Invertible binary algebras principally isotopic to a group

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Quasigroup, Invertible algebra

Definition

A binary groupoid $Q(A)$ is a non-empty set Q together with a binary operation A. Binary groupoid $Q(A)$ is called quasigroup if for all ordered pairs $(a, b) \in Q^2$ exists unique solutions $x, y \in Q$ of the following equations:

$$A(a, x) = b, A(y, a) = b.$$

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A binary algebra $(Q; \Sigma)$ is called invertible algebra or system of quasigroups if each operation in Σ is a quasigroup operation.

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Parastrophies

With each invertible algebra $(Q; \Sigma)$ the next five invertible algebras are connected:

$$(Q; \Sigma^{-1}),\ \ (Q; {}^{-1}\Sigma),\ \ (Q; {}^{-1}(\Sigma^{-1})),\ \ (Q; ({}^{-1}\Sigma)^{-1}),\ \ (Q; \Sigma^{*}),$$

where

$$\begin{aligned} \Sigma^{-1} &= \{A^{-1} \mid A \in \Sigma\}, \\ {}^{-1}\Sigma &= \{{}^{-1}A \mid A \in \Sigma\}, \\ {}^{-1}(\Sigma^{-1}) &= \{{}^{-1}(A^{-1}) \mid A \in \Sigma\}, \\ ({}^{-1}\Sigma)^{-1} &= \{({}^{-1}A)^{-1} \mid A \in \Sigma\}, \\ \Sigma^{*} &= \{A^{*} \mid A \in \Sigma\}. \end{aligned}$$

Each of these invertible algebras are called parastrophies of the algebra $(Q; \Sigma)$.

Second-order formula

Let us recall that the following absolutely closed second-order formula:

$$\forall X_1, \ldots, X_m \forall x_1, \ldots, x_n \quad (\omega_1 = \omega_2),$$
$$\forall X_1, \ldots, X_k \exists X_{k+1} \ldots, X_m \forall x_1, \ldots, x_n \quad (\omega_1 = \omega_2),$$

where ω_1, ω_2 are words written in the functional variables, $X_1, \ldots, X_m$, and in the objective variables, $x_1, \ldots, x_n$, are called $\forall(\forall)$-identity or hyperidentity and $\forall\exists(\forall)$-identity.

The satisfiability (truth) of these second order formula in the algebra $(Q; \Sigma)$ is in the sense of functional quantifiers $(\forall X_i)$ and $(\exists X_j)$ meaning: "for every value $X_i = A \in \Sigma$ of the corresponding arity" and "there exists a value $X_j = A \in \Sigma$ of the corresponding arity".

Isotopy

Definition

The groupoid $Q(A)$ is called isotopic to the groupoid $Q(B)$ if exist three maps α, β, γ of Q to Q such that

$$\gamma B(x, y) = A(\alpha x, \beta y)$$

for all $x, y \in Q$. The isotopy of the form $T = (\alpha, \beta, \varepsilon)$, where ε is the identity map, is called principal isotope.

Auxiliary results

In 1961 V.D. Belousov characterised quasigroups isotopic to groups and abelian groups.

Theorem

Let the nonempty set Q form a quasigroup under four operations A_i (i=1,2,3,4). If these operations satisfy the following identity:

$$A_1(A_2(x,y),z) = A_3(x, A_4(y,z)),$$

then there exists an operation $(\cdot)$ under which Q forms a group isotopic to all these four quasigroups.

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Auxiliary results

Theorem

Let the nonempty set Q form a quasigroup under six operations A_i (i=1,2,3,4,5,6). If these operations satisfy the following identity:

$$A_1(A_2(x,y),A_3(z,u)) = A_4(A_5(x,z),A_6(y,u)),$$

then there exists an operation $(\cdot)$ under which Q forms an abelian group isotopic to all these six quasigroups, i.e.

$$\begin{aligned} A_1(x,y) &= \alpha x \cdot \beta y, & A_4(x,y) &= \chi x \cdot \varphi y, \\ A_2(x,y) &= \alpha^{-1}(\gamma x \cdot \delta y), & A_5(x,y) &= \chi^{-1}(\gamma x \cdot \theta y), \\ A_3(x,y) &= \beta^{-1}(\theta x \cdot \psi y), & A_6(x,y) &= \varphi^{-1}(\delta x \cdot \psi y), \end{aligned}$$

where $\alpha,\beta,\gamma,\delta,\chi,\varphi,\psi,\theta$ are permutations of Q.

Principal Isotopy

We say that a binary algebra $(Q; \Sigma)$ is isotopic to the groupoid $Q(\cdot)$, if each operation in Σ is isotopic to the groupoid $Q(\cdot)$, i.e. for every operation $A \in \Sigma$ there exists permutations $\alpha_A, \beta_A, \gamma_A$ of Q, that:

$$\gamma_A A(x, y) = \alpha_A x \cdot \beta_A y,$$

for every $x, y \in Q$. Isopoty is called principal if $\gamma_A = \epsilon(\epsilon$ - unit permutation) for every $A \in \Sigma$.

Characterization of invertible binary algebras

We obtained characterizations of invertible algebras principally isotopic to a group or an abelian group by second-order formulas.

Main results

Theorem

The invertible algebra $(Q;\Sigma)$ *is principally isotopic to a group, if and only if the following second-order formula*

$$A({}^{-1}A(B(x, B^{-1}(y,z)), u), v) = B(x, B^{-1}(y, A({}^{-1}A(z,u), v))),$$

is valid in the algebra $(Q;\Sigma \cup \Sigma^{-1} \cup^{-1} \Sigma)$ *for all* $A, B \in \Sigma$.

Main results

Theorem

The invertible algebra $(Q;\Sigma)$ *is principally isotopic to an abelian group if and only if the following second-order formula:*

$$A({}^{-1}A(B(x,z),y), A^{-1}(u, B(w,y))) =$$

$$= A({}^{-1}A(B(w,z),y), A^{-1}(u, B(x,y))).$$

is valid in the algebra $(Q;\Sigma \cup \Sigma^{-1} \cup^{-1} \Sigma)$ *for all* $A, B \in \Sigma$.

Corollaries

Corollary

The class of quasigroups isotopic to groups is characterized by the following identity:

$$x(y\backslash((z/u)v)) = ((x(y\backslash z))/u)v.$$

Corollary

The class of quasigroups isotopic to abelian groups is characterized by the following identity:

$$((xz)/y)(u\backslash(wy)) = ((wz)/y)(u\backslash(xy)).$$

Quasigroups which are isotopic to abelian groups can be characterized by the following identity with four variables:

$$x\backslash(y(u\backslash v)) = u\backslash(y(x\backslash v)).$$

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Proposition

$$X(Y(x,y),z) = Y(X(z,y),x) \tag{1}$$

$$X(Y(x,y),z) = X(Y(z,y),x) \tag{2}$$

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The invertible algebra $(Q;\Sigma)$ with hyperidentity either (1), (2) or (3) is isotopic to an abelian group.

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