Representations of Boolean lattices by annihilators in associative rings

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All rings are associative with identity $1 \neq 0$, unless otherwise stated.

Let R be a ring and let $I \subseteq R$ be a subgroup of $(R, +)$.
I is a left ideal — $\forall_{r \in R} \forall_{i \in I} \quad ri \in I \quad (RI \subseteq I)$
I is a right ideal — $\forall_{r \in R} \forall_{i \in I} \quad ir \in I \quad (IR \subseteq I)$
I is a two-sided ideal (an ideal) — I is a left ideal that is also a right ideal.
The fact that I is a two-sided ideal of R will be denoted by $I \lhd R$.

$\mathcal{I}_l(R)$ – the set of all left ideals in R
$\mathcal{I}_r(R)$ – the set of all right ideals in R
$\mathcal{I}(R)$ – the set of two-sided ideals in R

$\mathcal{I}(R) = \mathcal{I}_l(R) \cap \mathcal{I}_r(R)$
If R is commutative ring then $\mathcal{I}_l(R) = \mathcal{I}_r(R) = \mathcal{I}(R)$.

For every ring R the sets $\mathcal{I}_l(R), \mathcal{I}_r(R), \mathcal{I}(R)$ ordered by inclusion are lattices with operations:

$$I \vee J = I + J \quad \text{and} \quad I \wedge J = I \cap J. \tag{1}$$

All these lattices have the smallest element 0 and the largest element R, and all $\mathcal{I}_l(R), \mathcal{I}_r(R), \mathcal{I}(R)$ are complete and modular.

The lattice $\mathcal{I}(R)$ is a sublattice of $\mathcal{I}_l(R)$ and also a sublattice of $\mathcal{I}_r(R)$.

Division ring — a ring R in which lattices $\mathcal{I}_l(R)$ and $\mathcal{I}_r(R)$ are a two-element chains $\{0, R\}$ (ring in which every nonzero element a has a multiplicative inverse, i.e., an element x with $ax = xa = 1$).
Examples: fields, ring of quaternions
Simple ring – a ring R in which the lattice $\mathcal{I}(R)$ is a two-element chain $\{0, R\}$}.
Examples: division rings, $M_n(K)$ a full ring of n-by-n matrices over field K

R. L. Blair, Ideal lattices and the structure of rings, Trans. Amer. Math. Soc. 75 (1953), 136-153.

Theorem

A ring R has a right ideal lattice $\mathcal{I}_r(R)$ which is a Boolean lattice if and only if R is isomorphic with a finite direct sum of division rings.

Theorem

1) A ring R has a two-sided ideal lattice $\mathcal{I}(R)$ which is a Boolean lattice if and only if R is isomorphic with a finite direct sum of simple rings.
2) The lattice $\mathcal{I}(R)$ is a Boolean lattice if and only if $\mathcal{I}(R)$ is complemented lattice.

Agreements

If $\emptyset \neq X \subseteq R$ is a subset, then let:
$\mathrm{l}(X)$ be the left annihilator of X in R

$$\mathrm{l}(X) = \{r \in R : \forall_{x \in X}\ rx = 0\},$$

$\mathrm{r}(X)$ be the right annihilator of X in R

$$\mathrm{r}(X) = \{r \in R : \forall_{x \in X}\ xr = 0\},$$

$\mathcal{A}_l(R)$ the set of all left annihilators in R,
$\mathcal{A}_r(R)$ the set of all right annihilators in R,

$$\mathcal{A}_l(R) \subseteq \mathcal{I}_l(R) \text{ and } \mathcal{A}_r(R) \subseteq \mathcal{I}_r(R)$$

$\mathcal{A}_l(R)$ ordered by inclusion is a complete lattice with operations:

$$I \vee J = \mathrm{l}(\mathrm{r}(I) \cap \mathrm{r}(J)) \quad \text{and} \quad I \wedge J = I \cap J \quad \text{for} \quad I, J \in \mathcal{A}_l(R)$$

$\mathcal{A}_r(R)$ ordered by inclusion is a complete lattice with operations:

$$I \vee J = \mathrm{r}(\mathrm{l}(I) \cap \mathrm{l}(J)) \quad \text{and} \quad I \wedge J = I \cap J \quad \text{for} \quad I, J \in \mathcal{A}_r(R).$$

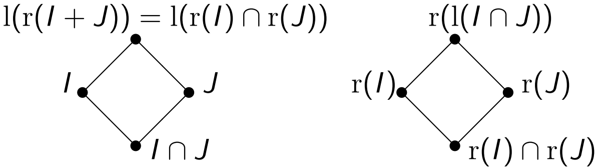
Properties:
1) If $I \lhd R$ then also $\mathrm{l}(I) \lhd R$ and $\mathrm{r}(I) \lhd R$.
2) For any $X \subseteq R$ the set $\mathrm{l}(\mathrm{r}(X))$ is the smallest left annihilator which contain X.
3) If $I \lhd R$ then also $\mathrm{l}(\mathrm{r}(I)) \lhd R$.
4) Both of lattices $\mathcal{A}_l(R)$ and $\mathcal{A}_r(R)$ have the smallest element 0 and the largest element R.

Theorem (J. Krempa, M.J.)

For every finite lattice L there exist noncommutative rings R_1, R_2 and commutative ring R_3 such that:
1) L can be represented as a sublattice of a lattice $\mathcal{A}_l(R_1)$.
2) L can be represented as a sublattice of a lattice $\mathcal{A}_r(R_2)$.
3) L can be represented as a sublattice of a lattice $\mathcal{A}_l(R_3) = \mathcal{A}_r(R_3)$.

Galois correspondence

For any ring R the mapping $\mathcal{A}_l(R) \to \mathcal{A}_r(R)$ given by $I \to \mathrm{r}(I)$ is an anti-isomorphism of complete lattices. The inverse function is given by $J \to \mathrm{l}(J)$ for $J \in \mathcal{A}_r(R)$.



The set of two-sided ideals which are left annihilators, denoted by $\mathcal{B}_l(R)$, is a sublattice of $\mathcal{A}_l(R)$.
The set of two-sided ideals which are right annihilators, denoted by $\mathcal{B}_r(R)$, is a sublattice of $\mathcal{A}_r(R)$.

Lattices of annihilators

$\mathcal{B}_l(R)$ $\mathcal{B}_r(R)$

$\mathcal{A}_l(R)$ $\mathcal{A}_r(R)$

For any ring R if $\mathcal{A}_l(R) = \mathcal{A}_r(R)$ (as subsets of 2^R) then $\mathcal{A}_l(R) = \mathcal{A}_r(R) = \mathcal{B}_l(R) = \mathcal{B}_r(R)$.

If R is commutative ring then $\mathcal{A}_l(R) = \mathcal{A}_r(R)$.

Statement

If $\mathcal{I}_l(R)$ is a Boolean lattice then $\mathcal{A}_l(R) = \mathcal{A}_r(R)$ is a Boolean lattice equal to $\mathcal{I}_l(R)$.
If $\mathcal{I}(R)$ is a Boolean lattice then $\mathcal{B}_l(R) = \mathcal{B}_r(R)$ is a Boolean lattice equal to $\mathcal{I}(R)$.

Example of a ring in which $\mathcal{A}_l(R) = \mathcal{A}_r(R)$ is Boolean lattice and this lattice is not equal to $\mathcal{I}_l(R)$

Let K be a field and let $R = K[x, y]/(xy)$ be a homomorphic image of the polynomial ring in commuting variables x,y. This ring is commutative hence $\mathcal{A}_l(R) = \mathcal{A}_r(R) = \mathcal{B}_l(R) = \mathcal{B}_r(R)$ and the set of annihilators is $\{0, R, (x), (y)\}$. For example $(x)^2$ is an ideal but not an annihilator in R.

An **element** $x \in R$ s said to be **nilpotent** if there exists a natural number n such that $x^n = 0$.
The **left (right) ideal** I in R is said to be **nilpotent** if there exists a natural number n such that $I^n = 0$.
A ring R is called **reduced** if has no nonzero nilpotent elements.

Let R be a ring with no nonzero nilpotent ideals. Then for each ideal I of R $\mathrm{l}(I) = \mathrm{r}(I)$. This implies that $\mathcal{B}_l(R) = \mathcal{B}_r(R)$.

Theorem (cf. S. Steinberg, *Lattice ordered Rings and Modules*)

Let R be a ring with no nonzero nilpotent ideals.Then $\mathcal{B}_l(R)$ is a Boolean lattice.

If R is a reduced ring then for every $x \in R$ we have $\mathrm{l}(x) = \mathrm{r}(x)$. Hence $\mathcal{A}_l(R) = \mathcal{A}_r(R)$ and R has no nonzero nilpotent ideals.
From this and the above theorem follows

Corollary

If R is a reduced ring then $\mathcal{A}_l(R) = \mathcal{A}_r(R)$ is a complete Boolean lattice.