Clones and Galois connections

Reinhard Pöschel

Institut für Algebra
Technische Universität Dresden

The 56th Summer School on Algebra and Ordered Sets
Špindlerův Mlýn, September 2 - 7, 2018

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(with 3 memorial intermissions)

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Outline

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(1811, Oct. 25 – 1832, May 31)

from Wikipedia:
Évariste Galois was a French mathematician. While still in his teens, he was able to determine a necessary and sufficient condition for a polynomial to be solvable by radicals, thereby solving a problem standing for 350 years. His work laid the foundations for Galois theory and group theory, two major branches of abstract algebra, and the subfield of Galois connections. He died at age 20 from wounds suffered in a duel.

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Galois connections

A *Galois connection* between two posets, here power set lattices $(\mathfrak{P}(M_1), \subseteq)$ and $(\mathfrak{P}(M_2), \subseteq)$, is a pair of mappings

$$\varphi : \mathfrak{P}(M_1) \to \mathfrak{P}(M_2) \text{ and } \psi : \mathfrak{P}(M_2) \to \mathfrak{P}(M_1)$$

satisfying (for $X, X_1, X_2 \in \mathfrak{P}(M_1)$, $Y, Y_1, Y_2 \in \mathfrak{P}(M_2)$):

(I) $X_1 \subseteq X_2 \implies \varphi X_1 \supseteq \varphi X_2$ (II) $Y_1 \subseteq Y_2 \implies \psi Y_1 \supseteq \psi Y_2$

(III) $X \subseteq \psi(\varphi X)$ (IV) $Y \subseteq \varphi(\psi Y)$

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(VII) $h_1 : X \mapsto \psi(\varphi X)$ is closure operator

(VIII) $h_2 : Y \mapsto \varphi(\psi Y)$ is closure operator

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(X) φ and ψ are order-reversing bijections between the Galois closures (= closures of h_1 and h_2)

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Galois connections via binary relations

Each Galois connection (φ, ψ) between $\mathfrak{P}(M_1)$ and $\mathfrak{P}(M_2)$ is uniquely determined by a binary relation

$$R \subseteq M_1 \times M_2$$

via

$$\varphi : \mathfrak{P}(M_1) \to \mathfrak{P}(M_2) : \varphi(X) := \{b \in M_2 \mid \forall a \in X : aRb\},$$
$$\psi : \mathfrak{P}(M_2) \to \mathfrak{P}(M_1) : \psi(Y) := \{a \in M_1 \mid \forall b \in Y : aRb\}.$$

We have:

$$aRb \iff b \in \varphi(\{a\}) \iff a \in \psi(\{b\}).$$

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"General Example": Formal concepts

In Formal Concept Analysis (*Rudolf Wille* ($\sim$ 1970)),

$$(G, M, I) := (M_1, M_2, R)$$

is called *formal context*.
($g \in G$ *objects* (Gegenstände), $m \in M$ *attributes* (Merkmale))
$I \subseteq G \times M$ *incidence relation*: $gIm :\iff$ object g has attribute m

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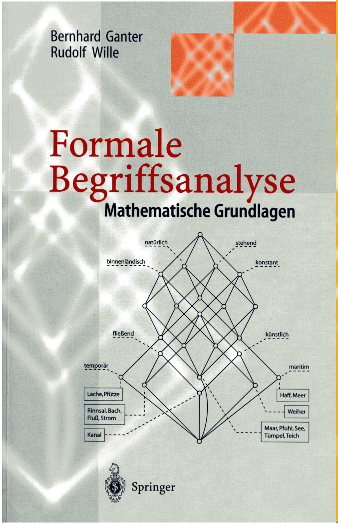

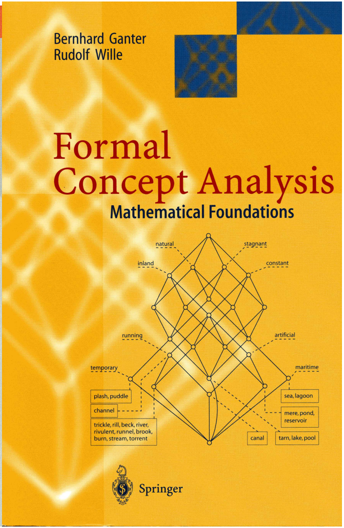

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Concepts (X, Y) (defined by the property $Y = X'$ and $X = Y'$)
have two components (*Galois closures*)

extent X and *intent* Y (dyadic view)

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The main problem for a Galois connection

The **main problem** for each Galois connection is the (internal)

Characterization of the Galois closures

i.e., sets satisfying $\psi(\varphi(X)) = X$ or $\varphi(\psi(Y)) = Y$

or, equivalently,

sets of the form $\psi(Y)$ and $\varphi(X)$ ($X \subseteq M_1$, $Y \subseteq M_2$).

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Example: The classical Galois connection

Let $E : K$ be a *field extension* (i.e., K is a subfield of a field E).

The classical Galois connection (in modern form) is given by:

$$M_1 := E,$$

$$M_2 := \mathrm{Aut}(E : K) := \{\sigma \in \mathrm{Aut}(E) \mid \forall a \in K : \sigma(a) = a\}$$

(the *Galois group* of $E : K$),

$aR\sigma :\iff \sigma(a) = a$ (fixed point).

Notation: $\mathrm{Aut}(E : L) := \varphi(L)$ for $L \subseteq E$,

$\mathrm{Fix}(E; U) := \psi(U)$ for $U \subseteq \mathrm{Aut}(E : K)$.

Theorem (Characterization of the Galois closures)

$L = \mathrm{Fix}(E; \mathrm{Aut}(E : L)) \iff K \leq L \leq E$ *(intermediate fields)*,

$U = \mathrm{Aut}(E : \mathrm{Fix}(E; U)) \iff U \leq \mathrm{Aut}(E : K)$

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($E : K$ *Galois extension* $\iff \exists$ finite $G \leq \mathrm{Aut}(E) : K = \mathrm{Fix}(E; G)$)

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Another well-known example

Galois connection given by $(M_1, M_2, \models)$ with

$M_1 := \mathrm{Alg}(\Omega)$ (*algebras* $\mathbf{A}$ of fixed type Ω),
$M_2 := T_\Omega(X) \times T_\Omega(X)$ (*identities* $s \approx t$, i.e., pairs of terms),
$\mathbf{A} \models s \approx t$ (algebra $\mathbf{A}$ satisfies an identity $s \approx t$).
Notation:
$\mathrm{Id}(\mathcal{K}) := \varphi(\mathcal{K})$ *identities* for a set $\mathcal{K}$ of algebras,
$\mathrm{Mod}(\Sigma) := \psi(\Sigma)$ *models* for a set Σ of identities.

Theorem (Characterization of the Galois closures)

Equational classes $\mathrm{Mod}\,\mathrm{Id}\,\mathcal{K} = \mathbf{HSP}\mathcal{K}$ *varieties*

Equational theories $\mathrm{Id}\,\mathrm{Mod}\,\Sigma = \langle\Sigma\rangle_{\textit{rules of equational logic}}$

fully invariant congruences of $T_\Omega(X)$.

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Galois connections: antitone vs. monotone

A monotone Galois connection between $(P_1, \leq)$ and $(P_2, \leq)$ is the same as a Galois connection between $(P_1, \leq)$ and $(P_2, \geq)$.

(φ^* upper adjoint, ψ^* lower adjoint)

Definition via a binary relation $R \subseteq M_1 \times M_2$:

$\varphi(X) := \{b \in M_2 \mid \forall a \in X : (a, b) \in R\}$, $\psi(Y) := \{a \in M_1 \mid \forall b \in Y : (a, b) \in R\}$,

$\varphi^*(X) := M_2 \setminus \{b \in M_2 \mid \forall a \in X : (a, b) \in R\} = \{b \mid \exists a \in X : (a, b) \notin R\}$,

Galois connections: antitone vs. monotone

Recall:

A pair (φ, ψ), $\varphi : \mathfrak{P}(M_1) \to \mathfrak{P}(M_2)$, $\psi : \mathfrak{P}(M_2) \to \mathfrak{P}(M_1)$ is a *Galois connection*,

iff $\forall\, X \subseteq M_1,\ Y \subseteq M_2$:

$$Y \subseteq \varphi(X) \iff \psi(Y) \supseteq X$$

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Outline

Galois connections

The “most basic Galois connection” in algebra

The clone lattice

Modifications of the Galois connection Pol – Inv

Some applications of THE Galois connection Pol – Inv

The "most basic Galois connection" in algebra

ALGEBRAS, LATTICES, VARIETIES VOLUME I

(1987)

Ralph N. McKenzie
University of California, Berkeley
George F. McNulty
University of South Carolina
Walter F. Taylor
University of Colorado

The most basic Galois connection in algebra is the one associated to the binary relation of preservation between operations and relations. (Nearly all of the most basic concepts in algebra can be defined in terms of this relation.)

Observe that the automorphisms, endomorphisms, subuniverses, and congruences of an algebra are defined by restricting the preservation relation to special types of relations. The congruences of an algebra, for example, are the equivalence relations that are preserved by the basic operations of the algebra.

4.1 Algebras and Clones p. 147

Notation

Notation f preserves ϱ:

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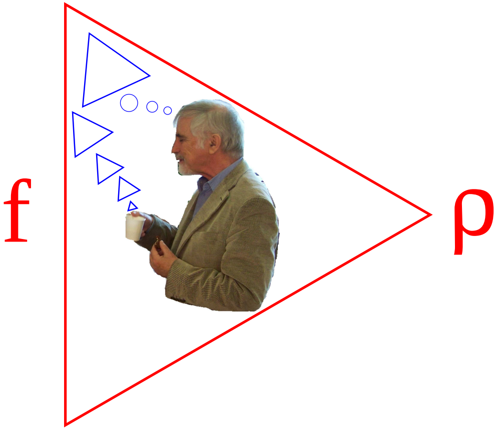

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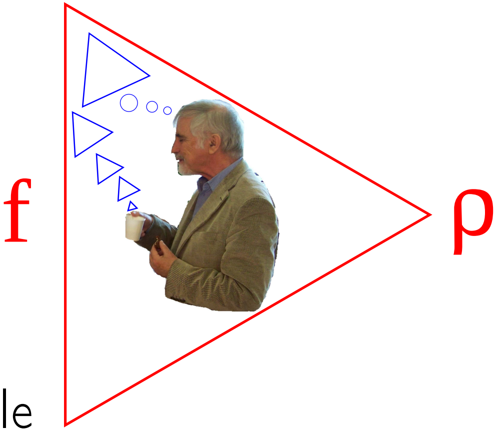

TECHNISCHE UNIVERSITÄT DRESDEN

The first memorial intermission

Rudolf Wille (1937–2017)

Rudolf Wille (2.11.1937 – 22.1.2017)

Memorial AAA86 in Darmstadt, June 1–3, 2018

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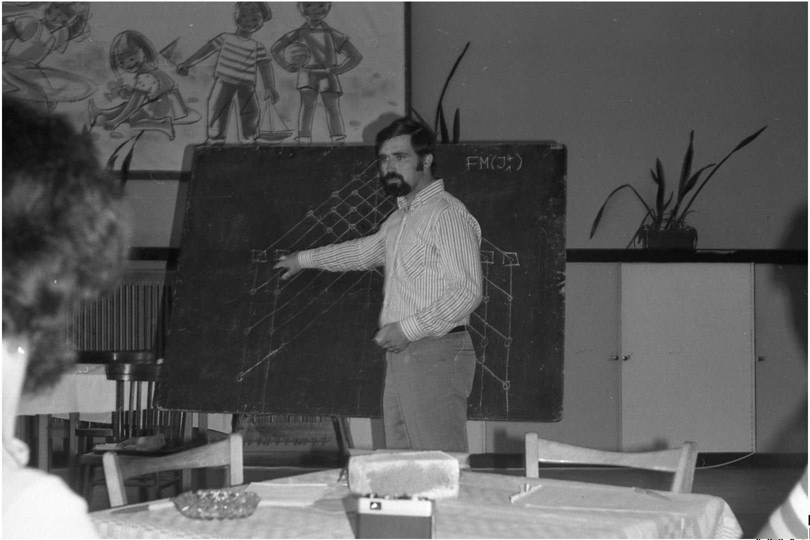
AAA series founded by Rudolf Wille 1971

Summerschool Rusava 1974



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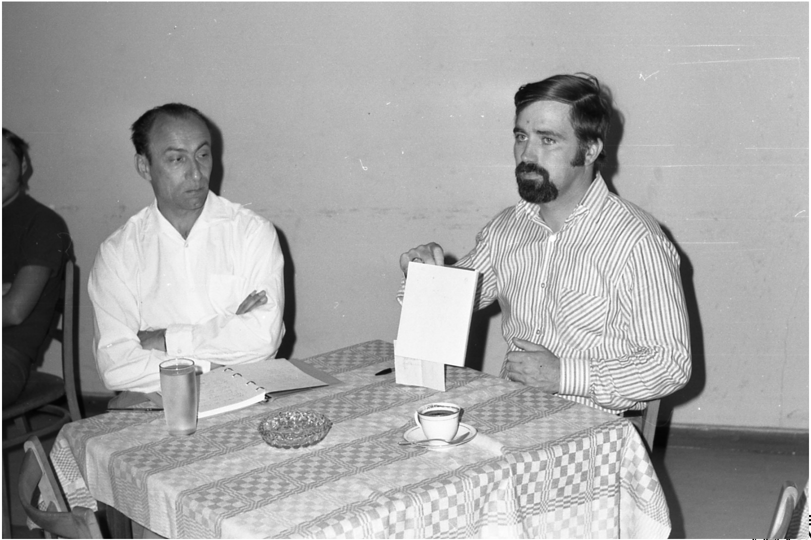

Summerschool Rusava 1974

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THE Galois connection Pol – Inv

induced by the relation **function f preserves relation ϱ**: $f \triangleright \varrho$

$F \subseteq \mathrm{Op}(A)$ (set of all finitary operations $f : A^n \to A$) ("objects")
$Q \subseteq \mathrm{Rel}(A)$ (set of all finitary relations $\varrho \subseteq A^m$) ("attributes")

$\mathrm{Inv}\, F := \{\varrho \in R_A \mid \forall f \in F : f \triangleright \varrho\}$ invariant relations

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Characterization of the Galois closures

Theorem

$\mathcal{A} = \langle A, F \rangle$ *finite algebra.*

- $\mathrm{Clo}(\mathcal{A}) = \langle F \rangle = \mathrm{Pol}\,\mathrm{Inv}\,F$ *(clone generated by F)* [1],
- $[Q]_{(\exists,\wedge,=)} = \mathrm{Inv}\,\mathrm{Pol}\,Q$ *(relational clone generated by Q).*

$\mathcal{A} = \langle A, F \rangle$ *(arbitrary, also infinite) algebra*

- $\mathrm{Loc}\,\mathrm{Clo}(\mathcal{A}) = \mathrm{Loc}\langle F \rangle = \mathrm{Pol}\,\mathrm{Inv}\,F$ *(locally closed clone generated by F)*
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[1] *D. Geiger* [Gei1968] (50 years ago!), *Lev Arkad'evic Kalužnin, Лев Аркадьевич Калужнин* [BodKKR1969]

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What are clones?

A set F of finitary functions $f : A^n \to A$ (on a base set A) is called *clone*[2] (*of operations*) if

- F contains all *projections* ($e_i^n(x_1, \dots, x_n) = x_i$)
- F is *closed under composition*[3] i.e., if $f, g_1, \dots, g_n \in F$ (f n-ary, g_i m-ary), then

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[2] *P.M. Cohn* (1965) attributes the notion to *Ph. Hall*

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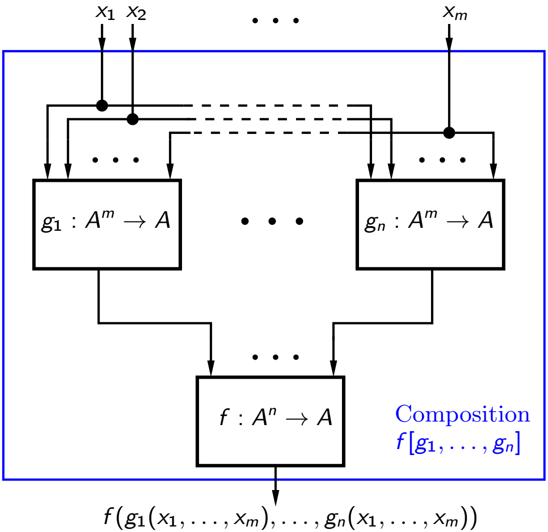

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$$f[g_1, \dots, g_n](x_1, \dots, x_m) := f(g_1(x_1, \dots, x_m), \dots, g_n(x_1, \dots, x_m))$$

[2] *P.M. Cohn* (1965) attributes the notion to *Ph. Hall*

[3] *K. Menger* (1961) composition = operation par excellence

What are clones?

A set Q of finitary relations on a set A is called a *clone* (*of relations*) or *relational clone, coclone* if it is closed with respect to primitive positive first order formulas $\varphi \in \Phi(\exists, \wedge, =)$,

i.e. for $\varphi(x_1 \ldots, x_m; \boldsymbol{\varrho_1}, \ldots, \boldsymbol{\varrho_n}) \in \Phi(\exists, \wedge, =)$ we have

$$\varrho_1, \ldots, \varrho_n \in Q \implies \{(a_1, \ldots, a_m) \in A^m \mid \models \varphi(a_1, \ldots, a_m; \varrho_1, \ldots, \varrho_n)\} \in Q.$$

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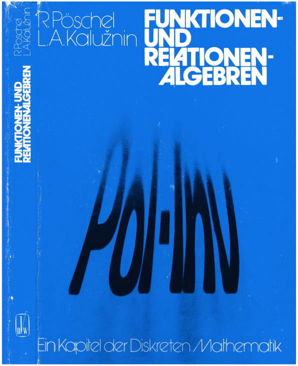
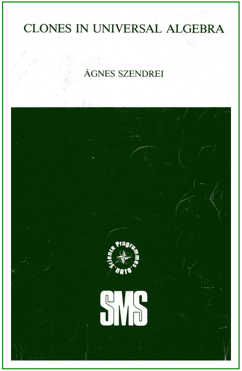
Some books about clones (and the Galois connection Pol – Inv)

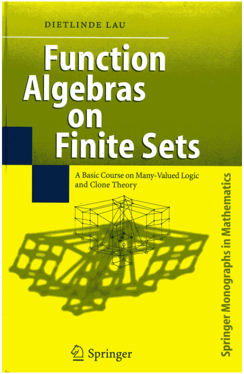
R. Pöschel
L.A. Kalužnin (1979)



A. Szendrei (1986)



D. Lau (2006)



TECHNISCHE UNIVERSITÄT DRESDEN

The second memorial intermission

Dietlinde Lau (1950–2018)

Dietlinde Lau (May 1, 1950 – June 12, 2018)

Das Institut für Mathematik der Universität Rostock trauert um

Prof. Dr. Dietlinde Lau

*01.05.1950 † 12.06.2018

Wir müssen Abschied nehmen von einer liebenswerten Kollegin und hervorragenden Hochschullehrerin. Frau Professor Dietlinde Lau hat ihre wissenschaftliche Karriere an der Universität Rostock absolviert. Durch eine Kinderlähmung war sie stark behindert, umso höher ist ihr äußerst erfolgreicher Weg wertzuschätzen. Sie begann ihr Mathematikstudium im Jahr 1969 und nach einem Forschungsstudium promovierte sie 1977 und habilitierte sich 1985 auf dem Gebiet der Diskreten Mathematik. Schließlich wurden ihre großen Leistungen bereits 1999 mit der Berufung zur außerplanmäßigen Professorin für Diskrete Mathematik gewürdigt.

Frau Professor Lau hat sich mit sehr großen Engagement für eine gut durchdachte, anspruchsvolle und erfolgsorientierte Lehre eingesetzt. Sie hat die Mathematik-Ausbildung für Informatiker aufgebaut und bis zu ihrem Ruhestand durchgeführt. Ein Denkmal hat sie sich mit ihren 2 Lehrbüchern und einem Übungsbuch zur Mathematik-Ausbildung für Informatiker gesetzt. Einzigartig hier in Rostock waren ihre hervorragenden Kenntnisse zur Geschichte der Mathematik, die sie in ihren Vorlesungen den Studierenden vermittelte.

Mit ihrer Forschung, insbesondere auch durch ein Buch auf dem Gebiet der Funktionenalgebren und der k-wertigen Logik, hat sie große internationale Anerkennung gefunden. Abschlussarbeiten hat sie sehr intensiv betreut, was u.a. auch zu einer erfolgreichen Promotion geführt hat.

Wir werden Frau Professor Lau als herzliche, sehr kollegiale und unermüdlich arbeitende Kollegin in Erinnerung behalten, die den größten Teil ihres Lebens dem Beruf, der Mathematik und dem Institut gewidmet hat. Wir werden ihr ein ehrendes Gedenken bewahren.

Prof. Dr. Konrad Engel

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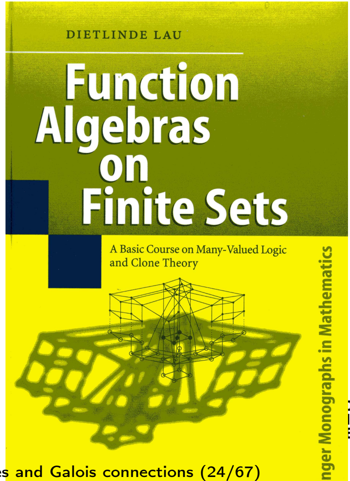


Dietlinde Lau (May 1, 1950 – June 12, 2018)

Outline

Galois connections

The “most basic Galois connection” in algebra

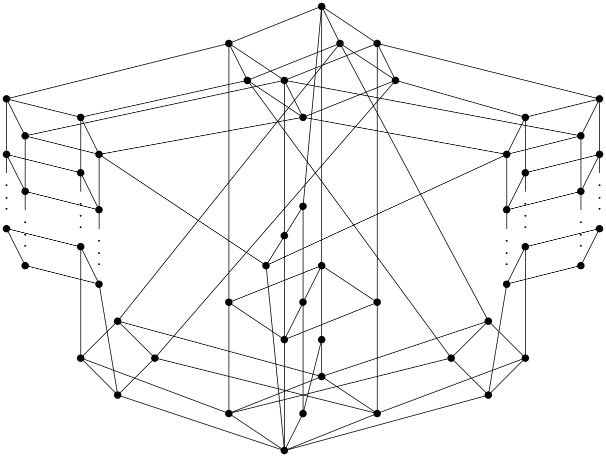
The clone lattice

Modifications of the Galois connection Pol – Inv

Some applications of THE Galois connection Pol – Inv

The lattice $\mathcal{L}_A$ of clones on A

The lattice $\mathcal{L}_A$ of all clones on base set $A = \{0, 1\}$ is countable (*E.L.Post* 1921/41)



The lattice $\mathcal{L}_A$ of clones on A

$|A| > 2$: The lattice of all clones on base set A is uncountable
$|\mathcal{L}_A| = 2^{\aleph_0}$ for $3 \leq |A| \in \mathbb{N}$, and $|\mathcal{L}_A| = 2^{2^{|A|}}$ for infinite A

The lattice $\mathcal{L}_A$ satisfies no nontrivial lattice identities
(*A. Bulatov* 1992,...)

every algebraic lattice (with at most $2^{|A|}$ compact elements) is (isomorphic to) a complete sublattice of $\mathcal{L}_A$ (*M. Pinsker*)

atomic and coatomic for finite A,
but not for infinite A (*M. Goldstern, Shelah* 2002)

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TECHNISCHE UNIVERSITÄT DRESDEN

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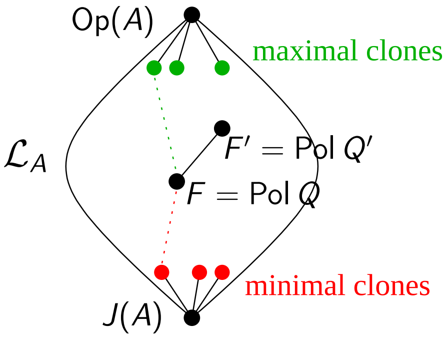

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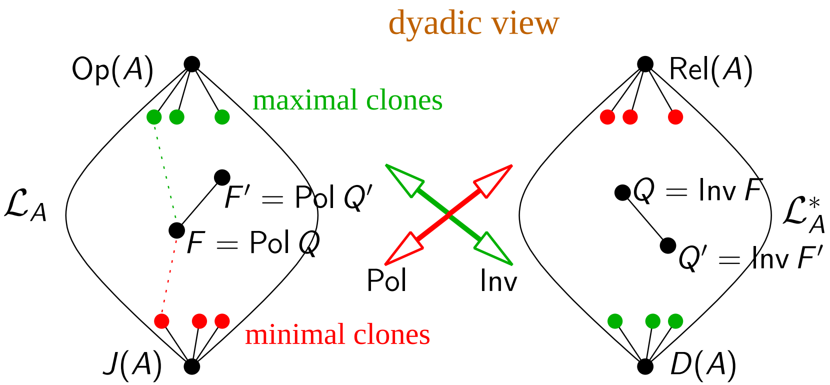

$F \vee F' = \mathrm{Pol}(Q \cap Q')$

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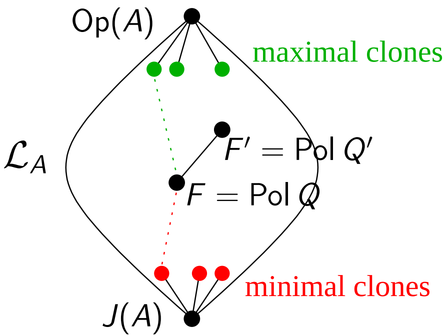

Maximal clones:

$|A| = 2$: *E.L. Post*
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$|A| = 4$: *A.I. Mal'cev* (≤ 1969)
$|A| \in \mathbb{N}$: *I.G. Rosenberg* (1970)
(of form $\mathrm{Pol}\, \varrho_i$, $i \in I$)
$|A|$ infinite: *I.G. Rosenberg, L. Heindorf, M. Goldstern, M. Pinsker ...*

Goldstern/Pinsker,
A survey of clones on infinite sets. 2007

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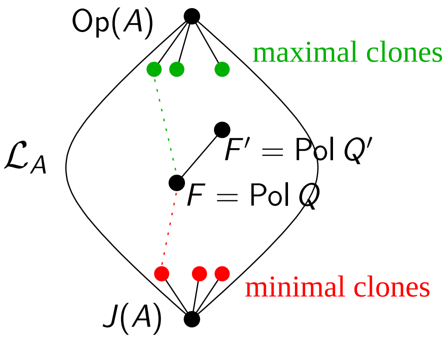

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submaximal clones: *D. Lau*
maximal clones C where $C \cap \text{Sym}(A)$ is maximal permutation group in $\text{Sym}(A)$:
P.P. Pálfy (2007)

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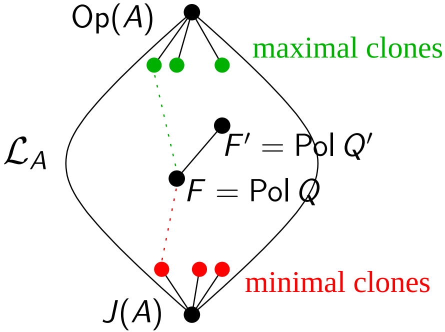

Minimal clones: complete description still open problem

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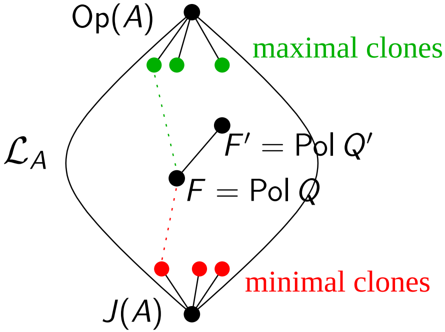

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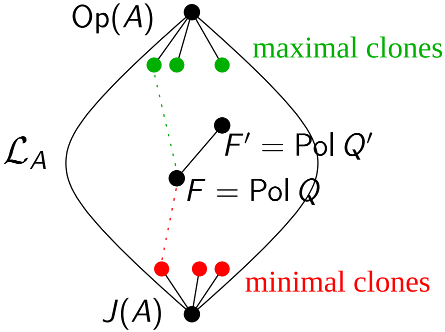

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further partial results: *G. Czédli, A. Szendrei, K. Kearnes, P.P. Pálfy, L. Szabo, B. Szszepara, T. Waldhauser,...*

essentially minimal clones: *I.G. Rosenberg, H. Machida, ...*

Ivo Rosenberg's 1970 publication on maximal clones

The third memorial intermission

Ivo Rosenberg (1934–2018)

Ivo Rosenberg (Dec. 13, 1934 – Jan. 18, 2018)

Memorial session for Ivo Rosenberg at AAA95 in Bratislava, March 2018

TECHNISCHE UNIVERSITÄT DRESDEN

Ivo Rosenberg (Dec. 13, 1934 – Jan. 18, 2018)

Summerschool in Štrbské Pleso 1997

Ivo Rosenberg (Dec. 13, 1934 – Jan. 18, 2018)

Summerschool in Stara Lesna 2009

Ivo Rosenberg (Dec. 13, 1934 – Jan. 18, 2018)

Summerschool in Podlesi 2011

Outline

What and how to modify?

(A) Restricting the operations and/or relations under consideration but keeping the invariance relation "ϱ is invariant for f".

(B) Generalizing the operations (e.g. to partial operations or multioperations) and/or relations with "canonically" modified invariance relation.

(C) Considering "natural" closure operators on operations and/or relations and trying to characterize the closed sets as Galois closures of a suitable Galois connection.

(D) Modifying the preservation property "$f \triangleright \varrho$".

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(A) Restrictions on operations and relations

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Let $E \subseteq \mathrm{Op}(A)$ and $R \subseteq \mathrm{Rel}(A)$ be "interesting" subsets.
Then ${}^E\mathrm{Pol} - {}^R\mathrm{Inv}$ defined by

$$\begin{aligned} {}^E\mathrm{Pol}\, Q &:= E \cap \mathrm{Pol}\, Q \text{ for } Q \subseteq R\,, \\ {}^R\mathrm{Inv}\, F &:= R \cap \mathrm{Inv}\, F \text{ for } F \subseteq E \end{aligned}$$

is a Galois connection (between $\mathfrak{P}(R)$ and $\mathfrak{P}(E)$).
Assume $E = \mathrm{Pol}\, Q_0$, $R = \mathrm{Inv}\, F_0$ (for "nice" sets Q_0, F_0) and, for simplicity, A finite. Then:

Theorem (Characterization of the Galois closures)

$$\begin{aligned} {}^E\mathrm{Pol}\, {}^R\mathrm{Inv}\, F &= E \cap \langle F_0 \cup F \rangle_{\mathrm{Op}(A)} \text{ for } F \subseteq \mathrm{Op}(A)\,, \\ {}^R\mathrm{Inv}\, {}^E\mathrm{Pol}\, Q &= R \cap [Q_0 \cup Q]_{(\exists,\wedge,=)} \text{ for } Q \subseteq \mathrm{Rel}(A). \end{aligned}$$

TECHNISCHE UNIVERSITÄT DRESDEN

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TECHNISCHE UNIVERSITÄT DRESDEN

The closure operator ${}^E\mathrm{Pol}\,{}^R\mathrm{Inv}$

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($\wedge, \vee$ in the lattice $\mathcal{L}_A$).

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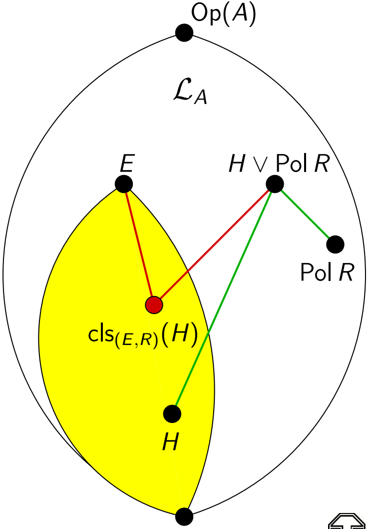

Restricting arities

$E := \mathrm{Op}^{(m)}(A)$ or $R := \mathrm{Rel}^{(m)}(A)$.

$\mathcal{A} = \langle A, F \rangle$ finite algebra.

Theorem (Characterization of the Galois closures)

- $m\text{-}\mathrm{Loc}\, F = \mathrm{Pol}\, \mathrm{Inv}^{(m)} F$ (*m-locally closed* clones, clones with *m-interpolation property*),
- $m\text{-}\mathrm{LOC}[Q]_{(\exists,\wedge,=)} = \mathrm{Inv}\, \mathrm{Pol}^{(m)} Q$ (*m-locally closed relational clones*).

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Restricting relations to graphs $g^\bullet$ of operations g leads to the Galois connection on operations induced by $\perp$: For $f, g \in \mathrm{Op}(A)$

$$f \perp g \text{ (}f \textit{ commutes with } g\text{)} \iff g \in \mathrm{Pol}\, f^\bullet \iff f \in \mathrm{Pol}\, g^\bullet.$$

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$$\begin{array}{c} \\ f \\ f \\ \vdots \\ f \end{array} \begin{array}{c} \begin{matrix} g & g & \cdots & g \end{matrix} \\ \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{pmatrix} \end{array} = \begin{array}{c} g \\ \begin{pmatrix} \circ \\ \circ \\ \vdots \\ \circ \end{pmatrix} \end{array}$$

$$\downarrow \quad \downarrow \qquad \quad \downarrow \qquad \downarrow$$

$$f\,(\circ \quad \circ \quad \cdots \quad \circ) = \square$$

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There are only finitely many centralizer clones.

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Arity bounds for centralizer clones

By the finiteness result [BurW1987], for any $k \in \mathbb{N}$, there must exist a bound $m \in \mathbb{N}$ such that every centralizer clone F on a k-element set is determined by the set $F^{(m)}$ of its m-ary operations, i.e.

$$F = \operatorname{Pol}\operatorname{Pol} F^{(m)}.$$

Upper bound: $m \leq k^{k^4 - k^3 + k^2}$ ([BurW1987], [Pö]),
improvement: $m \leq k^k$

Challenging conjecture: $m = k$
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TECHNISCHE UNIVERSITÄT DRESDEN

The Galois connections Aut – Maj and Aut – Spr

Keeping the relation $f \perp g$, further restrictions to permutations and majority functions Maj (or semiprojections Spr, resp.)

Proposition (*Behrisch/Pö* Aug. 2018)

For $G \subseteq \mathrm{Sym}(A)$ *we have:*

$$4\text{-}\mathrm{Loc}\, G \quad \mathrm{Aut}\,\mathrm{Maj}\, G \quad \mathrm{Aut}\,\mathrm{Spr}^{(3)}\, G \quad 3\text{-}\mathrm{Loc}\, G.$$

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Proof: : The graph of a ternary function is a quaternary relation.
: Encoding ternary irreflexive relations ϱ by majority operations (semiprojections, resp.) f:

$$f(x,y,z) = \begin{cases} x & \text{if } (x,y,z) \in \varrho \\ y & \text{if } (x,y,z) \in A^3_{\neq} \setminus \varrho \\ \ldots & \text{according to majority conditions,} \\ & \text{resp. } z, \text{ otherwise} \end{cases}$$

Then $\sigma \perp f \iff \sigma \rhd \varrho$, which implies $\mathrm{Aut}\,\mathrm{Maj}\, G \subseteq \mathrm{Aut}\,\mathrm{Inv}^{(3)}\, G$.

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Remark

In general, both inclusions can be strict.

$$4\text{-}\operatorname{Loc} G \subseteq \operatorname{Aut}\operatorname{Maj} G = \operatorname{Aut}\operatorname{Spr}^{(3)} G \subseteq 3\text{-}\operatorname{Loc} G.$$

"right $\subseteq$": Let $G \leq \operatorname{Sym}(A)$ be a sharply 3-transitive but not 4-transitive permutation group,
e.g., $\operatorname{PGL}(2,5)$ acting on the 6-element set A of 1-dimensional subspaces of the 2-dimensional vector space over $\operatorname{GF}(5)$, thanks to *Sven Reichard*,
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Define a majority operation f on $A^3_{\neq}$ via $f(b_1, b_2, b_3) = b_4$ whenever $(b_1, b_2, b_3, b_4) \in a^G$ (f is well-defined because G is sharply 3-transitive).
Clearly, $f \in \operatorname{Maj} G$.
$\operatorname{Inv}^{(3)} G$ is trivial (by 3-transitivity of G), thus $\operatorname{Aut}\operatorname{Inv}^{(3)} G = \operatorname{Sym}(A)$,
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(e.g., $G = \operatorname{P\Gamma L}(2,8)$, thanks to *Sven Reichard*), then
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Further Galois connections by restricting to E and R

E	R	Galois closure	References
$\mathrm{Op}(A)$	$\mathrm{Rel}(A)$	$\mathrm{Pol\,Inv}\,F$	[BodKKR69a], [BodKKR69b], [Gei68], [BakP75], [Rom76], [Rom77a], [Rom77b], [PösK79], cf. 2.3, 2.5
		$\mathrm{Inv\,Pol}\,Q$	[Gei68], [BodKKR69a], [BodKKR69b], [Sza78], [PösK79], [Pös79], [Pös80a], cf. 2.3, 2.5
		Generalization to infinitary relations or operations	
		$\mathrm{Pol\,Inv}^{\infty}\,F$	[Ros72], [KraP76], [Poi81]
		$\mathrm{Inv}^{\infty}\,\mathrm{Pol}\,Q$	[Ros79]
		$\mathrm{Pol}^{\infty}\,\mathrm{Inv}^{\infty}\,F$ } $\mathrm{Inv}^{\infty}\,\mathrm{Pol}^{\infty}\,Q$ }	[Kra76b], [KraP76], [Poi81]
		arity restrictions	
$\mathrm{Op}(A)$	$\mathrm{Rel}^{(m)}(A)$	$\mathrm{Pol\,Inv}^{(m)}\,F$	[Gei68], [BakP75], [Pös80a]
		$\mathrm{Inv}^{(m)}\,\mathrm{Pol}\,Q$	[Ros78]
$\mathrm{Op}(A)$	$\mathrm{Rel}^{(1)}(A)$	$\mathrm{Pol\,Sub}\,F$ $\mathrm{Sub\,Pol}\,Q$	[Sch82, Thm. 1.6], [Pös80a] see below
$\mathrm{Op}^{(m)}(A)$	$\mathrm{Rel}(A)$	$\mathrm{Inv\,Pol}^{(m)}\,Q$	[Sza78], [Pös80a]
$\mathrm{Tr}(A)$	$\mathrm{Rel}(A)$	$\mathrm{Inv\,End}\,Q$ } $\mathrm{End\,Inv}\,F$ }	[Kra38], [Kra50], [Kra76a], [Kra86], [Gou68], [BodKKR69a], [BodKKR69b], [Pös80a], [Bör00]
$\mathrm{Sym}(A)$	$\mathrm{Rel}(A)$	$\mathrm{Inv\,Aut}\,Q$ $(\mathrm{sInv\,Aut}\,Q)$ $\mathrm{wAut\,Inv}\,F$ $\mathrm{Aut\,sInv}\,F$	[Kra38], [BodKKR69a], [BodKKR69b], [Gou72a], [Pös80a], [Bör00]
$\mathrm{Sym}(A)$	$\mathrm{Rel}^{(m)}(A)$	$\mathrm{Aut\,Inv}^{(m)}\,F$	[Wie69]
		restriction to (graphs of) operations only	
$\mathrm{Op}(A)$	$\mathrm{Op}(A)^{\bullet}$	$\mathrm{Pol\,Pol}\,F$	[Sza78, Thm. 13], [Faj77], [Dan77](for $\lvert A\rvert = 3$), (also Kuznecov, cf. [Val76])
$\mathrm{Op}(A)$	$\mathrm{Tr}(A)^{\bullet}$	$\mathrm{Pol\,End}\,F$ $\mathrm{End\,Pol}\,Q$	[SauS82], ([Rei82] implicit operations) see below

concrete characterization problems

E	R	Galois closure	References
		concrete characterization of $\mathrm{Aut}\,\mathcal{A}$	
$\mathrm{Op}(A)$	$\mathrm{Sym}(A)^{\bullet}$	$\mathrm{Aut\,Pol}\,Q$	[Jón68] (cf. [Jón72, (2.4.3)]), [Kra50], [ArmS64], [Sza75], [Bre76]
		concrete characterization of $\mathrm{Aut}\,\mathcal{A}$ *for algebras* $\mathcal{A}$ *with at most m-ary operations*	
$\mathrm{Op}^{(m)}(A)$	$\mathrm{Sym}(A)^{\bullet}$	$\mathrm{Aut\,Pol}^{(m)}\,Q$	[Pło68], [Jón72, (2.4.1)], [Gou72a]
		concrete characterization of $\mathrm{Sub}\,\mathcal{A}$	
$\mathrm{Op}(A)$	$\mathfrak{P}(A)$	$\mathrm{Sub\,Pol}\,Q$	[BirF48] (cf. [Jón72, (3.6.4)]), [Gou68], [Gou72b] for unary algebras: [Jón72, (3.6.7)], [JohS67]
		concrete characterization of $\mathrm{Con}\,\mathcal{A}$	
$\mathrm{Op}(A)$	$\mathrm{Eq}(A)$	$\mathrm{Con\,Pol}\,Q$ $\mathrm{Pol\,Con}\,F$	[Arm70](partial solution), [Jón72, (4.4.1)], [QuaW71], [Wer74], [Dra74] for p-rings $\langle A; F\rangle$ [Isk72]
		concrete characterization of $\mathrm{End}\,\mathcal{A}$	
$\mathrm{Op}(A)$	$\mathrm{Tr}(A)^{\bullet}$	$\mathrm{End\,Pol}\,Q$	[Lam68], [GräL68], [SauS77a], [Sto69], [Sto75], [Jež72], [Sza78, Thm. 15]
		concrete characterization of	
$\mathrm{Op}(A)$	$\mathrm{Sym}(A)^{\bullet} \cup \mathfrak{P}(A)$	$\mathrm{Aut}\,\mathcal{A}$ & $\mathrm{Sub}\,\mathcal{A}$	[Sto72], [Gou72b], cf. 3.5
	$\mathrm{Sym}(A)^{\bullet} \cup \mathrm{Eq}(A)$	$\mathrm{Aut}\,\mathcal{A}$ & $\mathrm{Con}\,\mathcal{A}$	[Wer74](conjecture) cf. [Pös80b], (for simple $\mathcal{A}$ [Sch64]), cf. 3.5
	$\mathrm{Op}(A)^{\bullet} \cup \mathrm{Sub}(A)$	$\mathrm{End}\,\mathcal{A}$ & $\mathrm{Sub}\,\mathcal{A}$	[SauS77b](cf. [Jón74])
	$\mathrm{Op}(A)^{\bullet} \cup \mathrm{Sub}(A)$ $\cup\,\mathrm{Eq}(A)$	$\mathrm{Aut}\,\mathcal{A}$ & $\mathrm{Sub}\,\mathcal{A}$ & $\mathrm{Con}\,\mathcal{A}$	[Sza78], [Pös80a], cf. 3.5

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(B) Generalizing operations and/or relations with "canonically" modified invariance relation

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Generalization to

- partial operations ($f : B \to A$ with $B \subseteq A^n$)
- power operations (multi-operations) ($f : A \to \mathfrak{P}(A)$)
- infinitary operations and/or relations [Ros1972]
- multisorted algebras $A = (A_s)_{s \in S}$ and relations [Pös1973]
- multisorted relation pairs $(\varrho, \varrho') = ((\varrho_s)_{s \in S}, (\varrho'_s)_{s \in S})$ and minor closed classes ("clonoids", "minions", "preclones") [LehPW2018]
-
-

(C) Characterizing closures as Galois closures

Minor closed classes of functions

$f : A^n \to A$ is a *minor* of $g : A^m \to A$ if there is a mapping $\lambda : \{1, \dots, m\} \to \{1, \dots, n\}$ such that

$$f(a_1, \dots, a_n) = g(a_{\lambda(1)}, \dots, a_{\lambda(m)})$$

(permuting/identifying variables, adding fictive variables)

Question: How to characterize the closure $\langle F\rangle_{\mathrm{mc}}$ of $F \subseteq \mathrm{Op}(A)$ under taking minors (*minor closed classes* (clonoids, minions)) as Galois closure of a suitable Galois connection?

Answer: Take the Galois connection mPol – mInv of the context $(\mathrm{Op}(A), \mathrm{Rel}(A) \times \mathrm{Rel}(A), \triangleright)$ where

$$f \triangleright (\varrho, \varrho') :\Longleftrightarrow f(r_1, \dots, r_n) \in \varrho' \text{ for } r_1, \dots, r_n \in \varrho$$

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The Galois connection mPol – mInv

$$f \triangleright (\varrho, \varrho') :\Longleftrightarrow \begin{array}{l} f(\; a_{11} \quad a_{12} \quad \dots \quad a_{1n}) = \bullet \\ f(\; a_{21} \quad a_{22} \quad \dots \quad a_{2n}) = \bullet \\ \\ f(\; a_{m1} \quad a_{m2} \quad \dots \quad a_{mn}) = \bullet \\ \quad \in \varrho \quad\; \in \varrho \quad \dots \quad \in \varrho \Rightarrow \in \varrho' \end{array}$$

Theorem (Characterization of the Galois closures)

$\langle F \rangle_{\mathrm{mc}} = \mathrm{mPol}\,\mathrm{mInv}\, F$ *for* $F \subseteq \mathrm{Op}(A)$,

$[Q]_{\mathrm{mc}} = \mathrm{mInv}\,\mathrm{mPol}\, Q$ *for* $Q \subseteq \mathrm{Rel}(A) \times \mathrm{Rel}(A)$.

[Pip2002] (*N. Pippenger*), [LehPW2018, Thm. 4.10]

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The Galois connection mPol – mInv

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Logical operations

Let $\varphi(x_1, \dots, x_m)$ be a first order formula containing quantifyers and connectives from Φ only (with relation symbols $\varrho_1, \dots, \varrho_n$ and free variables $x_1, \dots, x_m$).

corresponding *logical operation* $\in \mathrm{Lop}_A(\Phi)$ on $\mathrm{Rel}(A)$:

$$L_\varphi(\varrho_1, \dots, \varrho_n) := \{(a_1, \dots, a_m) \mid \models \varphi(a_1, \dots, a_m)\}$$

Example: $\varrho_1 \circ \varrho_2 = \{(x, y) \mid \exists z : (x, z) \in \varrho_1 \wedge (z, y) \in \varrho_2\}$

$$= L_\varphi(\varrho_1, \varrho_2) = \{(x, y) \in A^2 \mid \models \varphi(\varrho_1, \varrho_2; x, y)\}$$

for the first order formula

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Question: How to characterize closures under logical operations $\mathrm{Lop}_A(\Phi)$ (for particular Φ) as Galois closures of a suitable Galois connection?

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$$= L_\varphi(\varrho_1, \varrho_2) = \{(x,y) \in A^2 \mid \models \varphi(\varrho_1, \varrho_2; x, y)\}$$

for the first order formula

$$\varphi(\varrho_1, \varrho_2; x, y) :\equiv \exists z : \varrho_1(x,z) \wedge \varrho_2(z,y) \quad \in \mathrm{Lop}_A(\exists, \wedge)$$

Question: How to characterize closures under logical operations $\mathrm{Lop}_A(\Phi)$ (for particular Φ) as Galois closures of a suitable Galois connection?

Logical operations

Let $\varphi(x_1, \dots, x_m)$ be a first order formula containing quantifyers and connectives from Φ only (with relation symbols $\varrho_1, \dots, \varrho_n$ and free variables $x_1, \dots, x_m$).
corresponding *logical operation* $\in \mathrm{Lop}_A(\Phi)$ on $\mathrm{Rel}(A)$:

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TECHNISCHE UNIVERSITÄT DRESDEN

Closures with logical operations

	$\mathrm{Lop}_A(\Phi)$-closure with $\Phi =$	Notation	Galois connection: closed relational system	Galois connection: closed operational system
for finite base set A:				
(1)	$(\exists, \wedge, \vee, \neg, =)$	$[Q]_{\mathrm{KA}}$	sInv – Aut: Krasner algebra	group of permutations
(2)	$(\exists, \wedge, \vee, =)$	$[Q]_{\mathrm{WKA}}$	Inv – End: weak Krasner algebra	monoid of unary functions
(3)	$(\exists, \wedge, =)$	$[Q]_{\mathrm{RA}}$	Inv – Pol: relational algebra	clone of finitary functions

(4)	$(\wedge,=)$		Inv – pPol	
			weak system with identity	down-closed clone of finitary partial functions
(5)	$(\wedge)$		Inv – mPol	
			weak system of relations	down-closed clone of finitary multifunctions
(6)	$(\exists,\wedge,\vee,\neg)$	$[Q]_{\mathrm{BSP}}$	sInv – sEnd (sbmEnd, resp.)	
			BSP Boolean system with projections	Special monoid of unary functions
(6')				(down-closed involuted monoid of bitotal multifunctions, resp.)

	$\mathrm{Lop}_A(\Phi)$-closure with $\Phi =$	Notation	Galois connection	
			closed relational system	closed operational system
for finite base set A:				
(7)	$(\wedge, \vee, \neg, =)$	$[Q]_{\mathrm{BSI}}$	$\mathrm{sInv} - \mathrm{spmEnd}$	
			BSI Boolean system with identity	down-closed involuted monoid of pp-multifunctions (partial permutations)
(8)	$(\wedge, \vee, \neg)$	$[Q]_{\mathrm{BS}}$	$\mathrm{sInv} - \mathrm{smEnd}$	
			BS Boolean system	down-closed involuted monoid of unary multifunctions

[BörPS2002]

(D) Modifying the preservation property

Strong preservation

Modifying the Galois connection of the context $(\mathrm{Sym}(A), \mathrm{Rel}(A), \triangleright)$:

$f \in \mathrm{Sym}(A)$ *strongly preserves* $\varrho \in \mathrm{Rel}(A)$:

$$f \triangleright_{\text{strong}} \varrho :\iff f \triangleright \varrho \text{ and } f^{-1} \triangleright \varrho.$$

The corresponding Galois connection (induced by $\triangleright_{\text{strong}}$):

$$\mathrm{sInv}\, G := \{\varrho \in \mathrm{Rel}(A) \mid \forall f \in G : f \triangleright_{\text{strong}} \varrho\},$$
$$\mathrm{Aut}\, Q := \{f \in \mathrm{Sym}(A) \mid \forall \varrho \in Q : f \triangleright_{\text{strong}} \varrho\}.$$

$G \subseteq \mathrm{Sym}(A)$, $Q \subseteq \mathrm{Rel}(A)$.
Note that in general $\mathrm{Aut}\, Q \neq \mathrm{wAut}\, Q = \{f \in \mathrm{Sym}(A) \mid \forall \varrho \in Q : f \triangleright \varrho\}$ (but equality holds for finite A).

Theorem (Characterization of the Galois closures)

$$\mathrm{Loc}\langle G\rangle_{\mathrm{Sym}(A)} = \mathrm{Aut}\,\mathrm{sInv}\, G$$
$$[Q]_{\mathrm{KC}} = \mathrm{sInv}\,\mathrm{Aut}\, Q = \mathrm{Inv}\,\mathrm{Aut}\, Q$$

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TECHNISCHE UNIVERSITÄT DRESDEN

The Galois connection e-Pol – e-Inv

induced by the relation f preserves ϱ at one place

function f e-preserves relation ϱ: $f \rhd_e \varrho$

i.e., $\exists i$: f preserves ϱ at i-th place

$$\text{e-Inv}\, F := \{\varrho \in R_A \mid \forall f \in F : f \rhd_e \varrho\}$$ *e-invariant relations*

$$\text{e-Pol}\, Q := \{f \in O_A \mid \forall \varrho \in Q : f \rhd_e \varrho\}$$ *e-polymorphisms*

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The Galois connection e-Pol – e-Inv

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$$\exists i : \quad \begin{array}{l} f(\, a_{11} \; \cdots \; a_{1i} \; \cdots \; a_{1n}\,) = \bullet \\ f(\, a_{21} \; \cdots \; a_{2i} \; \cdots \; a_{2n}\,) = \bullet \\ \\ f(\, a_{m1} \; \cdots \; a_{mi} \; \cdots \; a_{mn}\,) = \bullet \end{array}$$

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$$a_{\cdot i} \in \varrho \quad \Rightarrow \quad \in \varrho$$

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TECHNISCHE UNIVERSITÄT DRESDEN

The Galois connection e-Pol – e-Inv

induced by the relation *f* preserves ϱ at one place

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Connection between e-Pol and Pol

For $\varrho \in R_A^{(m)}$ let $\widehat{\varrho} := \{\varrho_{(n)} \mid n \in \mathbb{N}_+\}$ where

$$\varrho_{(n)} := \bigcup_{i=1}^{n} A^m \times \ldots \times \underbrace{\varrho}_{i\text{-th place}} \times \ldots \times A^m \in R_A^{(nm)} (= (A^m)^n \setminus (A^m \setminus \varrho)^n)$$

e.g., $\varrho_{(2)} = \{(a_1, \ldots, a_m, b_1, \ldots, b_m) \mid (a_1, \ldots, a_m) \in \varrho \text{ or } (b_1, \ldots, b_m) \in \varrho\}$.

Proposition (Characterization of e-Pol Q [Ros75], [Pös75])

Let $f \in O_A^{(n)}$, $\varrho \in R_A^{(m)}$, $Q \subseteq R_A$. Then

- $f \rhd_e \varrho \iff f \rhd \varrho_{(n)}$,
- $\mathrm{Pol}\,\varrho = \mathrm{Pol}\,\varrho_{(1)} \supseteq \cdots \supseteq \mathrm{Pol}\,\varrho_{(n)} \supseteq \mathrm{Pol}\,\varrho_{(n+1)} \supseteq \cdots \supseteq \text{e-Pol}\,\varrho$,
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- $\text{e-Pol}\,Q = \mathrm{Pol}\,\widehat{Q}$ *(where $\widehat{Q} := \bigcup\{\widehat{\varrho} \mid \varrho \in Q\}$), in particular it is a clone (e-clone).*

Connection between e-Pol and Pol

For $\varrho \in R_A^{(m)}$ let $\widehat{\varrho} := \{\varrho_{(n)} \mid n \in \mathbb{N}_+\}$ where (cube term blocker)

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TECHNISCHE UNIVERSITÄT DRESDEN

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$$\varrho_{(n)} := \bigcup_{i=1}^{n} A^m \times \ldots \times \underbrace{\varrho}_{i\text{-th place}} \times \ldots \times A^m \in R_A^{(nm)} (= (A^m)^n \setminus (A^m \setminus \varrho)^n)$$

e.g., $\varrho_{(2)} = \{(a_1, \ldots, a_m, b_1, \ldots, b_m) \mid$
$(a_1, \ldots, a_m) \in \varrho$ or $(b_1, \ldots, b_m) \in \varrho\}$.

Proposition (Characterization of e-Pol Q [Ros75], [Pös75])

Let $f \in O_A^{(n)}$, $\varrho \in R_A^{(m)}$, $Q \subseteq R_A$. Then

- $f \rhd_e \varrho \iff f \rhd \varrho_{(n)}$,
- $\mathrm{Pol}\,\varrho = \mathrm{Pol}\,\varrho_{(1)} \supseteq \cdots \supseteq \mathrm{Pol}\,\varrho_{(n)} \supseteq \mathrm{Pol}\,\varrho_{(n+1)} \supseteq \cdots \supseteq \text{e-Pol}\,\varrho$,
- $\mathrm{Pol}\,\widehat{\varrho} = \bigcap_{i=1}^{\infty} \mathrm{Pol}\,\varrho_{(n)} = \text{e-Pol}\,\varrho$,
- $\text{e-Pol}\,Q = \mathrm{Pol}\,\widehat{Q}$ *(where* $\widehat{Q} := \bigcup\{\widehat{\varrho} \mid \varrho \in Q\}$*),*
  in particular it is a clone (e-clone).

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e-clones

$F \subseteq O_A$ *e-clone* (Galois closed) if $F = \text{e-Pol}\,\text{e-Inv}\,F$.

Which clones are e-clones?

Trivial observation:
Each clone of essentially unary operations is an e-clone.

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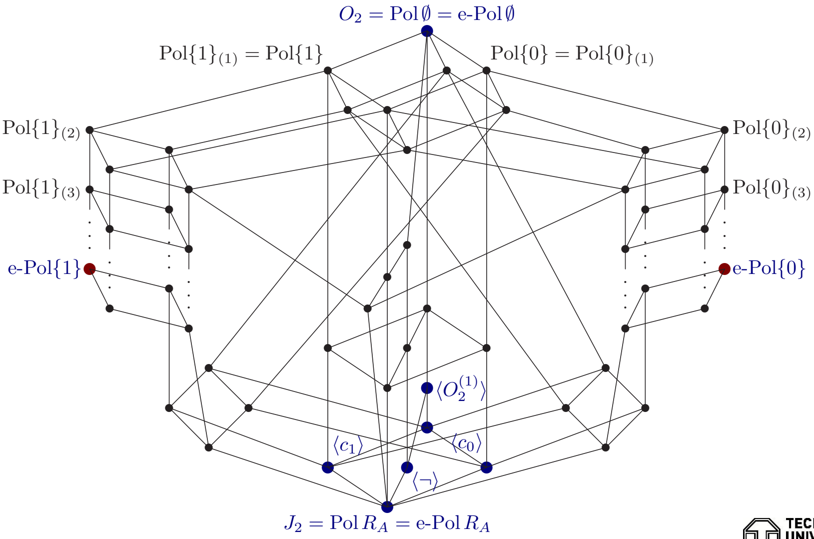
The 9 e-clones on a 2-element set



Non-finitely related e-clones

If the descending chain

$$\operatorname{Pol}\varrho = \operatorname{Pol}\varrho_{(1)} \supseteq \cdots \supseteq \operatorname{Pol}\varrho_{(n)} \supseteq \cdots \supseteq \bigcap_{i=1}^{\infty} \operatorname{Pol}\varrho_{(n)} = \text{e-Pol}\,\varrho \quad (*)$$

does not terminate, then e-Pol ϱ is non-finitely related.

Under which conditions this happens? Partial answer:

Proposition (Pös1975, Satz 15)

Let $\varrho \in R_A^{(1)}$ be a nonempty proper subset of A. Then $(*)$ is a non-terminating chain. In particular, e-Pol ϱ is non-finitely related. Moreover, $(*)$ is non-refinable.

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Some open problems for e-Pol – e-Inv

- Characterize relations $\varrho \in \mathrm{Rel}^{(m)}(A)$ with non-finitely related e-clone e-Pol ϱ.

- For finite A, are there finitely or infinitely many e-clones?

- Find an "inner" characterization of the Galois closures, i.e., of
  - e-Pol e-Inv F,
  - e-Inv e-Pol Q.

- Study analogously the other Galois connections introduced by *Ivo G. Rosenberg* in his 1975 paper.

I.G. Rosenberg, *Special types of universal algebras preserving a relation*, Portugaliae Mathematica 34(1975), 173 – 188.
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Thank you for your ATTENTION!

Outline

Galois connections

The "most basic Galois connection" in algebra

The clone lattice

Modifications of the Galois connection Pol – Inv

Some applications of THE Galois connection Pol – Inv

Problem 1

Can one express exponentiation

$$f(x, y) := x^y$$

as composition of addition and multiplication

$$g_+(x, y) := x + y, \quad g_\cdot(x, y) := x \cdot y$$

?

$(x, y \in \mathbb{N}_+)$

Answer to Problem 1

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$(x, y \in \mathbb{N}_+)$

Problem: $f \in \langle g_+, g_\cdot \rangle$?

Answer: No.

Proof: Idea: Find $\varrho \in \mathrm{Rel}(A)$ such that $g_+ \triangleright \varrho$, $g_\cdot \triangleright \varrho$ but not $f \triangleright \varrho$, because this would contradict to $f \in \langle g_+, g_\cdot \rangle$ $\subseteq \mathrm{Pol}\,\mathrm{Inv}\{g_+, g_\cdot\}$, i.e. $f \triangleright \mathrm{Inv}\{g_+, g_\cdot\}$.

Take $\varrho := \{(x, x') \in \mathbb{N}_+^2 \mid 3 \text{ divides } x - x'\}$

ϱ is invariant for $g_+, g_\cdot$, but not for f:

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ϱ is invariant for $g_+, g_\cdot$, but not for f:

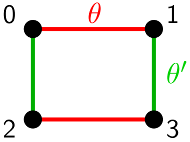
$$f(2, 4) = 2^4 = 16$$
$$f(2, 1) = 2^1 = 2$$

$\in \varrho$ (columns $(2,2)$ and $(4,1)$) $\quad \notin \varrho$ (column $(16, 2)$)

Problem 3

Let

$$A := \{0, 1, 2, 3\}$$
$$G := \{e, g\} := \{(0), (03)(12)\} \leq S_4 \text{ (permutation group)}$$
$$L := \{\theta_0, \theta, \theta', \theta_1\} \text{ (lattice of equivalence relations on } A)$$



θ_0, θ_1 trivial equivalence relations

Does there exists an algebra

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such that

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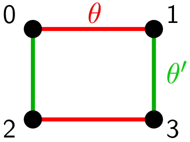

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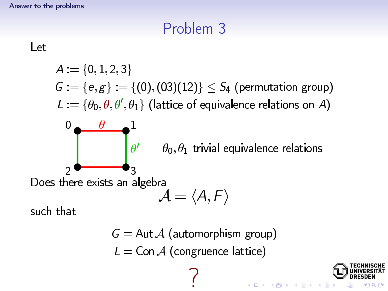
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Problem:

(*) $\exists F : G = \mathrm{Aut}\, F,\ L = \mathrm{Con}\, F$?

Answer: No.

Proof:

(*) $\Longrightarrow\ F \rhd Q := \{g^\bullet, \theta, \theta'\}$

w.l.o.g. $F = \mathrm{Pol}\, Q$. Then

$\mathrm{Inv}\, F = \mathrm{Inv}\,\mathrm{Pol}\, Q =_{Thm.} [Q]_{\mathrm{RA}}$

and we get from (*):

$$G^\bullet = (\mathrm{Aut}\, F)^\bullet = S_A^\bullet \cap \mathrm{Inv}\, F = S_A^\bullet \cap [Q]_{\mathrm{RA}}$$

$$L = \mathrm{Con}\, F = \mathrm{Eq}(A) \cap \mathrm{Inv}\, F = \mathrm{Eq}(A) \cap [Q]_{\mathrm{RA}}$$

in contradiction to $\exists h^\bullet \in S_A^\bullet \cap [Q]_{\mathrm{RA}} : h \notin G$

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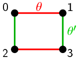

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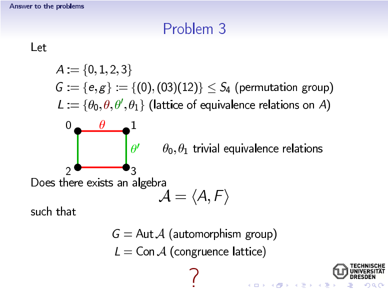
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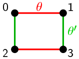
$A := \{0,1,2,3\}$
$G := \{e, g\} := \{(0), (03)(12)\} \leq S_4$ (permutation group)
$L := \{\theta_0, \theta, \theta', \theta_1\}$ (lattice of equivalence relations on A)



θ_0, θ_1 trivial equivalence relations

Does there exists an algebra

$$\mathcal{A} = \langle A, F \rangle$$

such that

$G = \operatorname{Aut} \mathcal{A}$ (automorphism group)
$L = \operatorname{Con} \mathcal{A}$ (congruence lattice)

?

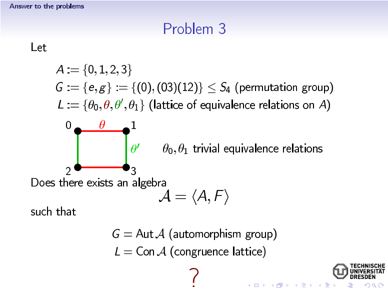
Problem:
(*) $\exists F : G = \operatorname{Aut} F, \ L = \operatorname{Con} F$?
Answer: No.
Proof:
(*) $\implies F \triangleright Q := \{g^\bullet, \theta, \theta'\}$
w.l.o.g. $F = \operatorname{Pol} Q$. Then
$\operatorname{Inv} F = \operatorname{Inv} \operatorname{Pol} Q =_{Thm.} [Q]_{\mathrm{RA}}$
and we get from (*):

$$G^\bullet = (\operatorname{Aut} F)^\bullet = S_A^\bullet \cap \operatorname{Inv} F = S_A^\bullet \cap [Q]_{\mathrm{RA}}$$

$$L = \operatorname{Con} F = \operatorname{Eq}(A) \cap \operatorname{Inv} F = \operatorname{Eq}(A) \cap [Q]_{\mathrm{RA}}$$

in contradiction to $\exists h^\bullet \in S_A^\bullet \cap [Q]_{\mathrm{RA}} : h \notin G$

Answer to Problem 3



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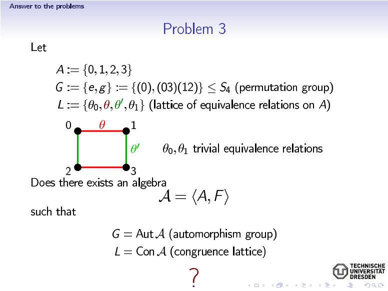

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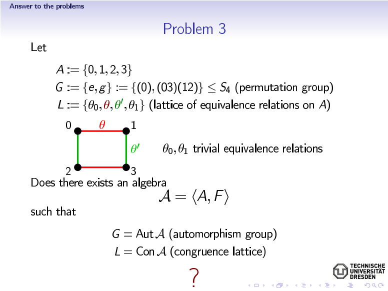
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Problem 3

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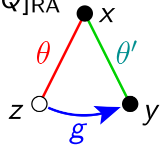
namely $h^\bullet := L_\varphi(\theta, \theta', g^\bullet)$

$$= \{(x,y) \in A^2 \mid \underbrace{\exists z : x\theta z \wedge z g^\bullet y \wedge x\theta' y}_{\varphi}\}$$



i.e. $h = (02)(13) \notin G$

Problem 4

How to recognize whether

$$\operatorname{Aut}\Gamma_1 \subseteq \operatorname{Aut}\Gamma_2 \qquad (*)$$

for graphs $\Gamma_1 = (V, E_1)$, $\Gamma_2 = (V, E_2)$?

e.g.

How to construct all graphs $\Gamma_2 = (V, E_2)$ satisfying $(*)$?

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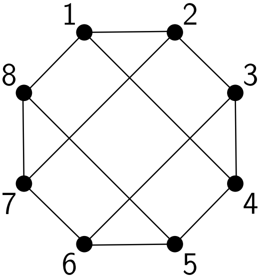

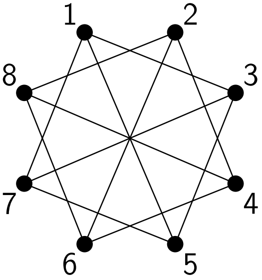

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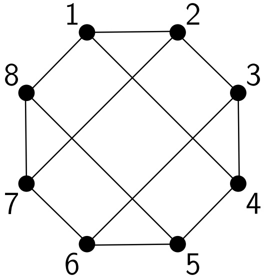
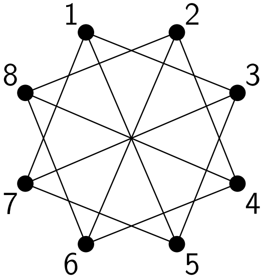
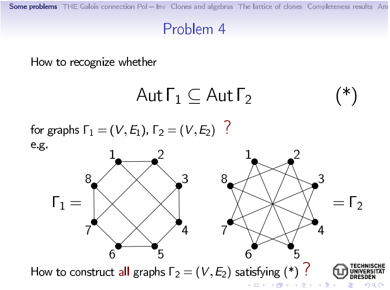




$= \Gamma_2$

How to construct all graphs $\Gamma_2 = (V, E_2)$ satisfying (*) ?

Answer to Problem 4



General answer:

$$\operatorname{Aut}\Gamma_1 = \operatorname{Aut} E_1 \subseteq \operatorname{Aut} E_2 = \operatorname{Aut}\Gamma_2$$

$$\iff \quad \operatorname{Inv}\operatorname{Aut} E_1 \supseteq \operatorname{Inv}\operatorname{Aut} E_2$$

$$\iff \quad [E_1]_{KA} \supseteq [E_2]_{KA} \text{ (by Theorem)}$$

$$\iff \quad \exists \text{ first order formula } \varphi \in \Phi(\exists, \wedge, \vee, \neg, =) : E_2 = L_\varphi(E_1)$$

Answer to Problem 4

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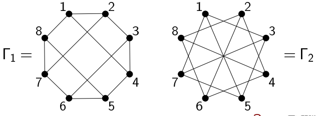

How to construct **all** graphs $\Gamma_2 = (V, E_2)$ satisfying (*) ?

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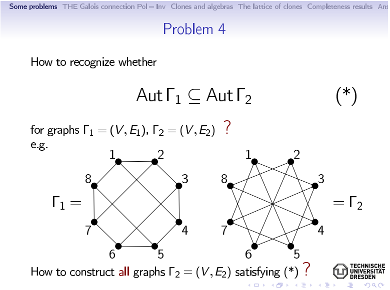

Concrete answer for example:
YES!

$E_2 = (E_1 \circ E_1) \setminus \Delta_V$, i.e.

$E_2 = L_\varphi(E_1) = \{(x,y) \mid \underbrace{\exists z : xE_1z \wedge zE_1y \wedge \neg(x{=}y)}_{\varphi}\}$

$\Longrightarrow \mathrm{Aut}\,\Gamma_1 \subseteq \mathrm{Aut}\,\Gamma_2.$

General answer:

$$\mathrm{Aut}\,\Gamma_1 = \mathrm{Aut}\,E_1 \subseteq \mathrm{Aut}\,E_2 = \mathrm{Aut}\,\Gamma_2$$

$\Longleftrightarrow$ $\mathrm{Inv}\,\mathrm{Aut}\,E_1 \supseteq \mathrm{Inv}\,\mathrm{Aut}\,E_2$

$\Longleftrightarrow$ $[E_1]_{\mathrm{KA}} \supseteq [E_2]_{\mathrm{KA}}$ (by Theorem)

$\Longleftrightarrow$ $\exists$ first order formula $\varphi \in \Phi(\exists, \wedge, \vee, \neg, =) : E_2 = L_\varphi(E_1)$

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Problem 5

What can be said about the computational complexity of

Constraint Satisfaction Problems (CSP)

Γ set of (finitary) relations on a domain D.

General (algebraic) definition of CSP:

CSP(Γ) := set of problems of the form

Does there exist a relational homomorphism

$$(V, \Sigma) \to (D, \Sigma')$$

(between relational systems of the same type) where $\Sigma' \subseteq \Gamma$?

Special CSP:
GRAPH COLORABILITY, *GRAPH ISOMORPHISM*, *SATISFIABILITY (SAT)*

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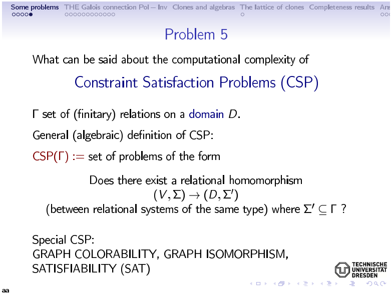
Computational complexity of CSP (cf. Problem 5)

Theorem (*P. Jeavons* 1998). $\Gamma_1, \Gamma_2 \subseteq \mathrm{Rel}(A)$ finite.

$\Gamma_1 \subseteq \mathrm{Inv}\,\mathrm{Pol}\,\Gamma_2 \implies$ CSP(Γ_1) can be reduced to CSP(Γ_2) in polynomial time.



in particular

$\mathbf{compl}(\mathrm{CSP}(\Gamma)) = \mathbf{compl}(\mathrm{CSP}([\Gamma]_{\mathrm{RA}}))$

depends only on the clone $\mathrm{Pol}\,\Gamma$.

large $\Gamma \longleftrightarrow$ small $\mathrm{Pol}\,\Gamma$

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