Ordered Kovács-Newman semigroups

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Motivation

Theory of Finite Semigroups

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- Pseudovarieties of finite semigroups

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- Lattice of Pseudovarieties and irreducibility

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Kovács-Newman groups and semigroups (Rhodes, Steinberg, 2004 and $\mathfrak{q}$-theory, 2009) and ordered case

Pseudovarieties

Recall operators $\mathbb{H}, \mathbb{S}, \mathbb{P}_{fin}$.

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Class of finite semigroups closed under finite direct products, subsemigroups and homomorphic images.

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Define operation of join in the lattice of pseudovarieties

Join

$\mathsf{V} \vee \mathsf{W} = \mathbb{HSP}_{fin}(\mathsf{V} \cup W)$

Irreducibility

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Tools to study irreducibility:

- Syntactic methods
- Profinite semigroups
- Join irreducible semigroups (e.g. Kovács-Newman)

Definitions

Preliminary definitions:

- G is **subdirect product** of G_1, G_2 ($G \ll G_1 \times G_2$) if $G \leq G_1 \times G_2$ and projections $\pi_i \colon G \to G_i$ are surjective
- G is **subdirectly indecomposable** if at least one of the projection in subdirect product is isomorphism

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- G is **subdirectly indecomposable** if at least one of the projection in subdirect product is isomorphism
- G is **monolithic** if it has unique minimal (non-trivial) normal subgroup (called monolith)

Note that subdirectly indecomposable $\Leftrightarrow$ monolithic.

Definition

Named after Kovács, Newman (1966) studying varieties of groups.

Kovács-Newman (KN) group (Rhodes, Steinberg, 2004)

Non-trivial finite group G is KN **if** whenever there is a diagram

$$G \overset{\varphi}{\twoheadleftarrow} H \ll G_1 \times G_2,$$

with H, G_1, G_2 finite groups, φ factors through one of the projections.

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Since $G \in \mathsf{H}_1 \vee \mathsf{H}_2$ if there is such diagram with $G_i \in \mathsf{H}_\mathsf{i}$, also pseudovariety $\mathbb{HSP}_{fin}(G)$, with G being KN group, is join irreducible pseudovariety.

Classification, Example, Application

Theorem ($\mathfrak{q}$-theory, 2009)

Finite group G is KN $\Leftrightarrow$ G is monolithic with non-abelian monolith.

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Symmetric groups $\mathbb{S}_n$ for $n \geq 5$ are KN. (unique normal subgroups $\mathbb{A}_n$ are non-abelian)

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Application

Pseudovariety G of finite groups is irreducible.

Semigroup Setting

Definition is analogic to that of KN group:

Kovács-Newman (KN) semigroup

Non-trivial finite semigroup S is KN **if** whenever there is a diagram

$$S \overset{\varphi}{\twoheadleftarrow} T \ll S_1 \times S_2,$$

with T, S_1, S_2 finite semigroups, φ factors through one of the projections.

The classification is more complicated. We need to define group mapping semigroups.

Semigroup Setting

Kernel of a semigroups S is a minimal two-sided ideal denoted $K(S)$.

Group mapping over kernel (GM) semigroups

Semigroup S is GM over its kernel $K(S)$ if S acts faithfully on both left and right of $K(S)$ and $K(S)$ contains non-trivial subgroup.

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Theorem

Finite semigroup is a Kovács-Newman semigroup if and only if it is group mapping semigroup over its kernel and the maximal subgroup of the kernel is KN group (monolithic with non-abelian monolith).

Applications

Non-isomorphic KN semigroups generate irreducible distinct pseudovarieties.

Theorem ($\mathfrak{q}$-theory)

Pseudovariety CS of completely simple semigroups is irreducible.

Theorem ($\mathfrak{q}$-theory)

Pseudovariety CR of completely regular semigroups is irreducible.

Ordered Case

Group mapping semigroups are unorderable.

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Theorem

KN ordered semigroups are precisely KN semigroups ordered by equality.

Application

Irreducibility of certain pseudovarieties in lattice of pseudovarieties of ordered semigroups.

Closure

Thank you so much for your attention.