphi)]$ (C)
- c $(\varphi \to (\psi \to \varphi)) \to (\varphi \to \varphi)$ a, b, and (MP)
- d $\varphi \to (\psi \to \varphi)$ (K)
- e $\varphi \to \varphi$ c, d, and (MP)

Now we prove in BCI $(\varphi \to \psi) \to ((\chi \to \varphi) \to (\chi \to \varphi))$ (prefixing):

- a $(\chi \to \varphi) \to ((\varphi \to \psi) \to (\chi \to \varphi))$ (B)
- b $[(\chi \to \varphi) \to ((\varphi \to \psi) \to (\chi \to \varphi))] \to [(\varphi \to \psi) \to ((\chi \to \varphi) \to (\chi \to \varphi))]$ (C)
- c $(\varphi \to \psi) \to ((\chi \to \varphi) \to (\chi \to \varphi))$ a, b, and (MP)

Examples of proofs – 2

$\varphi \to \psi, \psi \to \chi \rhd \varphi \to \chi$ (transitivity) is a derivable rule in BCI and BCK:

a	$\varphi \to \psi$	hypothesis
b	$\psi \to \chi$	hypothesis
c	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$	(B)
d	$(\psi \to \chi) \to (\varphi \to \chi)$	a, c, and (MP)
e	$\varphi \to \chi$	b, d, and (MP)

Basic syntactical notions – 6

Let $\mathcal{L}_1 \subseteq \mathcal{L}_2$ be propositional languages, L_i a logic in $\mathcal{L}_i$, and $\mathcal{S}$ a set of consecutions in $\mathcal{L}_2$.

- L_2 is the expansion of L_1 by $\mathcal{S}$ if it is the weakest logic in $\mathcal{L}_2$ containing L_1 and $\mathcal{S}$, i.e. the logic axiomatized by all $\mathcal{L}_2$-substitutional instances of consecutions from $\mathcal{S} \cup \mathcal{AS}$, for any presentation $\mathcal{AS}$ of L_1.
- L_2 is an expansion of L_1 if $\mathrm{L}_1 \subseteq \mathrm{L}_2$, i.e. it is the expansion of L_1 by $\mathcal{S}$, for some set of consecutions $\mathcal{S}$.
- L_2 is an axiomatic expansion of L_1 if it is an expansion obtained by adding a set of *axioms*.
- L_2 is a conservative expansion of L_1 if it is an expansion and for each consecution $\Gamma \rhd \varphi$ in $\mathcal{L}_1$ we have that $\Gamma \vdash_{\mathrm{L}_2} \varphi$ entails $\Gamma \vdash_{\mathrm{L}_1} \varphi$.

If $\mathcal{L}_1 = \mathcal{L}_2$, we use ‘extension’ instead ‘expansion’.

Even more interesting examples

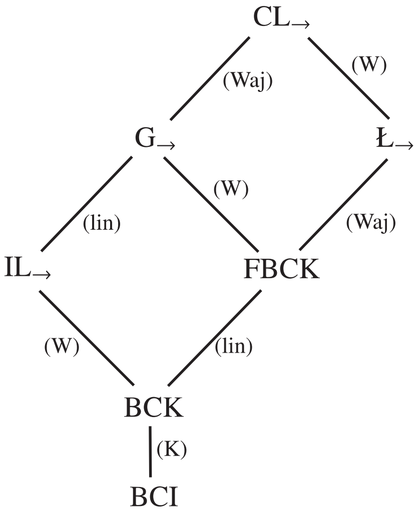
Consider the following axioms in $\mathcal{L}_{\to}$:

(W)	$(\varphi \to (\varphi \to \psi)) \to (\varphi \to \psi)$	contraction
(P)	$((\varphi \to \psi) \to \varphi) \to \varphi$	Peirce's law
(Waj)	$((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$	Wajsberg axiom
(lin)	$((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$	linearity

and define the following logics:

Logic	Presentation
FBCK	BCK extended by (lin)
$\mathrm{IL}_{\to}$	BCK extended by (W)
$\mathrm{G}_{\to}$	BCK extended by (W) and (lin)
$\mathrm{CL}_{\to}$	BCK extended by (W) and (P)
$\text{Ł}_{\to}$	BCK extended by (lin) and (Waj)

Prominent axiomatic extensions of BCI



Famous examples

$\mathcal{L}_{\mathrm{CL}} = \{\to, \wedge, \vee, \overline{0}\}$.

IL (intuitionistic logic): axiomatic expansion of $\mathrm{IL}_{\to}$
CL (classical logic): axiomatic expansion of $\mathrm{CL}_{\to}$
$\mathrm{Ł}$ (Łukasiewicz logic): axiomatic expansion of $\mathrm{Ł}_{\to}$
G (Gödel–Dummett logic): axiomatic expansion of $\mathrm{G}_{\to}$

by the axioms:

$(\bot)$	$\overline{0} \to \varphi$
(axAdj)	$\varphi \to (\psi \to \varphi \wedge \psi)$
(LB_1)	$\varphi \wedge \psi \to \varphi$
(LB_2)	$\varphi \wedge \psi \to \psi$
(axInf)	$(\varphi \to \psi) \wedge (\varphi \to \chi) \to (\varphi \to \psi \wedge \chi)$
(UB_1)	$\varphi \to \varphi \vee \psi$
(UB_2)	$\psi \to \varphi \vee \psi$
(axSup)	$(\varphi \to \chi) \wedge (\psi \to \chi) \to (\varphi \vee \psi \to \chi)$

Remarks on the famous examples – 1

- Classical logic has numerous other presentations more common than the one used here.
- Gödel–Dummett is usually presented in a language where $\vee$ is a defined connective:

$$\varphi \vee \psi = ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi)$$

  Then, Gödel–Dummett logic is the axiomatic extension of IL by the axiom of *prelinearity*:

$$(\varphi \to \psi) \vee (\psi \to \varphi)$$

Remarks on the famous examples – 2

- Łukasiewicz logic is usually presented in a language where $\wedge$ and $\vee$ are defined connectives:

  $$\varphi \vee \psi = (\varphi \to \psi) \to \psi \qquad \varphi \wedge \psi = \neg(\neg\varphi \vee \neg\psi)$$

  with an axiomatic system consisting of *modus ponens* and axioms (B), (K), (Waj), and

  $$(\neg\varphi \to \neg\psi) \to (\psi \to \varphi).$$

  Also, the following two connectives are usually defined in Łukasiewicz logic:

  $$\varphi \,\&\, \psi = \neg(\varphi \to \neg\psi) \qquad \varphi \oplus \psi = \neg\varphi \to \psi.$$

An infinitary example

A prominent extension of Łukasiewicz logic, denoted as Ł_∞, is obtained by adding the following infinitary rule:

$$\{\neg\varphi \to \varphi \mathbin{\&} \overset{n}{\ldots} \mathbin{\&} \varphi \mid n \geq 1\} \rhd \varphi$$

Basic semantical notions – 1

$\mathcal{L}$-algebra: $\boldsymbol{A} = \langle A, \langle c^{\boldsymbol{A}} \mid c \in C_{\mathcal{L}} \rangle \rangle$, where $A \neq \emptyset$ (universe) and $c^{\boldsymbol{A}} \colon A^n \to A$ for each $\langle c, n \rangle \in \mathcal{L}$.

Algebra of formulas: the algebra $\boldsymbol{Fm}_{\mathcal{L}}$ with domain $Fm_{\mathcal{L}}$ and operations $c^{\boldsymbol{Fm}_{\mathcal{L}}}$ for each $\langle c, n \rangle \in \mathcal{L}$ defined as:

$$c^{\boldsymbol{Fm}_{\mathcal{L}}}(\varphi_1, \ldots, \varphi_n) = c(\varphi_1, \ldots, \varphi_n).$$

$\boldsymbol{Fm}_{\mathcal{L}}$ is the absolutely free algebra in language $\mathcal{L}$ with generators Var.

Homomorphism of algebras: a mapping $f \colon A \to B$ such that for every $\langle c, n \rangle \in \mathcal{L}$ and every $a_1, \ldots, a_n \in A$,

$$f(c^{\boldsymbol{A}}(a_1, \ldots, a_n)) = c^{\boldsymbol{B}}(f(a_1), \ldots, f(a_n)).$$

Note that substitutions are exactly endomorphisms of $\boldsymbol{Fm}_{\mathcal{L}}$.

Examples of algebras – 1

Boolean algebra: $\boldsymbol{A} = \langle A, \wedge, \vee, \neg, \overline{0}, \overline{1}\rangle$, where $\langle A, \wedge, \vee, \overline{0}, \overline{1}\rangle$ is a bounded distributive lattice and for every $a \in A$:

$$a \wedge \neg a = \overline{0} \text{ and } a \vee \neg a = \overline{1} \qquad (\textit{complement})$$

Prototypical example: power set algebra of a set A, i.e. the structure $\langle P(A), \cap, \cup, -, \emptyset, A\rangle$, where for every $X \subseteq A$ we have $-X = A \setminus X$.

Stone's representation theorem: each Boolean algebra can be embedded into a Boolean algebra defined over the power set algebra of some set.

We denote the class of all Heyting algebras as $\mathbb{HA}$.

Examples of algebras – 2

Heyting algebra: $\boldsymbol{A} = \langle A, \wedge, \vee, \to, \overline{0}, \overline{1} \rangle$, where $\boldsymbol{A} = \langle A, \wedge, \vee, \overline{0}, \overline{1} \rangle$ is a bounded distributive lattice and for every $a, b, c \in A$:

$$a \wedge b \leq c \text{ if, and only if, } a \leq b \to c \qquad \textit{(residuation)}$$

where $\leq$ is the canonical lattice order.

$\to$ is called the residuum of $\wedge$.

Pseudocomplement: $\neg a = a \to \overline{0}$ for $a \in A$.

We denote the class of all Heyting algebras as $\mathbb{HA}$.

Each Boolean algebra can be seen as a Heyting algebra where the residuum is defined as $a \to b = \neg a \vee b$. Therefore, Boolean algebras turn out to be exactly the Heyting algebras in which $\neg$ satisfies the complement condition.

Examples of algebras – 3

Gödel algebra or G-algebra: A Heyting algebra $\boldsymbol{A} = \langle A, \wedge, \vee, \to, \overline{0}, \overline{1} \rangle$ such that for every $a, b \in A$:

$$(a \to b) \vee (b \to a) = \overline{1}. \qquad \textit{(prelinearity)}$$

We denote the class of all G-algebras as $\mathbb{G}$.

$\mathbb{BA} \subseteq \mathbb{G} \subseteq \mathbb{HA}$

Examples of algebras – 4

MV-algebra: $\langle A, \oplus, \neg, \overline{0} \rangle$, where $\oplus$ is a binary operation, $\neg$ is a unary operation and $\overline{0}$ is a constant such that the following are satisfied for any $a, b, c \in A$:

1. $a \oplus (b \oplus c) = (a \oplus b) \oplus c$
2. $a \oplus b = b \oplus a$
3. $a \oplus \overline{0} = \overline{0}$
4. $\neg\neg a = a$,
5. $\neg\overline{0} \oplus a = \neg\overline{0}$,
6. $\neg(\neg a \oplus b) \oplus b = \neg(\neg b \oplus a) \oplus a$.

We denote the class of all MV-algebras as $\mathbb{MV}$.

Lattice operations

Proposition

Let $\langle A, \oplus, \neg, \overline{0}\rangle$ *be an* MV*-algebra. For each* $a, b \in A$ *we define:*

- $a \,\&\, b = \neg(\neg a \oplus \neg b)$
- $a \to b = \neg(a \,\&\, \neg b)$
- $\overline{1} = \neg\overline{0}$
- $a \vee b = a \oplus (b \,\&\, \neg a)$
- $a \wedge b = a \,\&\, (b \oplus \neg a)$

  Then:

  1. $\langle A, \wedge, \vee, \overline{0}, \overline{1}\rangle$ *is a bounded distributive lattice, and*
  2. *for each* $a, b \in A$*, we have:* $a \,\&\, b \leq c$ *iff* $a \leq b \to c$.

Examples of algebras – 5

- the standard G-algebra: $[0,1]_{\mathrm{G}} = \langle [0,1], \wedge, \vee, \to, 0, 1\rangle$, where $\wedge$ and $\vee$ are the lattice operations given by the natural order in $[0,1]$, and for each $a, b \in [0,1]$:
$$a \to b = \begin{cases} 1 & \text{if } a \leq b, \\ b & \text{otherwise.} \end{cases}$$
- the standard MV-algebra: $[0,1]_{Ł} = \langle [0,1], \oplus, \neg, 0\rangle$, where for each $a, b \in [0,1]$, $a \oplus b = \min\{a+b, 1\}$ and $\neg a = 1 - a$. The lattice operations defined in the previous proposition coincide with the lattice operations given by the natural order in $[0,1]$.

Basic semantical notions – 2

$\mathcal{L}$-matrix: a pair $\mathbf{A} = \langle \boldsymbol{A}, F\rangle$ where $\boldsymbol{A}$ is an $\mathcal{L}$-algebra called the algebraic reduct of $\mathbf{A}$, and F is a subset of A called the filter of $\mathbf{A}$. The elements of F are called designated elements of $\mathbf{A}$.

A matrix $\mathbf{A} = \langle \boldsymbol{A}, F\rangle$ is

- trivial if $F = A$.
- finite if A is finite.
- Lindenbaum if $\boldsymbol{A} = \boldsymbol{Fm}_{\mathcal{L}}$.

$\boldsymbol{A}$-evaluation: a homomorphism from $\boldsymbol{Fm}_{\mathcal{L}}$ to $\boldsymbol{A}$, i.e. a mapping $e\colon Fm_{\mathcal{L}} \to A$, such that for each $\langle c, n\rangle \in \mathcal{L}$ and each n-tuple of formulas $\varphi_1, \ldots, \varphi_n$ we have:

$$e(c(\varphi_1, \ldots, \varphi_n)) = c^{\boldsymbol{A}}(e(\varphi_1), \ldots, e(\varphi_n)).$$

Basic semantical notions – 3

Semantical consequence: A formula φ is a semantical consequence of a set Γ of formulas w.r.t. a class $\mathbb{K}$ of $\mathcal{L}$-matrices if for each $\langle \boldsymbol{A}, F \rangle \in \mathbb{K}$ and each $\boldsymbol{A}$-evaluation e, we have $e(\varphi) \in F$ whenever $e[\Gamma] \subseteq F$; we denote it by $\Gamma \models_{\mathbb{K}} \varphi$.

Exercise 1

Let $\mathbb{K}$ be a class of $\mathcal{L}$-matrices. Then $\models_{\mathbb{K}}$ is a logic in $\mathcal{L}$.

Lemma (Tabular logics)

Furthermore, if $\mathbb{K}$ is a finite class of finite matrices, then the logic $\models_{\mathbb{K}}$ is finitary.

L-matrix: Let L be a logic in $\mathcal{L}$ and $\mathbf{A}$ an $\mathcal{L}$-matrix. We say that $\mathbf{A}$ is an L-matrix if $\mathrm{L} \subseteq \models_{\mathbf{A}}$. We denote the class of L-matrices by $\mathbf{MOD}(\mathrm{L})$.

Basic semantical notions – 4

Lemma (Images and preimages of models)

Let L *be a logic in* $\mathcal{L}$ *and a mapping* $g\colon \boldsymbol{A} \to \boldsymbol{B}$ *be a homomorphism of* $\mathcal{L}$*-algebras* $\boldsymbol{A}, \boldsymbol{B}$*. Then:*

- $\langle \boldsymbol{A}, g^{-1}[G]\rangle \in \mathbf{MOD}(\mathrm{L})$, *whenever* $\langle \boldsymbol{B}, G\rangle \in \mathbf{MOD}(\mathrm{L})$.
- $\langle \boldsymbol{B}, g[F]\rangle \in \mathbf{MOD}(\mathrm{L})$, *whenever* $\langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}(\mathrm{L})$ *and* g *is surjective and* $g(x) \in g[F]$ *implies* $x \in F$.

Basic semantical notions – 5

Logical filter: Given a logic L in $\mathcal{L}$ and an $\mathcal{L}$-algebra $\boldsymbol{A}$, a subset $F \subseteq A$ is an L-*filter* if $\langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}(\mathrm{L})$. By $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ we denote the set of all L-filters over $\boldsymbol{A}$.

$\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ is a closure system and can be given a lattice structure by defining for any $F, G \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$, $F \wedge G = F \cap G$ and $F \vee G = \mathrm{Fi}_{\mathrm{L}}^{\boldsymbol{A}}(F \cup G)$.

Generated filter: Given a set $X \subseteq A$, the logical filter generated by X is $\mathrm{Fi}_{\mathrm{L}}^{\boldsymbol{A}}(X) = \bigcap\{F \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A}) \mid X \subseteq F\}$.

$$\mathcal{F}i_{\mathrm{Min}}(\boldsymbol{A}) = \mathcal{P}(A) \qquad \mathcal{F}i_{\mathrm{AInc}}(\boldsymbol{A}) = \{\emptyset, A\} \qquad \mathcal{F}i_{\mathrm{Inc}}(\boldsymbol{A}) = \{A\}$$

Examples of logical filters – 1

Exercise 2

- Let $\boldsymbol{A}$ be a Heyting algebra. Then $F \in \mathcal{F}i_{\mathrm{IL}}(\boldsymbol{A})$ iff F is a lattice filter on $\boldsymbol{A}$.
- Let $\boldsymbol{A}$ be a G-algebra. Then $F \in \mathcal{F}i_{\mathrm{G}}(\boldsymbol{A})$ iff F is a lattice filter on $\boldsymbol{A}$.
- Let $\boldsymbol{A}$ be a Boolean algebra. Then $F \in \mathcal{F}i_{\mathrm{CL}}(\boldsymbol{A})$ iff F is a lattice filter on $\boldsymbol{A}$.
- Let $\boldsymbol{A}$ be an MV-algebra. Then $F \in \mathcal{F}i_{\text{Ł}}(\boldsymbol{A})$ iff F is a lattice filter on $\boldsymbol{A}$ and for each $x, y \in A$ such that $x, x \to y \in F$ we have $y \in F$.

Examples of logical filters – 2

$\mathbf{A} = \langle [0,1]_{\mathrm{G}}, (0,1] \rangle \in \mathbf{MOD}(\mathrm{CL})$.

Indeed, we know that $\mathbf{A} \in \mathbf{MOD}(\mathrm{IL})$ and, hence, we only need to show that $\models_{\mathbf{A}} ((\varphi \to \psi) \to \varphi) \to \varphi$.

Clearly if $e(\varphi) > 0$, then $e(((\varphi \to \psi) \to \varphi) \to \varphi) > 0$.
If $e(\varphi) = 0$, then $e(\varphi \to \psi) = 1$, $e((\varphi \to \psi) \to \varphi) = 0$, and so $e(((\varphi \to \psi) \to \varphi) \to \varphi) = 1 > 0$.

Examples of logical filters – 3

Now we can show that Ł_∞ is not finitary (hence, a proper extension of Łukasiewicz logic).

$\mathbf{M}_{\text{Ł}} = \langle [0,1]_{\text{Ł}}, \{1\} \rangle \in \mathbf{MOD}(\text{Ł}_\infty)$. However, for each positive $k \in \mathrm{N}$

$$\{\neg\varphi \to \varphi^n \mid 1 \leq n < k\} \not\models_{\mathbf{M}_{\text{Ł}}} \varphi,$$

where by φ^n we denote $\varphi \,\&\, .\overset{n}{.}. \,\&\, \varphi$. Indeed, it suffices to take the evaluation $e(\varphi) = \frac{k}{k+1}$ and note that $e(\varphi)^n = \frac{k-n}{k+1} \geq \frac{1}{k+1} = e(\neg\varphi)$ for $n < k$

Examples of logical filters – 4

A model of Ł which is not a model of $Ł_\infty$.

$\boldsymbol{C} = \langle C, \oplus, \neg, \overline{0} \rangle$ (Chang algebra):

- $C = \{\langle 0, i\rangle \mid i \in \mathrm{N}\} \cup \{\langle 1, -i\rangle \mid i \in \mathrm{N}\}$
- $\overline{0} = \langle 0, 0\rangle$
- $\langle x, i\rangle + \langle y, j\rangle = \begin{cases} (1,0) & \text{if } x+y=2 \\ (1,0) & \text{if } x+y=1 \text{ and } i+j \geq 0 \\ (x+y, i+j) & \text{otherwise} \end{cases}$
- $\neg\langle x, i\rangle = \langle 1-x, -i\rangle$.

$$\langle x, i\rangle \,\&\, \langle y, j\rangle = \begin{cases} \langle 0,0\rangle & \text{if } x+y=0 \\ \langle 0,0\rangle & \text{if } x+y=1 \text{ and } i+j \leq 0 \\ \langle x+y, i+j\rangle & \text{otherwise} \end{cases}$$

Now we consider the matrix $\mathbf{C} = \langle \boldsymbol{C}, \{\langle 1, 0\rangle\}\rangle$ and show that

$$\{\neg\varphi \to \varphi^n \mid n \geq 1\} \not\models_{\mathbf{C}} \varphi.$$

Indeed, $e(\varphi) = \langle 1, -1\rangle$, and compute by induction that $\langle 1, -1\rangle^n = \langle 1, -n\rangle$ and so $e(\neg\varphi \to \varphi^n) = \langle 1, -1\rangle \oplus \langle 1, -n\rangle = \langle 1, 0\rangle$.

Examples of logical filters – 6

For each $n \geq 2$, take the subalgebra $\boldsymbol{MV_n}$ of $[0,1]_{\text{Ł}}$ with the n-element domain $\{0, \frac{1}{n-1}, \ldots, 1\}$ and the matrix $\mathbf{Ł}_n = \langle \boldsymbol{MV_n}, \{1\} \rangle$.

$\models_{\mathbf{Ł}_n}$ is a finitary logic (by the lemma on tabular logics).

$\mathbf{Ł}_n \in \mathbf{MOD}(\text{Ł})$ (by the lemma on preimages of models).

$\mathbf{Ł}_n \in \mathbf{MOD}(\text{Ł}_\infty)$ (checking the semantical validity of the infinitary rule).

$\text{Ł}_\infty \subsetneq \models_{\{\mathbf{Ł}_n \mid n \geq 2\}}$. Consider the rule $\{(p_i \to p_{i+1})^i \to q \mid i > 0\} \rhd q$

Take $e(q) < 1$ and $(p_i \to p_{i+1})^i \to q = 1$ for each i. Then we must have $e((p_i \to p_{i+1})^i < 1$, i.e., $e(p_i) > e(p_{i+1})$. Thus there is an infinite decreasing chain in $\boldsymbol{MV_n}$, a contradiction!!
On the other hand, consider the $\mathbf{M}_{\text{Ł}}$-evaluation $e(p_i) = \frac{1}{2^i}$.

Examples of logical filters – 7

Exercise 3

The logic BCI: By $\boldsymbol{M}$ we denote the $\mathcal{L}_{\to}$-algebra with domain $\{\bot, \top, t, f\}$ and:

$\to^{\boldsymbol{M}}$	$\top$	t	f	$\bot$
$\top$	$\top$	$\bot$	$\bot$	$\bot$
t	$\top$	t	f	$\bot$
f	$\top$	$\bot$	t	$\bot$
$\bot$	$\top$	$\top$	$\top$	$\top$

Check that

$$\mathcal{F}i_{\mathrm{BCI}}(\boldsymbol{M}) = \{\{t, \top\}, \{t, f, \top\}, M\}.$$

The first completeness theorem

Proposition

For any logic L *in a language* $\mathcal{L}$, $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{Fm}_{\mathcal{L}}) = \mathrm{Th}(\mathrm{L})$.

Theorem

Let L *be a logic. Then for each set* Γ *of formulas and each formula* φ *the following holds:* $\Gamma \vdash_{\mathrm{L}} \varphi$ *iff* $\Gamma \models_{\mathbf{MOD}(\mathrm{L})} \varphi$.

Outline

Completeness theorem for classical logic

- Suppose that $T \in \mathrm{Th}(\mathrm{CL})$ and $\varphi \notin T$ ($T \nvdash_{\mathrm{CL}} \varphi$). We want to show that $T \not\models \varphi$ in some meaningful semantics.
- $T \not\models_{\langle \boldsymbol{Fm}_{\mathcal{L}}, T\rangle} \varphi$. 1st completeness theorem
- $\langle \alpha, \beta\rangle \in \Omega(T)$ iff $\alpha \leftrightarrow \beta \in T$ (congruence relation on $\boldsymbol{Fm}_{\mathcal{L}}$ compatible with T: if $\alpha \in T$ and $\langle \alpha, \beta\rangle \in \Omega(T)$, then $\beta \in T$).
- Lindenbaum–Tarski algebra: $\boldsymbol{Fm}_{\mathcal{L}}/\Omega(T)$ is a Boolean algebra and $T \not\models_{\langle \boldsymbol{Fm}_{\mathcal{L}}/\Omega(T), T/\Omega(T)\rangle} \varphi$.

  2nd completeness theorem
- Lindenbaum Lemma: If $\varphi \notin T$, then there is a maximal consistent $T' \in \mathrm{Th}(\mathrm{CL})$ such that $T \subseteq T'$ and $\varphi \notin T'$.
- $\boldsymbol{Fm}_{\mathcal{L}}/\Omega(T') \cong \boldsymbol{2}$ (subdirectly irreducible Boolean algebra) and $T \not\models_{\langle \boldsymbol{2}, \{1\}\rangle} \varphi$. 3rd completeness theorem

Weakly implicative logics

Definition

A logic L in a language $\mathcal{L}$ is weakly implicative if there is a binary connective $\to$ (primitive or definable) such that:

$$
\begin{array}{ll}
(\mathrm{R}) & \vdash_{\mathrm{L}} \varphi \to \varphi \\
(\mathrm{MP}) & \varphi, \varphi \to \psi \vdash_{\mathrm{L}} \psi \\
(\mathrm{T}) & \varphi \to \psi, \psi \to \chi \vdash_{\mathrm{L}} \varphi \to \chi \\
(\mathrm{sCng}) & \varphi \to \psi, \psi \to \varphi \vdash_{\mathrm{L}} c(\chi_1, \dots, \chi_i, \varphi, \dots, \chi_n) \to \\
& \qquad c(\chi_1, \dots, \chi_i, \psi, \dots, \chi_n)
\end{array}
$$

for each $\langle c, n\rangle \in \mathcal{L}$ and each $0 \leq i < n$.

Examples of (non-)weakly implicative logics – 1

- Min and AInc are not weakly implicative because they have no theorems (and hence no connective can satisfy the reflexivity requirement).
- Inc is weakly implicative (any binary connective works).
- Since prefixing is a theorem of BCI, in particular we obtain

  $$\varphi \to \psi, \psi \to \varphi \vdash_{\mathrm{BCI}} (\varphi \to \chi) \to (\psi \to \chi)$$
  $$\varphi \to \psi, \psi \to \varphi \vdash_{\mathrm{BCI}} (\chi \to \varphi) \to (\chi \to \psi)$$

  Thus all extensions of BCI are weakly implicative.

Examples of (non-)weakly implicative logics – 2

The axiomatic expansions of BCK we have seen are weakly implicative.
It is enough to show:

$$\varphi \to \psi, \psi \to \varphi \vdash \varphi \vee \chi \to \psi \vee \chi$$
$$\varphi \to \psi, \psi \to \varphi \vdash \varphi \wedge \chi \to \psi \wedge \chi$$
$$\varphi \to \psi, \psi \to \varphi \vdash \chi \vee \varphi \to \chi \vee \psi$$
$$\varphi \to \psi, \psi \to \varphi \vdash \chi \wedge \varphi \to \chi \wedge \psi$$

Observe that the equivalence connective $\equiv$ (defined as $(\varphi \to \psi) \wedge (\psi \to \varphi)$) is also a weak implication, though it differs substantially from $\to$ in logical behavior, for instance we have $\varphi \vdash \psi \to \varphi$ but not $\varphi \vdash \psi \equiv \varphi$.

Modal logics – 1

$\mathcal{L}_\Box$: $\mathcal{L}_{\mathrm{CL}}$ with an additional unary connective $\Box$.

$$
\begin{array}{ll}
(\mathrm{K}_\Box) & \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi) \\
(\mathrm{T}_\Box) & \Box\varphi \to \varphi \\
(4_\Box) & \Box\varphi \to \Box\Box\varphi \\
(\mathrm{Nec}_\Box) & \varphi \rhd \Box\varphi
\end{array}
$$

Global modal logics:

- K is the expansion of CL by $(\mathrm{K}_\Box)$ and $(\mathrm{Nec}_\Box)$.
- T: axiomatic extension of K by $(\mathrm{T}_\Box)$
- $\mathrm{K4}$: axiomatic extension of K by $(4_\Box)$
- $\mathrm{S4}$: axiomatic extension of T by $(4_\Box)$

Modal logics – 2

Local modal logics:

If L is a global modal logic, its local variant can be defined in two equivalent ways:

1. as the axiomatic expansion of CL by all the theorems of L,
2. by taking as axioms all the formulas $\Box \overset{n}{\ldots} \Box\varphi$ for each $n \geq 0$ and each axiom φ of L and *modus ponens* as the only inference rule.

Examples of (non-)weakly implicative logics – 3

- Global modal logics are weakly implicative (using the axiom $(\mathrm{K}_\Box)$ and the rule of necessitation).
- Local modal logics are not weakly implicative. Indeed, let L be any such logic and assume that $\overline{1} \to \varphi, \varphi \to \overline{1} \vdash_{\mathrm{L}} \Box\overline{1} \to \Box\varphi$. Since L expands CL, we know that

$$\vdash_{\mathrm{L}} \varphi \to \overline{1} \qquad \varphi \vdash_{\mathrm{L}} \overline{1} \to \varphi \qquad \vdash_{\mathrm{L}} \overline{1}.$$

  Thus also $\vdash_{\mathrm{L}} \Box\overline{1}$ and so $\varphi \vdash_{\mathrm{L}} \Box\varphi$, i.e., L is equal to its global variant, which is known not be the case.

Congruence Property – 1

Conventions

Unless said otherwise, L is a weakly implicative in a language $\mathcal{L}$ with an implication $\to$. We write:

- $\varphi \leftrightarrow \psi$ instead of $\{\varphi \to \psi, \psi \to \varphi\}$
- $\Gamma \vdash \Delta$ whenever $\Gamma \vdash \chi$ for each $\chi \in \Delta$
- $\Gamma \dashv\vdash \Delta$ whenever $\Gamma \vdash \Delta$ and $\Delta \vdash \Gamma$.

Theorem

Let φ, ψ, χ be formulas. Then:

- $\vdash_{\mathrm{L}} \varphi \leftrightarrow \varphi$
- $\varphi \leftrightarrow \psi \vdash_{\mathrm{L}} \psi \leftrightarrow \varphi$
- $\varphi \leftrightarrow \delta, \delta \leftrightarrow \psi \vdash_{\mathrm{L}} \varphi \leftrightarrow \psi$
- $\varphi \leftrightarrow \psi \vdash_{\mathrm{L}} \chi \leftrightarrow \hat{\chi}$, *where $\hat{\chi}$ is obtained from χ by replacing some occurrences of φ in χ by ψ.*

Congruence Property – 2

Corollary

Let $\to'$ *be a connective satisfying* (R), (MP), (T), (sCng). *Then*

$$\varphi \leftrightarrow \psi \dashv\vdash_{\mathrm{L}} \varphi \leftrightarrow' \psi.$$

Leibniz congruence – 1

Let us fix a weakly implicative logic L.

Definition

Let $\mathbf{A} = \langle \boldsymbol{A}, F \rangle$ be an L-matrix. We define:

- the matrix preorder $\leq_{\mathbf{A}}$ of $\mathbf{A}$ as

$$a \leq_{\mathbf{A}} b \qquad \text{iff} \qquad a \to^{\boldsymbol{A}} b \in F$$

- the Leibniz congruence $\Omega_{\boldsymbol{A}}(F)$ of $\mathbf{A}$ as

$$\langle a, b \rangle \in \Omega_{\boldsymbol{A}}(F) \qquad \text{iff} \qquad a \leq_{\mathbf{A}} b \text{ and } b \leq_{\mathbf{A}} a.$$

A congruence θ of $\boldsymbol{A}$ is logical in a matrix $\langle \boldsymbol{A}, F \rangle$ if for each $a, b \in A$ if $a \in F$ and $\langle a, b \rangle \in \theta$, then $b \in F$.

Leibniz congruence – 2

Theorem

Let $\mathbf{A} = \langle A, F\rangle$ *be an* L*-matrix. Then:*

1. $\leq_{\mathbf{A}}$ *is a preorder.*
2. $\Omega_{\boldsymbol{A}}(F)$ *is the largest logical congruence of* $\boldsymbol{A}$*.*
3. $\langle a, b\rangle \in \Omega_{\boldsymbol{A}}(F)$ *iff for each* $\chi \in Fm_{\mathcal{L}}$ *and each* $\boldsymbol{A}$*-evaluation* e*:*

$$e[p{\to}a](\chi) \in F \qquad \text{iff} \qquad e[p{\to}b](\chi) \in F.$$

Leibniz congruence – 2

Theorem

Let $\mathbf{A} = \langle A, F \rangle$ *be an* L*-matrix. Then:*

1. $\leq_{\mathbf{A}}$ *is a preorder.*
2. $\Omega_{\mathbf{A}}(F)$ *is the largest logical congruence of* $\mathbf{A}$*.*
3. $\langle a, b \rangle \in \Omega_{\mathbf{A}}(F)$ *iff for each* $\chi \in Fm_{\mathcal{L}}$ *and each* $\mathbf{A}$*-evaluation* e*:*

$$e[p \to a](\chi) \in F \qquad \textit{iff} \qquad e[p \to b](\chi) \in F.$$

Proof.

1. Take A-evaluation e such that $e(p) = a$, $e(q) = b$, and $e(r) = c$. Recall that in L we have: $\vdash_{\mathrm{L}} p \to p$ and $p \to q, q \to r \vdash_{\mathrm{L}} p \to r$. As $\mathbf{A} = \mathbf{MOD}(\mathrm{L})$ we have: $e(p \to p) \in F$, i.e., $a \leq_{\mathbf{A}} a$ and
if $e(p \to q), e(q \to r) \in F$, then $e(p \to r) \in F$ i.e., if $a \leq_{\mathbf{A}} b$ and $b \leq_{\mathbf{A}} c$, then $a \leq_{\mathbf{A}} c$.

Leibniz congruence – 2

Theorem

Let $\mathbf{A} = \langle A, F \rangle$ *be an* L*-matrix. Then:*

1. $\leq_{\mathbf{A}}$ *is a preorder.*
2. $\Omega_{\mathbf{A}}(F)$ *is the largest logical congruence of* $\mathbf{A}$*.*
3. $\langle a, b \rangle \in \Omega_{\mathbf{A}}(F)$ *iff for each* $\chi \in Fm_{\mathcal{L}}$ *and each* $\mathbf{A}$*-evaluation* e*:*

$$e[p \to a](\chi) \in F \qquad \textit{iff} \qquad e[p \to b](\chi) \in F.$$

Proof.

2. $\Omega_{\mathbf{A}}(F)$ is obviously an equivalence relation. It is a congruence due to (sCng) and logical due to (MP).
Take a logical congruence θ and $\langle a, b \rangle \in \theta$. Since $\langle a, a \rangle \in \theta$, we have $\langle a \to^{\mathbf{A}} a, a \to^{\mathbf{A}} b \rangle \in \theta$. As $a \to^{\mathbf{A}} a \in F$ and θ is logical we get $a \to^{\mathbf{A}} b \in F$, i.e., $a \leq_{\mathbf{A}} b$. The proof of $b \leq_{\mathbf{A}} a$ is analogous.

Leibniz congruence – 2

Theorem

Let $\mathbf{A} = \langle A, F\rangle$ *be an* L*-matrix. Then:*

1. $\leq_{\mathbf{A}}$ *is a preorder.*
2. $\Omega_{\boldsymbol{A}}(F)$ *is the largest logical congruence of* $\boldsymbol{A}$*.*
3. $\langle a, b\rangle \in \Omega_{\boldsymbol{A}}(F)$ *iff for each* $\chi \in Fm_{\mathcal{L}}$ *and each* $\boldsymbol{A}$*-evaluation* e*:*

$$e[p{\to}a](\chi) \in F \qquad \textit{iff} \qquad e[p{\to}b](\chi) \in F.$$

Proof.

3. One direction is a corollary of the congruence property and (MP). The converse one: set $\chi = p \to q$ and $e(q) = b$: then $a \to^{\boldsymbol{A}} b \in F$ iff $b \to^{\boldsymbol{A}} b \in F$, thus $a \leq_{\mathbf{A}} b$. The proof of $b \leq_{\mathbf{A}} a$ is analogous (using $e(q) = a$). □

Algebraic counterpart

Definition

An L-matrix $\mathbf{A} = \langle \boldsymbol{A}, F \rangle$ is reduced, $\mathbf{A} \in \mathbf{MOD}^*(\mathrm{L})$ in symbols, if $\Omega_{\boldsymbol{A}}(F)$ is the identity relation $\mathrm{Id}_{\boldsymbol{A}}$.

An algebra $\boldsymbol{A}$ is L-algebra, $\boldsymbol{A} \in \mathbf{ALG}^*(\mathrm{L})$ in symbols, if there is a set $F \subseteq A$ such that $\langle \boldsymbol{A}, F \rangle \in \mathbf{MOD}^*(\mathrm{L})$.

Note that $\Omega_{\boldsymbol{A}}(A) = A^2$. Thus from $\mathcal{F}i_{\mathrm{Inc}}(\boldsymbol{A}) = \{A\}$ we obtain:

$$\boldsymbol{A} \in \mathbf{ALG}^*(\mathrm{Inc}) \qquad \text{iff} \qquad A \text{ is a singleton}$$

Examples: classical logic CL and logic BCI

Exercise 4

Classical logic: prove that for any Boolean algebra $\boldsymbol{A}$:

$$\Omega_{\boldsymbol{A}}(\{1\}) = \mathrm{Id}_A \qquad \text{i.e., } \boldsymbol{A} \in \mathbf{ALG}^*(\mathrm{CL}).$$

On the other hand, show that:

$$\Omega_{\boldsymbol{4}}(\{a,1\}) = \mathrm{Id}_A \cup \{\langle 1, a\rangle, \langle 0, \neg a\rangle\} \text{ i.e. } \langle \boldsymbol{4}, \{a,1\}\rangle \notin \mathbf{MOD}^*(\mathrm{CL}).$$

BCI: recall the algebra $\boldsymbol{M}$ defined via:

$\to^{\boldsymbol{M}}$	$\top$	t	f	$\bot$
$\top$	$\top$	$\bot$	$\bot$	$\bot$
t	$\top$	t	f	$\bot$
f	$\top$	$\bot$	t	$\bot$
$\bot$	$\top$	$\top$	$\top$	$\top$

Show that:

$$\Omega_{\boldsymbol{M}}(\{t, \top\}) = \Omega_{\boldsymbol{M}}(\{t, f, \top\}) = \mathrm{Id}_M \qquad \text{i.e.} \quad \boldsymbol{M} \in \mathbf{ALG}^*(\mathrm{BCI}).$$

Factorizing matrices – 1

Let us take $\mathbf{A} = \langle \boldsymbol{A}, F \rangle \in \mathbf{MOD}(\mathrm{L})$. We write:

- $\boldsymbol{A}^*$ for $\boldsymbol{A}/\Omega_{\boldsymbol{A}}(F)$
- $[\cdot]_F$ for the canonical epimorphism of $\boldsymbol{A}$ onto $\boldsymbol{A}^*$ defined as:

$$[a]_F = \{b \in A \mid \langle a, b \rangle \in \Omega_{\boldsymbol{A}}(F)\}$$

- $\mathbf{A}^*$ for $\langle \boldsymbol{A}^*, [F]_F \rangle$.

Lemma

Let $\mathbf{A} = \langle \boldsymbol{A}, F \rangle \in \mathbf{MOD}(\mathrm{L})$ *and* $a, b \in A$. *Then:*

1. $a \in F$ *iff* $[a]_F \in [F]_F$.
2. $\mathbf{A}^* \in \mathbf{MOD}(\mathrm{L})$.
3. $[a]_F \leq_{\mathbf{A}^*} [b]_F$ *iff* $a \to^{\boldsymbol{A}} b \in F$.
4. $\mathbf{A}^* \in \mathbf{MOD}^*(\mathrm{L})$.

Factorizing matrices – 2

Proof.

1. One direction is trivial. Conversely: $[a]_F \in [F]_F$ implies that $[a]_F = [b]_F$ for some $b \in F$; thus $\langle a, b\rangle \in \Omega_{\boldsymbol{A}}(F)$ and, since $\Omega_{\boldsymbol{A}}(F)$ is a logical congruence, we obtain $a \in F$.
2. Recall that the second claim of Lemma 1.12 says that for a surjective $g\colon \boldsymbol{A} \to \boldsymbol{B}$ and $F \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ we get $g[F] \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{B})$, whenever $g(x) \in g[F]$ implies $x \in F$.
3. $[a]_F \leq_{\mathbf{A}^*} [b]_F$ iff $[a]_F \to^{\boldsymbol{A}^*} [b]_F \in [F]_F$ iff $[a \to^{\boldsymbol{A}} b]_F \in [F]_F$ iff $a \to^{\boldsymbol{A}} b \in F$.
4. Assume that $\langle [a]_F, [b]_F\rangle \in \Omega_{\boldsymbol{A}^*}([F]_F)$, i.e., $[a]_F \leq_{\mathbf{A}^*} [b]_F$ and $[b]_F \leq_{\mathbf{A}^*} [a]_F$. Therefore $a \to^{\boldsymbol{A}} b \in F$ and $b \to^{\boldsymbol{A}} a \in F$, i.e., $\langle a, b\rangle \in \Omega_{\boldsymbol{A}}(F)$. Thus $[a]_F = [b]_F$. □

Lindenbaum–Tarski matrix

Let L be a weakly implicative logic in $\mathcal{L}$ and $T \in Th(\mathrm{L})$. For every formula φ, we define the set

$$[\varphi]_T = \{\psi \in Fm_{\mathcal{L}} \mid \varphi \leftrightarrow \psi \subseteq T\}.$$

The Lindenbaum–Tarski matrix with respect to L and T, $\mathbf{LindT}_T$, has the filter $\{[\varphi]_T \mid \varphi \in T\}$ and algebraic reduct with the domain $\{[\varphi]_T \mid \varphi \in Fm_{\mathcal{L}}\}$ and operations:

$$c^{\mathbf{LindT}_T}([\varphi_1]_T, \dots, [\varphi_n]_T) = [c(\varphi_1, \dots, \varphi_n)]_T$$

Clearly, for every $T \in \mathrm{Th}(\mathrm{L})$ we have:

$$\mathbf{LindT}_T = \langle \boldsymbol{Fm}_{\mathcal{L}}, T\rangle^*.$$

The second completeness theorem

Theorem

Let L be a weakly implicative logic. Then for any set Γ of formulas and any formula φ the following holds:

$$\Gamma \vdash_{\mathrm{L}} \varphi \quad \textit{iff} \quad \Gamma \models_{\mathbf{MOD}^*(\mathrm{L})} \varphi.$$

The second completeness theorem

Theorem

Let L *be a weakly implicative logic. Then for any set* Γ *of formulas and any formula* φ *the following holds:*

$$\Gamma \vdash_{\mathrm{L}} \varphi \quad \textit{iff} \quad \Gamma \models_{\mathbf{MOD}^*(\mathrm{L})} \varphi.$$

Proof.

Using just the soundness part of the first completeness theorem it remains to prove:

$$\Gamma \models_{\mathbf{MOD}^*(\mathrm{L})} \varphi \quad \text{implies} \quad \Gamma \vdash_{\mathrm{L}} \varphi.$$

Take Lindenbaum–Tarski matrix $\mathbf{LindT}_{\mathrm{Th_L}(\Gamma)} = \langle \boldsymbol{Fm}_{\mathcal{L}}, \mathrm{Th_L}(\Gamma)\rangle^*$ and evaluation $e(\psi) = [\psi]_{\mathrm{Th_L}(\Gamma)}$. As clearly $e[\Gamma] \subseteq e[\mathrm{Th_L}(\Gamma)] = [\mathrm{Th_L}(\Gamma)]_{\mathrm{Th_L}(\Gamma)}$, then, as $\mathbf{LindT}_{\mathrm{Th_L}(\Gamma)}$ is an L-model, we have:
$e(\varphi) = [\varphi]_{\mathrm{Th_L}(\Gamma)} \in [\mathrm{Th_L}(\Gamma)]_{\mathrm{Th_L}(\Gamma)}$, and so $\varphi \in \mathrm{Th_L}(\Gamma)$ i.e., $\Gamma \vdash_{\mathrm{L}} \varphi$. □

Closure systems and closure operators – 1

Closure system over a set A: a collection of subsets $\mathcal{C} \subseteq \mathcal{P}(A)$ closed under arbitrary intersections and such that $A \in \mathcal{C}$. The elements of $\mathcal{C}$ are called closed sets.

Closure operator over a set A: a mapping $C\colon \mathcal{P}(A) \to \mathcal{P}(A)$ such that for every $X, Y \subseteq A$:

1. $X \subseteq C(X)$,
2. $C(X) = C(C(X))$, and
3. if $X \subseteq Y$, then $C(X) \subseteq C(Y)$.

Exercise 5

If C is a closure operator, $\{X \subseteq A \mid C(X) = X\}$ is a closure system.

If $\mathcal{C}$ is closure system, $C(X) = \bigcap\{Y \in \mathcal{C} \mid X \subseteq Y\}$ is a closure operator.

Closure systems and closure operators – 2

A closure operator C is finitary if for every $X \subseteq A$, $C(X) = \bigcup\{C(Y) \mid Y \subseteq X \text{ and } Y \text{ is finite}\}$.

A closure system $\mathcal{C}$ is called inductive if it is closed under unions of upwards directed families (i.e. families $\mathcal{D} \neq \emptyset$ such that for every $A, B \in \mathcal{D}$, there is $C \in \mathcal{D}$ such that $A \cup B \subseteq C$).

Theorem (Schmidt Theorem)

A closure operator C is finitary if, and only if, its associated closure system $\mathcal{C}$ is inductive.

Closure systems and closure operators – 3

Each logic L determines a closure system $\mathrm{Th}(\mathrm{L})$ and a closure operator Th_{L}.

Conversely, given a structural closure operator C over $Fm_{\mathcal{L}}$ (for every σ, if $\varphi \in C(\Gamma)$, then $\sigma(\varphi) \in C(\sigma[\Gamma])$), there is a logic L such that $C = \mathrm{Th}_{\mathrm{L}}$.

L is a finitary logic iff Th_{L} is a finitary closure operator.

The set of all L-filters over a given algebra $\boldsymbol{A}$, $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ is a closure system over A. Its associated closure operator is $\mathrm{Fi}_{\mathrm{L}}^{\boldsymbol{A}}$.

Transfer theorem for finitarity

Corollary

Given a logic L *in a language* $\mathcal{L}$*, the following conditions are equivalent:*

1. L *is finitary.*
2. $\mathrm{Fi}_{\mathrm{L}}^{\boldsymbol{A}}$ *is a finitary closure operator for any* $\mathcal{L}$*-algebra* $\boldsymbol{A}$*.*
3. $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ *is an inductive closure system for any* $\mathcal{L}$*-algebra* $\boldsymbol{A}$*.*

Closure systems and closure operators – 4

A base of a closure system $\mathcal{C}$ over A is any $\mathcal{B} \subseteq \mathcal{C}$ satisfying one of the following equivalent conditions:

1. $\mathcal{C}$ is the coarsest closure system containing $\mathcal{B}$.
2. For every $T \in \mathcal{C} \setminus \{A\}$, there is a $\mathcal{D} \subseteq \mathcal{B}$ such that $T = \bigcap \mathcal{D}$.
3. For every $T \in \mathcal{C} \setminus \{A\}$, $T = \bigcap \{B \in \mathcal{B} \mid T \subseteq B\}$.
4. For every $Y \in \mathcal{C}$ and $a \in A \setminus Y$ there is $Z \in \mathcal{B}$ such that $Y \subseteq Z$ and $a \notin Z$.

Exercise 6

Show that the four definitions are equivalent.

An element X of a closure system $\mathcal{C}$ over A is called (finitely) $\cap$-irreducible if for each (finite non-empty) set $\mathcal{Y} \subseteq \mathcal{C}$ such that $X = \bigcap_{Y \in \mathcal{Y}} Y$, there is $Y \in \mathcal{Y}$ such that $X = Y$.

Abstract Lindenbaum Lemma

An element X of a closure system $\mathcal{C}$ over A is called maximal w.r.t. an element a if it is a maximal element of the set $\{Y \in \mathcal{C} \mid a \notin Y\}$ w.r.t. the order given by inclusion.

Proposition

Let $\mathcal{C}$ be a closure system over a set A and $T \in \mathcal{C}$. Then, T is maximal w.r.t. an element if, and only if, T is $\cap$-irreducible.

Lemma

Let C be a finitary closure operator and $\mathcal{C}$ its corresponding closure system. If $T \in \mathcal{C}$ and $a \notin T$, then there is $T' \in \mathcal{C}$ such that $T \subseteq T'$ and T' is maximal with respect to a. $\cap$-irreducible closed sets form a base.

Operations on matrices – 1

$\langle \boldsymbol{A}, F\rangle$: first-order structure in the equality-free predicate language with function symbols from $\mathcal{L}$ and a unique unary predicate symbol interpreted by F.

Submatrix: $\langle \boldsymbol{A}, F\rangle \subseteq \langle \boldsymbol{B}, G\rangle$ if $\boldsymbol{A} \subseteq \boldsymbol{B}$ and $F = A \cap G$. Operator: $\mathbf{S}(\langle \boldsymbol{A}, F\rangle)$ is the class of all subalgebras of $\langle \boldsymbol{A}, F\rangle$.
Homomorphic image: $\langle \boldsymbol{B}, G\rangle$ is a homomorphic image of $\langle \boldsymbol{A}, F\rangle$ if it exists $h\colon \boldsymbol{A} \to \boldsymbol{B}$ homomorphism of algebras such that $h[F] \subseteq G$. Operator $\mathbf{H}$.
Strict homomorphic image: $\langle \boldsymbol{B}, G\rangle$ is a strict homomorphic image of $\langle \boldsymbol{A}, F\rangle$ if it exists $h\colon \boldsymbol{A} \to \boldsymbol{B}$ homomorphism of algebras such that $h[F] \subseteq G$ and $h[A \setminus F] \subseteq B \setminus G$. Operator $\mathbf{H}_{\mathrm{S}}$.
Isomorphic image: Image by a bijective strict homomorphism. Operator $\mathbf{I}$.

Operations on matrices – 2

Direct product: Given matrices $\{\langle \boldsymbol{A}_i, F_i\rangle \mid i \in I\}$, their direct product is $\langle \boldsymbol{A}, F\rangle$, where $\boldsymbol{A} = \prod_{i\in I} \boldsymbol{A}_i$, $f^{\boldsymbol{A}}(a_1, \ldots, a_n)(i) = f^{\boldsymbol{A}_i}(a_1(i), \ldots, a_n(i))$. $F = \prod_{i\in I} F_i$. $\pi_j : \boldsymbol{A} \twoheadrightarrow \boldsymbol{A}_j$. Operator $\mathbf{P}$.

Exercise 7

Let L be a weakly implicative logic. Then:

1. $\mathbf{SP}(\mathbf{MOD}(\mathrm{L})) \subseteq \mathbf{MOD}(\mathrm{L})$.
2. $\mathbf{SP}(\mathbf{MOD}^*(\mathrm{L})) \subseteq \mathbf{MOD}^*(\mathrm{L})$.

Subdirect products and subdirect irreducibility

A matrix $\mathbf{A}$ is said to be representable as a subdirect product of the family of matrices $\{\mathbf{A}_i \mid i \in I\}$ if there is an embedding homomorphism α from $\mathbf{A}$ into the direct product $\prod_{i\in I} \mathbf{A}_i$ such that for every $i \in I$, the composition of α with the i-th projection, $\pi_i \circ \alpha$, is a surjective homomorphism. In this case, α is called a subdirect representation, and it is called finite if I is finite.

Operator $\mathbf{P}_{\mathrm{SD}}(\mathbb{K})$.

A matrix $\mathbf{A} \in \mathbb{K}$ is (finitely) subdirectly irreducible relative to $\mathbb{K}$ if for every (finite non-empty) subdirect representation α of $\mathbf{A}$ with a family $\{\mathbf{A}_i \mid i \in I\} \subseteq \mathbb{K}$ there is $i \in I$ such that $\pi_i \circ \alpha$ is an isomorphism. The class of all (finitely) subdirectly irreducible matrices relative to $\mathbb{K}$ is denoted as $\mathbb{K}_{\mathrm{R(F)SI}}$.

$\mathbb{K}\mathrm{RSI} \subseteq \mathbb{K}\mathrm{RFSI}$.

Characterization of RSI and RFSI reduced models

Theorem

Given a weakly implicative logic L *and* $\mathbf{A} = \langle A, F \rangle \in \mathbf{MOD}^*(\mathrm{L})$, *we have:*

1. $\mathbf{A} \in \mathbf{MOD}^*(\mathrm{L})\mathrm{RSI}$ *iff* F *is* $\cap$*-irreducible in* $\mathcal{F}i_{\mathrm{L}}(A)$.
2. $\mathbf{A} \in \mathbf{MOD}^*(\mathrm{L})\mathrm{RFSI}$ *iff* F *is finitely* $\cap$*-irreducible in* $\mathcal{F}i_{\mathrm{L}}(A)$.

Subdirect representation

Theorem

If L *is a finitary weakly implicative logic, then*

$$\mathbf{MOD}^*(\mathrm{L}) = \mathbf{P}_{\mathrm{SD}}(\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}),$$

in particular every matrix in $\mathbf{MOD}^*(\mathrm{L})$ *is representable as a subdirect product of matrices in* $\mathbf{MOD}^*(\mathrm{L})\mathrm{RSI}$.

The third completeness theorem

Theorem

Let L *be a finitary weakly implicative logic. Then*

$$\vdash_{\mathrm{L}} \,=\, \models_{\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}}.$$

Leibniz operator

Leibniz operator: the function giving for each $F \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ the Leibniz congruence $\Omega_{\boldsymbol{A}}(F)$.

Proposition

Let L *be a weakly implicative logic* L *and* $\boldsymbol{A}$ *an* $\mathcal{L}$*-algebra. Then*

1. $\Omega_{\boldsymbol{A}}$ *is monotone: if* $F \subseteq G$ *then* $\Omega_{\boldsymbol{A}}(F) \subseteq \Omega_{\boldsymbol{A}}(G)$.
2. $\Omega_{\boldsymbol{A}}$ *commutes with inverse images by homomorphisms: for every* $\mathcal{L}$*-algebra* $\boldsymbol{B}$*, homomorphism* $h\colon \boldsymbol{A} \to \boldsymbol{B}$*, and* $F \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{B})$:

$$\Omega_{\boldsymbol{A}}(h^{-1}[F]) = h^{-1}[\Omega_{\boldsymbol{B}}(F)] = \{\langle a, b\rangle \mid \langle h(a), h(b)\rangle \in \Omega_{\boldsymbol{B}}(F)\}.$$

3. $\Omega_{\boldsymbol{A}}[\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})] = \boldsymbol{Con}_{\mathbf{ALG}^*(\mathrm{L})}(\boldsymbol{A})$.

$\boldsymbol{Con}_{\mathbf{ALG}^*(\mathrm{L})}(\boldsymbol{A})$ is the set ordered by inclusion of congruences of $\boldsymbol{A}$ giving a quotient in $\mathbf{ALG}^*(\mathrm{L})$.

Example

Recall that for the algebra $\boldsymbol{M} \in \mathbf{ALG}^*(\mathrm{BCI})$ defined via:

$\to^{\boldsymbol{M}}$	$\top$	t	f	$\bot$
$\top$	$\top$	$\bot$	$\bot$	$\bot$
t	$\top$	t	f	$\bot$
f	$\top$	$\bot$	t	$\bot$
$\bot$	$\top$	$\top$	$\top$	$\top$

we have

$$\Omega_{\boldsymbol{M}}(\{t, \top\}) = \Omega_{\boldsymbol{M}}(\{t, f, \top\}) = \mathrm{Id}_{\boldsymbol{M}}$$ i.e., $\Omega_{\boldsymbol{M}}$ is not injective

Outline

Non-associative residuated lattices [Galatos–Ono. APAL 2010]

A pointed residuated lattice-ordered groupoid with unit $\boldsymbol{A}$ is algebra of a type $\mathcal{L}_{\mathrm{SL}} = \{\&, \backslash, /, \wedge, \vee, \overline{0}, \overline{1}\}$:

- $\langle A, \wedge, \vee \rangle$ is a lattice
- $\langle A, \&, \overline{1} \rangle$ is a groupoid with unit $\overline{1}$
- for each $x, y, z \in A$:

$$x \,\&\, y \leq z \quad \text{IFF} \quad x \leq z \,/\, y \quad \text{IFF} \quad y \leq x \setminus z$$

For simplicity we will speak about SL-algebras

SL-algebras form a variety, we will denote it as $\mathbb{SL}$.

Notable examples

- FL-algebras = pointed residuated lattices = 'associative' SL-algberas
- Algebras of relations, where $\&$ is relational composition and

$$R \setminus S = (R \mathbin{\&} R^c)^c \qquad S \mathbin{/} R = (S^c \mathbin{\&} R)^c$$

- ℓ-groups, where $a \setminus b = a^{-1} \mathbin{\&} b$ and $b \mathbin{/} a = b \mathbin{\&} a^{-1}$
- Powersets of monoids, where

$$X \setminus Y = \{z \mid X \mathbin{\&} \{z\} \subseteq Y\} \qquad Y \mathbin{/} X = \{z \mid \{z\} \mathbin{\&} X \subseteq Y\}$$

- Ideals of a ring ...

Classes of residuated structures

Any quasivariety of $\mathbb{SL}$-algbebras with possible additional operators will be called a class of residuated structures

Classes of residuated structures

Any quasivariety of SL-algberas with possible additional operators will be called a class of residuated structures

- Subvarieties of $\mathbb{SL}$, where $\&$ is associative, commutative, idempotent, divisible, etc.
- Integral SL-algebras: those where $\overline{1}$ is a top element of A
- Semilinear classes (those generated by their linearly ordered members)
- Hájek's BL-algebras (associative, commutative, integral, divisible, semilinear SL-algebras)
- MV-algebras (BL-algebras where $(x \to \overline{0}) \to \overline{0} = x$)
- Boolean algebras (idempotent MV-algebras)

Plus any of these with additional operators ...

The logic of $\mathbb{SL}$-algebras

Theorem

The relation $\vdash_{\mathrm{SL}}$ *defined as:*

$$T \vdash_{\mathrm{SL}} \varphi \quad \textit{iff} \quad \{\psi \wedge \overline{1} \approx \overline{1} \mid \psi \in T\} \models_{\mathbb{SL}} \varphi \wedge \overline{1} \approx \overline{1}$$

is a logic.

The logic of $\mathbb{SL}$-algebras

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is a logic.

Axiomatization $\mathcal{SL}$ for SL [Galatos–Ono. APAL, 2010]

Axioms:

$$
\begin{array}{lll}
\varphi \wedge \psi \setminus \varphi & \varphi \wedge \psi \setminus \psi & (\chi \setminus \varphi) \wedge (\chi \setminus \psi) \setminus (\chi \setminus \varphi \wedge \psi) \\
\varphi \setminus \varphi \vee \psi & \psi \setminus \varphi \vee \psi & (\varphi \setminus \chi) \wedge (\psi \setminus \chi) \setminus (\varphi \vee \psi \setminus \chi) \\
\varphi \setminus ((\psi \,/\, \varphi) \setminus \psi) & \psi \setminus (\varphi \setminus \varphi \,\&\, \psi) & (\chi \,/\, \varphi) \wedge (\chi \,/\, \psi) \setminus (\chi \,/\, \varphi \vee \psi) \\
\overline{1} & \overline{1} \setminus (\varphi \setminus \varphi) & \varphi \setminus (\overline{1} \setminus \varphi)
\end{array}
$$

Rules:

$$
\begin{array}{ll}
\{\varphi, \varphi \setminus \psi\} \rhd \psi & \{\varphi\} \rhd (\varphi \setminus \psi) \setminus \psi \\
\{\varphi \setminus (\psi \setminus \chi)\} \rhd \psi \setminus (\chi \,/\, \varphi) & \{\psi \,/\, \varphi\} \rhd \varphi \setminus \psi \\
\{\varphi \setminus \psi\} \rhd (\psi \setminus \chi) \setminus (\varphi \setminus \chi) & \{\psi \setminus \chi\} \rhd (\varphi \setminus \psi) \setminus (\varphi \setminus \chi) \\
\{\varphi, \psi\} \rhd \varphi \wedge \psi & \{\psi \setminus (\varphi \setminus \chi)\} \rhd \varphi \,\&\, \psi \setminus \chi
\end{array}
$$

A formal definition of substructural logics

We write $\varphi \to \psi$ instead of $\varphi \setminus \psi$

$\varphi \leftrightarrow \psi$ instead of $(\varphi \to \psi) \wedge (\psi \to \varphi)$

Definition

A logic L in a language $\mathcal{L}$ is a substructural logic if

- $\mathcal{L} \supseteq \mathcal{L}_{\mathrm{SL}}$
- If $T \vdash_{\mathrm{SL}} \varphi$, then $T \vdash_{\mathrm{L}} \varphi$
- for each n, $i < n$, and each n-ary connective $c \in \mathcal{L} \setminus \mathcal{L}_{\mathrm{SL}}$ holds:

$$\varphi \leftrightarrow \psi \vdash_{\mathrm{L}} c(\chi_1, \dots \chi_i, \varphi, \dots, \chi_n) \leftrightarrow c(\chi_1, \dots \chi_i, \psi, \dots, \chi_n)$$

Note: the last condition can be proven for all connectives of $\mathcal{L}_{\mathrm{SL}}$

From substructural logics to classes of residuated structures

Theorem

Let L *be a substructural logic. We say that an* $\mathcal{L}$*-algebra* $\boldsymbol{A}$ *is an* L*-algebra, whenever*

1. *its* $\mathcal{L}_{\mathrm{SL}}$*-reduct is an* SL*-algebra and*
2. $T \vdash_{\mathrm{L}} \varphi$ *implies that* $\{\psi \geq \overline{1} \mid \psi \in T\} \models_{\boldsymbol{A}} \varphi \geq \overline{1}$

The class of all L*-algebras, denoted as* $\mathbb{Q}_{\mathrm{L}}$*, is a class of residuated structures and*

$$T \vdash_{\mathrm{L}} \varphi \quad \textit{iff} \quad \{\psi \geq \overline{1} \mid \psi \in T\} \models_{\mathbb{Q}_{\mathrm{L}}} \varphi \geq \overline{1}$$

From substructural logics to classes of residuated structures and back

Theorem

Let $\mathbb{Q}$ be a class of residuated structures of type $\mathcal{L} \supseteq \mathcal{L}_{\mathrm{SL}}$. Then the relation $\mathrm{L}_{\mathbb{Q}}$ defined as:

$$T \vdash_{\mathrm{L}_{\mathbb{Q}}} \varphi \quad \textit{iff} \quad \{\psi \geq \overline{1} \mid \psi \in T\} \models_{\mathbb{Q}} \varphi \geq \overline{1}$$

is a substructural logic. And

$$E \models_{\mathbb{Q}} \alpha \approx \beta \quad \textit{iff} \quad \{\varphi \leftrightarrow \psi \mid \varphi \approx \psi \in E\} \vdash_{\mathrm{L}_{\mathbb{Q}}} \alpha \leftrightarrow \beta$$

It gets even better

Theorem

The operators $\mathbb{Q}_\star$ and $\mathrm{L}_\star$ are dual-lattice isomorphisms between the lattice of substructural logics in language $\mathcal{L}$ and the lattice of subquasivarieties of SL-algebras with operators $\mathcal{L} \setminus \mathcal{L}_{\mathrm{SL}}$.

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$$\varphi \vdash_{\mathrm{L}} \varphi \wedge \overline{1} \leftrightarrow \overline{1} \qquad \varphi \wedge \overline{1} \leftrightarrow \overline{1} \vdash_{\mathrm{L}} \varphi$$

$$\varphi \approx \psi \models_{\mathbb{Q}} (\varphi \leftrightarrow \psi) \wedge \overline{1} \approx \overline{1} \qquad (\varphi \leftrightarrow \psi) \wedge \overline{1} \approx \overline{1} \models_{\mathbb{Q}} \varphi \approx \psi$$

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The operators $\mathbb{Q}_\star$ and $\mathrm{L}_\star$ are dual-lattice isomorphisms between the lattice of substructural logics in language $\mathcal{L}$ and the lattice of subquasivarieties of SL-algebras with operators $\mathcal{L} \setminus \mathcal{L}_{\mathrm{SL}}$.

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Note: all these results are just particularizations of known facts of Abstract Algebraic Logic (AAL)

Examples of substructural logics

- Ono's substructural logics including classical and intuitionistic logic
- expansions by additional connectives, e.g. (classical) modalities, exponentials in linear logic and Baaz's Delta in fuzzy logics
- the fragments of the logics above to languages containing implication, such as BCK, BCI, psBCK, BCC, hoop logics, etc.

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Special axioms:

usual name	**s**	axioms
associativity	a	$(\varphi \mathbin{\&} \psi) \mathbin{\&} \chi \leftrightarrow \varphi \mathbin{\&} (\psi \mathbin{\&} \chi)$
exchange	e	$\varphi \mathbin{\&} \psi \to \psi \mathbin{\&} \varphi$
contraction	c	$\varphi \to \varphi \mathbin{\&} \varphi$
weakening	w	$\varphi \mathbin{\&} \psi \to \psi$ and $\overline{0} \to \varphi$

Logic given by these axioms; let $X \subseteq \{\mathrm{e}, \mathrm{c}, \mathrm{w}\}$ we define logics

- SL_X axiomatized by adding axioms from X of those of SL
- FL_X axiomatized by adding associativity to SL_X

Proof by cases

For classical or intuitionistic logic we have:

$$\frac{\Gamma, \varphi \vdash_{\mathrm{L}} \chi \qquad\qquad \Gamma, \psi \vdash_{\mathrm{L}} \chi}{\Gamma \cup \{\varphi \vee \psi\} \vdash_{\mathrm{L}} \chi}$$

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But in $\mathrm{FL_e}$ it would entail $\varphi \vee \psi \vdash_{\mathrm{FL_e}} (\varphi \wedge \overline{1}) \vee (\psi \wedge \overline{1})$, i.e.,

$$(\varphi \vee \psi) \wedge \overline{1} \approx \overline{1} \models_{\mathbb{Q}_{\mathrm{FL_e}}} (\varphi \wedge \overline{1}) \vee (\psi \wedge \overline{1}) \approx \overline{1}$$

which can be easily refuted

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On the other hand we can show that:

$$\frac{\Gamma, \varphi \vdash_{\mathrm{FL_e}} \chi \qquad \qquad \Gamma, \psi \vdash_{\mathrm{FL_e}} \chi}{\Gamma \cup \{(\varphi \wedge \overline{1}) \vee (\psi \wedge \overline{1})\} \vdash_{\mathrm{FL_e}} \chi}$$

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Results in this section are from: Czelakowski. *Protoalgebraic Logic*, 2000 and P. Cintula, C. Noguera. The proof by cases property and its variants in structural consequence relations. *Studia Logica*, 2013.

Generalized disjunctions

Let $\nabla(p, q, \overrightarrow{r})$ be a set of formulas. We write

$$\varphi \nabla \psi = \bigcup \{\nabla(\varphi, \psi, \overrightarrow{\alpha}) \mid \overrightarrow{\alpha} \in \mathrm{Fm}_{\mathcal{L}}^{\leq\omega}\}.$$

Definition

∇ is a p-disjunction if:

(PD)	$\varphi \vdash_{\mathrm{L}} \varphi \nabla \psi$	and	$\psi \vdash_{\mathrm{L}} \varphi \nabla \psi$		
PCP	$\Gamma, \varphi \vdash_{\mathrm{L}} \chi$	and	$\Gamma, \psi \vdash_{\mathrm{L}} \chi$	implies	$\Gamma, \varphi \nabla \psi \vdash_{\mathrm{L}} \chi$

Definition

A logic L is a p-disjunctional if it has a p-disjunction.

We drop the prefix ‘p-’ if there are no parameters $\overrightarrow{r}$ in ∇

Separating examples

Example

- $\vee$ is a disjunction in $\mathrm{FL_{ew}}$
- $\vee$ is not a disjunction in $\mathrm{FL_{e}}$,

Separating examples

Example

- $\vee$ is a disjunction in $\mathrm{FL_{ew}}$
- $\vee$ is not a disjunction in $\mathrm{FL_e}$, but $(p \wedge \overline{1}) \vee (q \wedge \overline{1})$ is
- No single formula is a disjunction in $\mathrm{G}_{\to}$

Separating examples

Example

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- $\vee$ is not a disjunction in FL_{e}, but $(p \wedge \overline{1}) \vee (q \wedge \overline{1})$ is
- No single formula is a disjunction in $\mathrm{G}_{\rightarrow}$
  but the set $\{(p \rightarrow q) \rightarrow q, (q \rightarrow p) \rightarrow p\}$ is
- No finite set of formulas is a disjunction in K

Separating examples

Example

- $\vee$ is a disjunction in $\mathrm{FL}_{\mathrm{ew}}$
- $\vee$ is not a disjunction in FL_{e}, but $(p \wedge \overline{1}) \vee (q \wedge \overline{1})$ is
- No single formula is a disjunction in $\mathrm{G}_{\to}$
  but the set $\{(p \to q) \to q, (q \to p) \to p\}$ is
- No finite set of formulas is a disjunction in K
  but the set $\{\Box^n p \vee \Box^m q \mid n, m \geq 0\}$ is
- No set of formulas in two variables is a disjunction in $\mathrm{IPC}_{\to}$

Separating examples

Example

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- No set of formulas in two variables is a disjunction in $\mathrm{IPC}_{\to}$
  but the formula $(p \to r) \to ((q \to r) \to r)$ is a p-disjunction.

Separating examples

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- No set of formulas in two variables is a disjunction in $\mathrm{IPC}_{\to}$
  but the formula $(p \to r) \to ((q \to r) \to r)$ is a p-disjunction.

Conjecture: The logics SL and FL are not disjunctional;
later we show that they are p-disjunctional

A little detour to AAL 1: filters

Definition

Let L be a substructural logic in $\mathcal{L}$ and $\boldsymbol{A}$ be an $\mathcal{L}$-algebra. A set $F \subseteq A$ is called L-filter on $\boldsymbol{A}$ if:

$$T \vdash_{\mathrm{L}} \varphi \text{ implies that for each } \boldsymbol{A}\text{-evaluation } e \text{ if } e[T] \subseteq F \text{ then } e(\varphi) \in F$$

- If the $\mathcal{L}_{\mathrm{SL}}$-reduct of $\boldsymbol{A}$ is an SL-algebra then:

  $\boldsymbol{A}$ is an L-algebra IFF the set $[\overline{1}\rangle$ is an L-filter
- If $\boldsymbol{A}$ is an L-algebra, then $[\overline{1}\rangle = \{x \in A \mid \overline{1} \leq x\}$ is the least L-filter
- Filters on $\boldsymbol{A}$ form an algebraic closure system

  by $\mathrm{Fi}(X)$ we denote the filter generated by X
- Filters on $\boldsymbol{Fm}_{\mathcal{L}}$ are the closure system corresponding to L
- When seen as a lattice they are isomorphic to the lattice of

  $\mathbb{Q}_{\mathrm{L}}$-relative congruences on $\boldsymbol{A}$

Filters in p-disjunctional logics

Theorem

Let L *be a logic with a p-disjunction* ∇. *Then for each* $\mathcal{L}$*-algebra* $\boldsymbol{A}$ *and each* $X, Y \cup \{x, y\} \subseteq A$*:*

$$\mathrm{Fi}(X, x) \cap \mathrm{Fi}(X, y) = \mathrm{Fi}(X, x \nabla^{\boldsymbol{A}} y)$$

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Theorem

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Filters in p-disjunctional logics

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Let L *be a logic with a p-disjunction* ∇*. Then for each* $\mathcal{L}$*-algebra* $\boldsymbol{A}$ *and each* $X, Y \cup \{x, y\} \subseteq A$*:*

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Theorem

Let L *be a substructural logic. TFAE:*

1. L *is p-disjunctional*
2. *The lattice of all* L*-filters on any* $\mathcal{L}$*-algebra is distributive*
3. $\mathbb{Q}_{\mathrm{L}}$ *is relative-congruence-distributive*

Corollary

For each subvariety $\mathbb{V}$ *of* $\mathbb{SL}$*,* $\mathrm{L}_{\mathbb{V}}$ *is p-disjunctional logic*

A little detour to AAL 2: RFSI algebras

Let us by $\mathbb{Q}_{\mathrm{RFSI}}$ denote that class of $\mathbb{Q}$-relatively finitely subdirectly irreducible (RFSI) L-algebras. We know that:

$$T \vdash_{\mathrm{L}} \varphi \qquad \text{iff} \qquad \{\psi \geq \overline{1} \mid \psi \in T\} \models_{(\mathbb{Q}_{\mathrm{L}})_{\mathrm{RFSI}}} \varphi \geq \overline{1}$$

$\boldsymbol{A} \in (\mathbb{Q}_{\mathrm{L}})_{\mathrm{RFSI}}$ iff the the filter $[\overline{1}\rangle$ is finitely meet irreducible, i.e.,
there is no pair of filters $F, G \supset [\overline{1}\rangle$ s.t. $F \cap G = [\overline{1}\rangle$.

∇-prime filters

Definition

A filter F on $\boldsymbol{A}$ is ∇-prime if for every $a, b \in A$, $a \nabla^{\boldsymbol{A}} b \subseteq F$ implies $a \in F$ or $b \in F$.

Theorem

Let ∇ be a p-disjunction in L and $\boldsymbol{A}$ and L-algebra. Then $\boldsymbol{A} \in (\mathbb{Q}_{\mathrm{L}})_{\mathrm{RFSI}}$ iff the filter $[\overline{1}\rangle$ is ∇-prime.

Proof:

Assume that $\boldsymbol{A}$ is not RFSI: there are $F_i \supset [\overline{1}\rangle$ s.t. $[\overline{1}\rangle = F_1 \cap F_2$. Let $a_i \in F_i \setminus [\overline{1}\rangle$. Thus $a_1 \nabla a_2 \subseteq F_i$, i.e., $[\overline{1}\rangle$ is not ∇-prime

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Assume that $[\overline{1}\rangle$ is not ∇-prime: there are $x, y \not\geq \overline{1}$ s.t. $x \nabla y \subseteq [\overline{1}\rangle$. Then $\mathrm{Fi}(x), \mathrm{Fi}(y) \supset [1\rangle$ and:

$$[\overline{1}\rangle = \mathrm{Fi}(x \nabla y) = \mathrm{Fi}(x) \cap \mathrm{Fi}(y) \qquad \text{i.e., } \boldsymbol{A} \text{ is not RFSI.}$$

A little detour to AAL 3: simple observations

Let $\mathcal{AX}$ be an axiomatic system of a logic L, then F is an L filter iff it is an upset containing $\overline{1}$ and for each rule $T \rhd \varphi$ we have:

$$\text{for each } \boldsymbol{A}\text{-evaluation } e \text{ if } e[T] \subseteq F \text{ then } e(\varphi) \in F$$

$\mathrm{L} + \mathcal{A}$ is the extension of L by axioms from $\mathcal{A}$.

$\mathbb{Q}_{\mathrm{L}+\mathcal{A}}$ is a relative subvariety of $\mathbb{Q}_{\mathrm{L}}$ axiomatized by $\{\varphi \geq \overline{1} \mid \varphi \in \mathcal{A}\}$

Positive universal formulas

A *positive universal formula* is built from equations using conjunction and disjunction.

Lemma (Galatos. Studia Logica, 2004)

A positive universal formula C is equivalent the formula $\bigvee_{\varphi \in F_C} \overline{1} \leq \varphi$

Lemma

Let L be a logic, ∇ a p-disjunction, C a positive universal formula, and $\boldsymbol{A}$ an L-algebra.

- If $\boldsymbol{A} \models C$, then $e[\nabla_{\varphi \in F_C} \varphi] \geq \overline{1}$ for each $\boldsymbol{A}$-evaluation e.

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- If $\boldsymbol{A} \models C$, then $e[\nabla_{\varphi \in F_C} \varphi] \geq \overline{1}$ for each $\boldsymbol{A}$-evaluation e.
- Furthermore, if $[\overline{1}\rangle$ a ∇-prime, then the converse holds as well.

Logics given by positive universal classes of algebras

Theorem

Let L *be a logic with a p-disjunction* ∇ *and* $\mathcal{C}$ *a set of positive universal formulas. Then:*

$$\mathrm{L}_{\mathbf{Q}(\{\boldsymbol{A} \text{ an } \mathrm{L}\text{-algebra} \mid \boldsymbol{A} \models \mathcal{C}\})} = \mathrm{L} + \{\nabla_{\varphi \in F_C} \varphi \mid C \in \mathcal{C}\}.$$

Proof

We set $\mathrm{L}' = \mathrm{L} + \{\nabla_{\varphi \in F_C} \varphi \mid C \in \mathcal{C}\}$; $\mathbb{U} = \{\boldsymbol{A} \text{ an } \mathrm{L}\text{-algebra} \mid \boldsymbol{A} \models \mathcal{C}\}$.

Clearly $\mathbb{U} \subseteq \mathbb{Q}_{\mathrm{L}'}$, so $\mathbf{Q}(\mathbb{U}) \subseteq \mathbb{Q}_{\mathrm{L}'}$ and so $\mathrm{L}' \subseteq \mathrm{L}_{\mathbf{Q}(\mathbb{U})}$.

Conversely, assume that $T \nvdash_{\mathrm{L}'} \varphi$. There is an $\boldsymbol{A} \in (\mathbb{Q}_{\mathrm{L}'})_{\mathrm{RFSI}}$ where $[\overline{1}\rangle$ a ∇-prime (because L' is axiomatic extension of L and so ∇ is p-disjunction in L') and an $\boldsymbol{A}$-model of T s.t. $e(\varphi) \not\geq \overline{1}$.
Then $\boldsymbol{A} \in \mathbb{U}$ and so $T \nvdash_{\mathrm{L}_{\mathbf{Q}(\mathbb{U})}} \varphi$, i.e. $\mathrm{L}_{\mathbf{Q}(\mathbb{U})} \subseteq \mathrm{L}'$.

Quasivarieties given by positive universal classes of algebras

Corollary

Let L *be a logic with a p-disjunction* ∇. *The quasivariety generated by the class of* L*-algebras satisfying a set of positive universal formulas* $\mathcal{C}$ *is axiomatized (relative to* $\mathbb{Q}_{\mathrm{L}}$*) by:*

$$\{\varphi \geq \overline{1} \mid C \in \mathcal{C} \text{ and } \varphi \in \nabla_{\psi \in F_C} \psi\}$$

Note that the axiomatized quasivariety is relative subvariety.

Quasivarieties given by positive universal classes of algebras

Corollary

Let L be a logic with a p-disjunction ∇. The quasivariety generated by the class of L-algebras satisfying a set of positive universal formulas $\mathcal{C}$ is axiomatized (relative to $\mathbb{Q}_{\mathrm{L}}$) by:

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Note that the axiomatized quasivariety is relative subvariety.

A remark: this result can be generalized to Qvs generated by classes of RFSI L-algebras satisfying a set of disjunctions of quasiequations.

Intersection of relative subvarieties

Corollary

Let L *be a logic with a p-disjunction* ∇*. The join of two relative subvarieties* $\mathbb{Q}_{\mathrm{L}}$ *axiomatized (relative to* $\mathbb{Q}_{\mathrm{L}}$*) by* $\mathcal{E}_1$ *and* $\mathcal{E}_2$ *is axiomatized (relative to* $\mathbb{Q}_{\mathrm{L}}$*) by:*

$$\{\chi \geq \overline{1} \mid \varphi_1 \approx \psi_1 \in \mathcal{E}_1, \varphi_2 \approx \psi_2 \in \mathcal{E}_2, \text{ and } \chi \in (\varphi_1 \leftrightarrow \psi_1) \nabla (\varphi_2 \leftrightarrow \psi_2)\}$$

Note that it is the join both in the lattice of subquasivarieties and relative subvarieties

Proof

Assume that the set of variables of $\mathcal{E}_1$ and $\mathcal{E}_2$ are disjoint.
Then $\boldsymbol{A} \in \mathbb{Q}_1 \cup \mathbb{Q}_2$ iff $\boldsymbol{A} \models (\varphi_1 \approx \psi_1) \vee (\varphi_2 \approx \psi_2)$ for each $\varphi_1 \approx \psi_1 \in \mathcal{E}_1, \varphi_2 \approx \psi_2 \in \mathcal{E}_2$.
Now all we need is: $\mathbb{SL} \models (\varphi \approx \psi) \Leftrightarrow (\varphi \leftrightarrow \psi) \geq \overline{1}$

First, the simple case

Theorem (P. Cintula, C. Noguera. Studia Logica, 2013)

Let L *be a substructural logic with an axiomatic system having rules* Ru *and let* $\nabla(p, q, \overrightarrow{r})$ *be a set of formulas such that*

$$\varphi \vdash_{\mathrm{L}} \varphi \nabla \psi \qquad \psi \vdash_{\mathrm{L}} \varphi \nabla \psi \qquad \psi \nabla \varphi \vdash_{\mathrm{L}} \varphi \nabla \psi \qquad \varphi \nabla \varphi \vdash_{\mathrm{L}} \varphi$$

Then ∇ *is a p-disjunction in* L *iff for each* χ *and each* $T \rhd \varphi \in$ Ru*:*

$$\{\psi \nabla \chi \mid \psi \in T\} \vdash_{\mathrm{L}} \varphi \nabla \chi$$

Corollary

Let L_0 *be a substructural logic with a p-disjunction* ∇ *and let* $\mathcal{L}$ *be axiomatized by adding rules* Ru *to any axiomatic system of* L_0*.*
Then ∇ *is a p-disjunction in* L *iff for each* χ *and each* $T \rhd \varphi \in$ Ru*:*

$$\{\psi \nabla \chi \mid \psi \in T\} \vdash_{\mathrm{L}} \varphi \nabla \chi$$

Second, a bit more tricky

Let us consider the following rules:

(MP)	$\varphi, \varphi \to \psi \rhd \psi$	*modus ponens*
(Adj)	$\varphi \rhd \varphi \wedge \overline{1}$	unit adjunction
(PN)	$\varphi \rhd \lambda_\alpha(\varphi) \qquad \varphi \rhd \rho_\alpha(\varphi)$	product normality

where

- a left conjugate of φ is $\lambda_\alpha(\varphi) = (\alpha \setminus \varphi \,\&\, \alpha) \wedge \overline{1}$
- a right conjugate of φ is $\rho_\alpha(\varphi) = (\alpha \,\&\, \varphi \,/\, \alpha) \wedge \overline{1}$

Theorem (Folklore)

Logic	The only rules needed in its axiomatization
$\mathrm{FL_{ew}}$	modus ponens
$\mathrm{FL_e}$	modus ponens *and unit adjunction*
FL	modus ponens *and product normality*

What about SL?

We need more conjugates:

$$\alpha_{\delta,\varepsilon}(\varphi) = (\delta \,\&\, \varepsilon \setminus \delta \,\&\, (\varepsilon \,\&\, \varphi))$$
$$\alpha'_{\delta,\varepsilon}(\varphi) = (\delta \,\&\, \varepsilon \setminus (\delta \,\&\, \varphi) \,\&\, \varepsilon)$$
$$\beta_{\delta,\varepsilon}(\varphi) = (\delta \setminus (\varepsilon \setminus (\varepsilon \,\&\, \delta) \,\&\, \varphi)$$
$$\beta'_{\delta,\varepsilon}(\varphi) = (\delta \setminus ((\delta \,\&\, \varepsilon) \,\&\, \varphi \,/\, \varepsilon)$$

And rules of the form:

$$\varphi \rhd \eta_{\delta,\varepsilon}(\varphi)$$

for $\eta \in \{\alpha, \alpha', \beta, \beta'\}$

What about SL?

We need more conjugates:

$$\begin{aligned}
\alpha_{\delta,\varepsilon}(\varphi) &= (\delta \mathbin{\&} \varepsilon \setminus \delta \mathbin{\&} (\varepsilon \mathbin{\&} \varphi)) \\
\alpha'_{\delta,\varepsilon}(\varphi) &= (\delta \mathbin{\&} \varepsilon \setminus (\delta \mathbin{\&} \varphi) \mathbin{\&} \varepsilon) \\
\beta_{\delta,\varepsilon}(\varphi) &= (\delta \setminus (\varepsilon \setminus (\varepsilon \mathbin{\&} \delta) \mathbin{\&} \varphi) \\
\beta'_{\delta,\varepsilon}(\varphi) &= (\delta \setminus ((\delta \mathbin{\&} \varepsilon) \mathbin{\&} \varphi \,/\, \varepsilon)
\end{aligned}$$

And rules of the form:

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for $\eta \in \{\alpha, \alpha', \beta, \beta'\}$

For the proof see: P. Cintula, R. Horčík, C. Noguera. Non-associative substructural logics and their semilinear extensions: Axiomatization and completeness properties. The Review of Symbolic Logic, 2013

Conventions

Let us consider a new propositional variable $\star$

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We write $\delta(\varphi)$ for a formula resulting from δ by replacing all $\star$ by φ.

Definition (Iterated Γ-formulas)

Let Γ be a set of $\star$-formulas. We define the sets of $\star$-formulas Γ^* as the smallest set s.t. :

- $\star \in \Gamma^*$,
- $\delta(\chi) \in \Gamma^*$ for each $\delta \in \Gamma$ and each $\chi \in \Gamma^*$.

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The rest of this section is based on P. Cintula, R. Horčík, C. Noguera. RSL, 2013

Main definition

Definition

L is almost (MP)-based w.r.t. a set of basic deduction terms bDT if it has an axiomatic system where

- there are no rules with three or more premises
- there is only one rule with two premises: *modus ponens*
- the remaining rules are from $\{\varphi \vdash \chi(\varphi) \mid \varphi \in Fm_{\mathcal{L}},\ \chi \in \mathrm{bDT}\}$
- for each $\beta \in \mathrm{bDT}$ there is $\beta' \in \mathrm{bDT}^*$ s.t.:

$$\vdash_{\mathrm{L}} \beta'(\varphi \to \psi) \to (\beta(\varphi) \to \beta(\psi)).$$

Almost-Implicational Deduction Theorem

Definition (Conjuncted Γ-formulas)

Let Γ be a set of $\star$-formulas. We define the sets of $\star$-formulas $\Pi(\Gamma)$ as the smallest set containing $\Gamma \cup \{\bar{1}\}$ and closed under $\&$.

Theorem

Let L *be almost* (MP)*-based w.r.t. a set of basic deductive terms* bDT*. Then for each set* $\Gamma \cup \{\varphi, \psi\}$ *of formulas:*

$$\Gamma, \varphi \vdash_{\mathrm{L}} \psi \qquad \textit{iff} \qquad \Gamma \vdash_{\mathrm{L}} \delta(\varphi) \to \psi \textit{ for some } \delta \in \Pi(\mathrm{bDT}^{*}).$$

Filter generation

Theorem

Let L *be almost* (MP)*-based w.r.t. a set of basic deductive terms* bDT*. Then for each set* $\Gamma \cup \{\varphi, \psi\}$ *of formulas:*

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Filter generation

Theorem

Let L *be almost* (MP)*-based w.r.t. a set of basic deductive terms* bDT*. Let* $\boldsymbol{A}$ *be an* $\mathcal{L}$*-algebra and* $X \cup \{x\} \subseteq A$*. Then*

$$y \in \mathrm{Fi}_{\mathrm{L}}^{\boldsymbol{A}}(X, x) \quad \textit{iff} \quad z \to y \in \mathrm{Fi}_{\mathrm{L}}^{\boldsymbol{A}}(X) \text{ for some } z \in (\Pi(\mathrm{bDT}^*))^{\boldsymbol{A}}(x).$$

$$\Gamma^{\boldsymbol{A}}(x) = \{\delta(x, a_1, \ldots, a_n) \mid \delta(\star, p_1, \ldots, p_n) \in \Gamma \text{ and } a_1, \ldots, a_n \in A\}$$

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$$\Gamma^{\boldsymbol{A}}(X) = \Pi(\bigcup\{\Gamma^{\boldsymbol{A}}(x) \mid x \in X\})$$

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$$\Gamma^{\boldsymbol{A}}(X) = \Pi(\bigcup\{\Gamma^{\boldsymbol{A}}(x) \mid x \in X\})$$

Corollary

Let L *be almost* (MP)*-based w.r.t. a set of basic deductive terms* bDT*. Let* $\boldsymbol{A}$ *be an* L*-algebra and* $X \subseteq A$*. Then*

$$\mathrm{Fi}_{\mathrm{L}}^{\boldsymbol{A}}(X) = \{a \in A \mid a \geq y \textit{ for some } y \in (\Pi(\mathrm{bDT}^*))^{\boldsymbol{A}}(X)\}$$

Disjunction in almost (MP)-based logics

Theorem

Let L *be almost* (MP)*-based w.r.t. a set of basic deductive terms* bDT*. Then*

$$\nabla_{\mathrm{L}} = \{\alpha(p) \vee \beta(q) \mid \alpha, \beta \in (\mathrm{bDT} \cup \{\star \wedge \overline{1}\})^{*}\}$$

is a (p-)disjunction in L*.*

Semilinear logics

Let us by $\mathbb{Q}^{\ell}_{\mathrm{L}}$ denote the class of linearly ordered L-algebras.

Definition

A substructural logic L is called semilinear if

$$T \vdash_{\mathrm{L}} \varphi \quad \text{iff} \quad \{\psi \geq \overline{1} \mid \psi \in T\} \models_{\mathbb{Q}^{\ell}_{\mathrm{L}}} \varphi \geq \overline{1}$$

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This section is based on P. Cintula, R. Horčík, C. Noguera. RSL, 2013

Note: some of the results hold in much wider setting.

Characterizations of substructural semilinear logics

Theorem

Let L be a substructural logic. TFAE:

1. L *is semilinear*
2. $\mathbb{Q}_{\mathrm{L}} = \mathbf{Q}(\mathbb{Q}^{\ell}_{\mathrm{L}})$
3. $\mathbb{Q}^{\ell}_{\mathrm{L}} = (\mathbb{Q}_{\mathrm{L}})_{\mathrm{RFSI}}$
4. *Each* L*-algebra is a subdirect product of* L*-chains*
5. *Any* L*-filter in an* $\mathcal{L}$*-algebra is an intersection of linear ones*

   a filter F is *linear* if $x \to y \in F$ or $y \to x \in F$, for each x, y
6. *The following metarule holds:*

$$\frac{T, \varphi \to \psi \vdash_{\mathrm{L}} \chi \qquad T, \psi \to \varphi \vdash_{\mathrm{L}} \chi}{T \vdash_{\mathrm{L}} \chi}$$

Characterizations of substructural semilinear logics

Theorem

Let L *be a substructural logic and an axiomatic system* $\mathcal{AX}$. *TFAE:*

1. L *is semilinear,*
2. L *proves* $(\varphi \to \psi) \vee (\psi \to \varphi)$ *and enjoys the metarule:*
$$\frac{T, \varphi \vdash_{\mathrm{L}} \chi \qquad T, \psi \vdash_{\mathrm{L}} \chi}{T, \varphi \vee \psi \vdash_{\mathrm{L}} \chi}$$
3. L *proves* $(\varphi \to \psi) \vee (\psi \to \varphi)$ *and any* L*-filter in an* $\mathcal{L}$*-algebra is an intersection of* $\vee$*-prime ones,*
4. L *proves* $(\varphi \to \psi) \vee (\psi \to \varphi)$ *and for every rule* $T \rhd \varphi$ *in* $\mathcal{AX}$ *and propositional variable* p *not occurring in* T, φ *we have*
$$\{\psi \vee \chi \mid \psi \in T\} \vdash_{\mathrm{L}} \varphi \vee \chi$$

Weakest semilinear extension

Theorem

- *There is the least semilinear logic extending* L*, denoted as* L^{ℓ}
- $\mathrm{L}^{\ell} = \mathrm{L}_{\mathbf{Q}(\mathbb{Q}^{\ell}_{\mathrm{L}})}$
- *If* L *is almost* (MP)*-based with* bDT*, then* L^{ℓ} *is axiomatized by adding axioms:*

$$((\varphi \to \psi) \wedge \overline{1}) \vee \delta((\psi \to \varphi) \wedge \overline{1}), \text{ for each } \delta \in \mathrm{bDT} \cup \{\star\}$$

Corollary

Let $\mathbb{Q}$ *be a class of residuated structures s.t.* $\mathrm{L}_{\mathbb{Q}}$ *is an almost* (MP)*-based with* bDT*. Then* $\mathbf{Q}(\{\boldsymbol{A} \in \mathbb{Q} \mid \boldsymbol{A} \text{ linear}\})$ *is a relative subvariety of* $\mathbb{Q}$ *axiomatized (relative to* $\mathbb{Q}$*) by*

$$((\varphi \to \psi) \wedge \overline{1}) \vee \delta((\psi \to \varphi) \wedge \overline{1}) \approx \overline{1}, \text{ for each } \delta \in \mathrm{bDT} \cup \{\star\}$$

Characterizations of completeness properties

Let L be substructural semilinear logic and $\mathbb{K}$ a class of L-chains.

Theorem (Characterization of strong $\mathbb{K}$-completeness)

1. *For each $T \cup \{\varphi\}$ holds: $T \vdash_{\mathrm{L}} \varphi$ iff $\{\psi \geq \overline{1} \mid \psi \in T\} \models_{\mathbb{K}} \varphi \geq \overline{1}$.*
2. $\mathbb{Q}_{\mathrm{L}} = \mathbf{ISP}_{\sigma\text{-}f}(\mathbb{K})$.
3. *Each countable L-chain is embeddable into some member of $\mathbb{K}$.*

Theorem (Characterization of finite strong $\mathbb{K}$-completeness)

1. *For each finite $T \cup \{\varphi\}$ holds: $T \vdash_{\mathrm{L}} \varphi$ iff $\{\psi \geq \overline{1} \mid \psi \in T\} \models_{\mathbb{K}} \varphi \geq \overline{1}$.*
2. $\mathbb{Q}_{\mathrm{L}} = \mathbf{Q}(\mathbb{K})$, *i.e., $\mathbb{K}$ generates $\mathbb{Q}_{\mathrm{L}}$ as a quasivariety.*
3. *Each finite subset of any L-chain is partially embeddable into an element of $\mathbb{K}$.*

Finite chain semantics

Let $\mathcal{F}$ be a class of finite chains

Theorem (Characterization of strong finite-chain completeness)

1. L *enjoys the* $\mathrm{S}\mathcal{F}\mathrm{C}$*,*
2. *All* L*-chains are finite,*
3. *There exists* $n \in$ N *such each* L*-chain has at most* n *elements,*
4. *There exists* $n \in$ N *such that* $\emptyset \vdash_{\mathrm{L}} \bigvee_{i<n}(x_i \to x_{i+1})$*.*

Known results: $\mathrm{FS}\mathcal{F}\mathrm{C}$ fails in FL^{ℓ} and $\mathrm{FL}^{\ell}_{\mathrm{e}}$
$\mathrm{FS}\mathcal{F}\mathrm{C}$ holds in $\mathrm{FL}^{\ell}_{X\cup\{\mathrm{w}\}}$ and SL^{ℓ}_{X}

Open problems: $\mathrm{FS}\mathcal{F}\mathrm{C}$ of $\mathrm{FL}^{\ell}_{\mathrm{c}}$ and $\mathrm{FL}^{\ell}_{\mathrm{ec}}$

Standard completeness

Let $\mathcal{R}$ be a class of chains with domain ((half)-open) real unit interval with usual lattice order

Known results: $\mathrm{FS}\mathcal{R}\mathrm{C}$ fails in FL^{ℓ} and $\mathrm{FL}^{\ell}_{\mathrm{c}}$
$\mathrm{S}\mathcal{R}\mathrm{C}$ holds in $\mathrm{FL}^{\ell}_{\mathrm{e}}$, $\mathrm{FL}^{\ell}_{\mathrm{w}}$, $\mathrm{FL}^{\ell}_{\mathrm{ew}}$, $\mathrm{FL}^{\ell}_{\mathrm{w,c}}$ and SL^{ℓ}_{X}
$\mathrm{S}\mathcal{R}\mathrm{C}$ fails but $\mathrm{FS}\mathcal{R}\mathrm{C}$ holds in logic of BL- and MV-alg.

Open problems: (F)S$\mathcal{R}$C of $\mathrm{FL}^{\ell}_{\mathrm{ec}}$

Implication gives a nice bridge between logic and algebra . . .

Wanna know more?

Forthcoming book:
P. Cintula, C. Noguera. *Logic and Implication: An introduction to the general algebraic study of non-classical logics*, Trends in Logic, Springer.