Lattice Congruences Preserving Involutions

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Integer Part, Cardinality, Smallest and Largest Equivalence, Atoms of a Lattice with Smallest Element

- $x \in \mathbb{R}$

Notation

$\lfloor x \rfloor = \max\{n \in \mathbb{Z} \mid n \leq x\}$

- S: a set

Notation

- $|S|$: the cardinality of S
- $\Delta_S = \{(x, x) \mid x \in S\}$
- $\nabla_S = S \times S$

- L: lattice with a 0

Notation

$\mathrm{At}(L)$: the set of the atoms of L

Isomorphisms, Congruences

- τ: a similarity type of universal algebras
- $n \in \mathbb{N}$; $\kappa_1, \dots, \kappa_n$: constants from τ
- $\mathbb{V}$: a variety of algebras of type τ
- A and B: algebras with reducts in $\mathbb{V}$
- $U \subseteq A^2$

Notation

- $A \cong_{\mathbb{V}} B$: the τ–reducts of A and B are isomorphic
- $\mathrm{Con}_{\mathbb{V}}(A)$: the lattice of congruences of the τ–reduct of A
- $\mathrm{Con}_{\mathbb{V}_{\kappa_1 \dots \kappa_n}}(A) = \{\theta \in \mathrm{Con}_{\mathbb{V}}(A) \mid (\forall\, i \in \overline{1,n})\,(\kappa_i^A/\theta = \{\kappa_i^A\})\}$: complete (thus bounded) lattice, sublattice of $\mathrm{Con}_{\mathbb{V}}(A)$
- $Cg_{A,\mathbb{V}}(U)$: the congruence of the τ–reduct of A generated by U

Exception: if $\mathbb{V}$ is the variety of (bounded) lattices, then we omit the index $_{\mathbb{V}}$.

If $A \times B$ has no skew congruences, in particular if $\mathbb{V}$ is congruence–distributive:

- $\mathrm{Con}_{\mathbb{V}_{\kappa_1 \dots \kappa_n}}(A \times B) \cong \mathrm{Con}_{\mathbb{V}_{\kappa_1 \dots \kappa_n}}(A) \times \mathrm{Con}_{\mathbb{V}_{\kappa_1 \dots \kappa_n}}(B)$

If $\mathbb{V}$: variety of bounded ordered structures:

- $\mathrm{Con}_{\mathbb{V}01}(A)$: the lattice of the *pseudo–identical* congruences of A

i–lattices and bi–lattices

- Algebras (and their reducts): designated by their underlying sets.

Definition ((bounded) involution lattice ((b)i–lattice))

$(L, \cdot')$, where:

- L: (bounded) lattice
- $\cdot'$: *involution* on L, that is:

$$\cdot' : L \to L$$
$$(\forall\, a \in L)\,(a'' = a)$$
$$(\forall\, a, b \in L)\,(a \leq b \Rightarrow b' \leq a')$$

Thus: (bounded) involution lattice: $(L, \cdot')$, where:

- L: self–dual (bounded) lattice
- $\cdot' : L \to L$: dual lattice automorphism of L

Notation

- $\mathbb{I}$: the variety of i–lattices
- $\mathbb{BI}$: the variety of bi–lattices

De Morgan Algebras and (Pseudo–)Kleene Algebras

Definition (De Morgan algebra)

- distributive bi–lattice

Definition (pseudo–Kleene algebra)

bi–lattice $(L, \cdot')$ fulfilling:

- for all $a, b \in L$, $a \wedge a' \leq b \vee b'$

$\cdot'$: *Kleene complement*

Definition (Kleene algebra (Kleene lattice) (bounded normal i–lattice))

- distributive pseudo–Kleene algebra

Notation

- $\mathbb{PKA}$: the variety of pseudo–Kleene algebras
- $\mathbb{KL}$: the variety of Kleene lattices

Paraorthomodularity, BZ–lattices

Definition (paraorthomodular bi–lattice)

bi–lattice $(L, \cdot')$ such that, for all $a, b \in L$:

- $a \leq b$ and $a' \wedge b = 0 \Rightarrow a = b$

Definition (Brouwer–Zadeh lattice (BZ–lattice))

$(L, \cdot^{\sim})$, where:

- L: bi–lattice
- $\cdot^{\sim} : L \rightarrow L$ such that, for all $a, b \in L$:

$$a \wedge a^{\sim} = 0$$
$$a \leq a^{\sim\sim}$$
$$a^{\sim\prime} = a^{\sim\sim}$$
$$a \leq b \Rightarrow b^{\sim} \leq a^{\sim}$$

$\cdot^{\sim}$: *Brouwer complement*

Notation

- $\mathbb{BZL}$: the variety of Brouwer–Zadeh lattices

Memo on Orthomodular Lattices, Antiortholattices

Remark (characterizations for orthomodular lattices)

L: bi–lattice. Then: L: orthomodular lattice iff:

- L: paraorthomodular
- L: *ortholattice*, that is: $\{a \in L \mid a \wedge a' = 0\} = L$

iff:

- L: paraorthomodular
- $(L, \cdot^{\sim} = \cdot')$: BZ–lattice

Remark (characterization for antiortholattices and other properties)

L: pseudo–Kleene algebra. Then $(L, ^{\sim})$: antiortholattice iff:

- $\{a \in L \mid a \wedge a' = 0\} = \{0, 1\}$
- $\cdot^{\sim} : L \to L$ is the *trivial Brouwer complement*: $\begin{cases} 0^{\sim} = 1 \\ (\forall\, a \in L \setminus \{0\})\,(a^{\sim} = 0) \end{cases}$

Note that:

- if L: antiortholattice, then L: paraorthomodular
- if L: BZ–lattice/pseudo–Kleene algebra with 0: meet–irreducible, then L: antiortholattice
- thus any self–dual bounded chain: antiortholattice

Ordinal/Horizontal Sum of (bi/BZ)–lattices

- H: lattice with a 1; K: lattice with a 0
- A: lattice with a 1; A': the dual of A; $f : A \to A'$: dual lattice isomorphism; B: bi–lattice
- L and M: bounded lattices / bi–lattices / BZ–lattices; $|L|, |M| > 2$

Notation (ordinal sum: $\oplus$, horizontal sum: $\boxplus$)

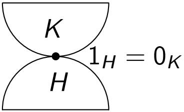
- $H \oplus K$: the ordinal sum of H with K
- $L \boxplus M$: the horizontal sum of L with M

$\cdot'^{A\oplus B\oplus A'}|_A = f$

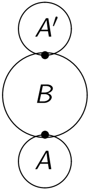
$\cdot'^{A\oplus B\oplus A'}|_B = \cdot'^B$

$\cdot'^{A\oplus B\oplus A'}|_{A'} = f^{-1}$

$H \oplus K$:



$A \oplus B \oplus A' \in \mathbb{I}$:



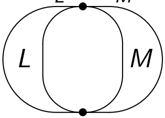
$A \oplus B \oplus A' \in \mathbb{PKA}$ iff:
A: bounded lattice and $B \in \mathbb{PKA}$

$\cdot'^{L\boxplus M}|_L = \cdot'^L$, $\cdot'^{L\boxplus M}|_M = \cdot'^M$

$\cdot^{\sim L\boxplus M}|_L = \cdot^{\sim L}$, $\cdot^{\sim L\boxplus M}|_M = \cdot^{\sim M}$

$L \boxplus M \in \mathbb{BI}$:

$1_L = 1_M$



$0_L = 0_M$

$L \boxplus M \in \mathbb{BZL}$ iff:
L or M: ortholattice

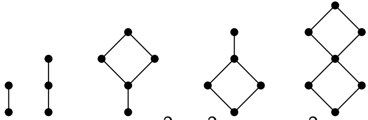
If:

- A: non–trivial bounded lattice
- $\cdot^{\sim} : A \oplus B \oplus A' \to A \oplus B \oplus A'$: the trivial Brouwer complement

then: $A \oplus B \oplus A'$: antiortholattice

Distinguished bi–lattices

- for any $n \in \mathbb{N}^*$, $\mathcal{L}_n$: the n–element chain
- M_3: the *diamond*
- N_5: the *pentagon*
- B_6: the *benzene ring*



$\mathcal{L}_n$: *antiortholattice*

$\mathrm{Con}_{\mathbb{I}}(M_3) = \mathrm{Con}(M_3) \cong \mathcal{L}_2$

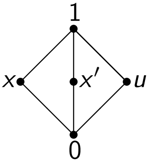
$N_5 \cong_{\mathbb{I}} \mathcal{L}_3 \boxplus \mathcal{L}_4 \in \mathbb{BI} \setminus \mathbb{PKA}$

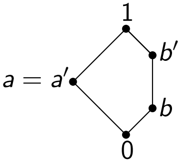
$B_6 \in \mathbb{PKA}$

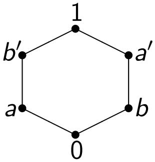
$B_6 \cong \mathcal{L}_4 \boxplus \mathcal{L}_4$

$M_3 \cong_{\mathbb{I}} \mathcal{L}_2^2 \boxplus \mathcal{L}_3 \in \mathbb{PKA}$

$B_6 \ncong_{\mathbb{I}} \mathcal{L}_4 \boxplus \mathcal{L}_4 \in \mathbb{BI} \setminus \mathbb{PKA}$







$\mathrm{Con}(\mathcal{L}_n) \cong \mathcal{L}_2^{n-1}$

$\mathrm{Con}_{\mathbb{I}}(\mathcal{L}_n) \cong \mathcal{L}_2^{\lfloor n \rfloor}$

if $n \geq 2$: $\mathrm{Con}_{\mathbb{BZL}}(\mathcal{L}_n) \cong \mathcal{L}_2^{\lfloor n \rfloor - 1}$

$\mathrm{Con}(N_5) \cong \mathcal{L}_2 \oplus \mathcal{L}_2^2$

$\mathrm{Con}_{\mathbb{I}}(N_5) \cong \mathcal{L}_3$

$\mathrm{Con}(B_6) \cong \mathcal{L}_2^2 \oplus \mathcal{L}_2^2$

$\mathrm{Con}_{\mathbb{I}}(B_6) \cong \mathcal{L}_2 \oplus \mathcal{L}_2^2$

$\mathrm{Con}_{\mathbb{I}}(\mathcal{L}_4 \boxplus \mathcal{L}_4) \cong \mathcal{L}_2^2 \oplus \mathcal{L}_2$

Congruences of i–lattices

Until mentioned otherwise:

- L: i–lattice
- $U \subseteq L^2$
- $\theta \in \mathrm{Con}(L)$

Notation

$U' = \{(a', b') \mid (a, b) \in U\}$

Proposition

- $Cg_{L,\mathbb{I}}(\theta) = \theta \vee \theta'$
- $Cg_L(U)' = Cg_L(U')$
- $Cg_{L,\mathbb{I}}(U) = Cg_L(U) \vee Cg_L(U')$
- $\mathrm{Con}_{\mathbb{I}}(L) = \{Cg_{L,\mathbb{I}}(\theta) \mid \theta \in \mathrm{Con}(L)\}$

Proposition (well known)

The variety of distributive lattices has the congruence extension property (CEP).

Corollary

The variety of distributive i–lattices has the CEP.

Congruences of i–lattices, Atoms, Subdirect Irreducibility

Proposition

If: $(\forall\, \alpha \in \mathrm{Con}(L))\, (\alpha \leq \theta$ or $\theta \leq \alpha)$, then $\theta \in \mathrm{Con}_{\mathbb{I}}(L)$.

Corollary

If $\mathrm{Con}(L)$ *is a chain, then* $\mathrm{Con}_{\mathbb{I}}(L) = \mathrm{Con}(L)$.

Proposition

If $\mathrm{Con}(L)$ is a Boolean lattice, then $\mathrm{Con}_{\mathbb{I}}(L)$ is a Boolean sublattice of $\mathrm{Con}(L)$.

Corollary (particular cases: finite De Morgan/Kleene algebras)

If L*: finite modular i–lattice, then* $\mathrm{Con}_{\mathbb{I}}(L)$*: Boolean lattice, in particular* $|\mathrm{Con}_{\mathbb{I}}(L)| \in \{2^n \mid n \in \mathbb{N}\}$.

Proposition

- $Cg_{L,\mathbb{I}}(\theta) \in \mathrm{At}(\mathrm{Con}_{\mathbb{I}}(L))$ iff $\theta \in \mathrm{At}(\mathrm{Con}(L))$
- $\mathrm{At}(\mathrm{Con}_{\mathbb{I}}(L)) = \mathrm{At}(\mathrm{Con}(L)) \cap \mathrm{Con}_{\mathbb{I}}(L) = \{Cg_{L,\mathbb{I}}(\theta) \mid \theta \in \mathrm{At}(\mathrm{Con}(L))\}$

Corollary (as an i–lattice)

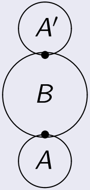
L is subdirectly irreducible iff $\mathrm{At}(\mathrm{Con}(L)) = \{\alpha, \alpha'\}$ *for some* $\alpha \in \mathrm{Con}(L)$.

Ordinal and Horizontal Sums, Recalling Full Congruences of Orthomodular Lattices

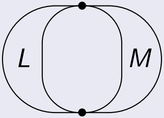
Proposition (the construction $A \oplus B \oplus A'$ and the horizontal sum (holds for finite horizontal sums of bi/BZ–lattices other than $\mathcal{L}_1$, $\mathcal{L}_2$))

- A: lattice with a 1, A': the dual of A
- B, L, M: bi–lattices, $|L|, |M| > 2$

$A \oplus B \oplus A' \in \mathbb{I}$



$L \boxplus M \in \mathbb{BI}$



Then:

- $\mathrm{Con}_{\mathbb{I}}(A \oplus B \oplus A') \cong \mathrm{Con}(A) \times \mathrm{Con}_{\mathbb{I}}(B)$
- $\mathrm{Con}_{\mathbb{I}}(L \boxplus M) = \mathrm{Con}_{\mathbb{BI}01}(L \boxplus M) \cup \{\nabla_{L \boxplus M}\} \cong (\mathrm{Con}_{\mathbb{BI}01}(L) \times \mathrm{Con}_{\mathbb{BI}01}(M)) \oplus \mathcal{L}_2$

Proposition (well known; particular case: L: Boolean algebra)

L: orthomodular lattice. Then:

$$\mathrm{Con}(L) = \mathrm{Con}_{\mathbb{I}}(L) = \mathrm{Con}_{\mathbb{BZL}}(L)$$

Congruences of Involution Lattices and Antiortholattices

Proposition

- L: antiortholattice

Then:

$$\begin{aligned}\mathrm{Con}_{\mathbb{BZL}}(L) &= \mathrm{Con}_{\mathbb{BZL}01}(L) \cup \{\nabla_L\}\\ &= \mathrm{Con}_{\mathbb{BI}01}(L) \cup \{\nabla_L\}\\ &\cong \mathrm{Con}_{\mathbb{BI}01}(L) \oplus \mathcal{L}_2\end{aligned}$$

Corollary ($A = \mathcal{L}_2$ in the construction recalled above)

- *B: pseudo–Kleene algebra*

$\mathcal{L}_2 \oplus B \oplus \mathcal{L}_2$: antiortholattice

Then:

- $\mathrm{Con}_{\mathbb{I}}(\mathcal{L}_2 \oplus B \oplus \mathcal{L}_2) \cong \mathrm{Con}_{\mathbb{I}}(\mathcal{L}_2) \times \mathrm{Con}_{\mathbb{I}}(B)$,

thus:

$$|\mathrm{Con}_{\mathbb{I}}(\mathcal{L}_2 \oplus B \oplus \mathcal{L}_2)| = 2 \cdot |\mathrm{Con}_{\mathbb{I}}(B)|$$

- $\mathrm{Con}_{\mathbb{BZL}01}(\mathcal{L}_2 \oplus B \oplus \mathcal{L}_2) = \mathrm{Con}_{\mathbb{BI}01}(\mathcal{L}_2 \oplus B \oplus \mathcal{L}_2) \cong \mathrm{Con}_{\mathbb{I}}(B)$,

thus:

$$|\mathrm{Con}_{\mathbb{BZL}}(\mathcal{L}_2 \oplus B \oplus \mathcal{L}_2)| = |\mathrm{Con}_{\mathbb{I}}(B)| + 1 = |\mathrm{Con}_{\mathbb{I}}(\mathcal{L}_2 \oplus B \oplus \mathcal{L}_2)|/2 + 1$$

Part of a result from:

G. Czédli, C. Mureşan, On Principal Congruences and the Number of Congruences of a Lattice with More Ideals than Filters, arXiv:1711.06394 [math.RA]:

Theorem (an infinite lattice can have any number of congruences between 2 and its number of subsets)

For any infinite cardinal numbers ν, κ with

$$2 \leq \kappa \leq 2^{\nu},$$

there exists a lattice L with:

$$|L| = \nu \text{ and } |\mathrm{Con}(L)| = \kappa$$

Corollary (an infinite i–lattice can have any number of congruences between 2 and its number of subsets)

For any infinite cardinal numbers ν, κ with

$$2 \leq \kappa \leq 2^{\nu},$$

there exists an i–lattice L with:

$$|L| = \nu \text{ and } |\mathrm{Con}_{\mathbb{I}}(L)| = \kappa$$

$n \in \mathbb{N}^*$. By:

G. Czédli, A Note on Finite Lattices with Many Congruences, arXiv:1712.06117 [math.RA].

R. Freese, Computing Congruence Lattices of Finite Lattices, *Proc. Amer. Math. Soc.* **125**, 3457–3463 (1997),

the n–element lattice with the largest number of congruences: $L_1(n) = \mathcal{L}_n$. By:

G. Czédli, A Note on Finite Lattices with Many Congruences, arXiv:1712.06117 [math.RA],

the n–element lattices with the second largest number of congruences: for $n \geq 4$ and every $k \in \overline{1, n-3}$, $L_2(n,k) = \mathcal{L}_k \oplus \mathcal{L}_2^2 \oplus \mathcal{L}_{n-k-2}$. By:

C. Mureșan, J. Kúlin, Some Extremal Values of the Number of Congruences of a Finite Lattice, arXiv:1801.05282 [math.RA]:

the n–element lattices with the third, fourth and fifth largest numbers of congruences:

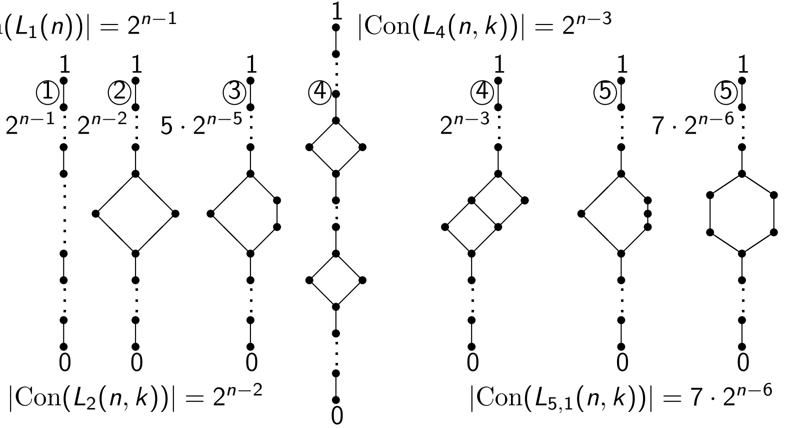
- for $n \geq 5$ and every $k \in \overline{1, n-4}$, $L_3(n,k) = \mathcal{L}_k \oplus N_5 \oplus \mathcal{L}_{n-k-3}$
- for $n \geq 6$ and every $k \in \overline{1, n-5}$, $L_4(n,k) = \mathcal{L}_k \oplus (\mathcal{L}_2 \times \mathcal{L}_3) \oplus \mathcal{L}_{n-k-4}$; for $n \geq 7$ and every $r, s \in \mathbb{N}^*$ such that $r + s \leq n - 5$: $L_4(n,r,s) = \mathcal{L}_r \oplus \mathcal{L}_2^2 \oplus \mathcal{L}_s \oplus \mathcal{L}_2^2 \oplus \mathcal{L}_{n-r-s-4}$
- for $n \geq 6$ and every $k \in \overline{1, n-5}$, $L_{5,1}(n,k) = \mathcal{L}_k \oplus (\mathcal{L}_3 \boxplus \mathcal{L}_5) \oplus \mathcal{L}_{n-k-4}$ and $L_{5,2}(n,k) = \mathcal{L}_k \oplus (\mathcal{L}_4 \boxplus \mathcal{L}_4) \oplus \mathcal{L}_{n-k-4} \cong \mathcal{L}_k \oplus B_6 \oplus \mathcal{L}_{n-k-4}$

$\mathrm{Con}(L_1(n)) \cong \mathcal{L}_2^{n-1}$ $\quad$ $\mathrm{Con}(L_4(n,k)) \cong \mathrm{Con}(L_4(n,r,s)) \cong \mathcal{L}_2^{n-3}$

$\mathrm{Con}(L_2(n,k)) \cong \mathcal{L}_2^{n-2}$

$|\mathrm{Con}(L_3(n,k))| = 5 \cdot 2^{n-5}$ $\quad$ $|\mathrm{Con}(L_{5,2}(n,k))| = 7 \cdot 2^{n-6}$

$|\mathrm{Con}(L_1(n))| = 2^{n-1}$ $\quad$ $|\mathrm{Con}(L_4(n,k))| = 2^{n-3}$



$|\mathrm{Con}(L_2(n,k))| = 2^{n-2}$ $\quad$ $|\mathrm{Con}(L_{5,1}(n,k))| = 7 \cdot 2^{n-6}$

$|\mathrm{Con}(L_4(n,r,s))| = 2^{n-3}$

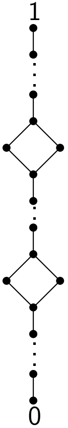
$\mathrm{Con}(L_3(n,k)) \cong (\mathcal{L}_2 \oplus \mathcal{L}_2^2) \times \mathcal{L}_2^{n-5}$

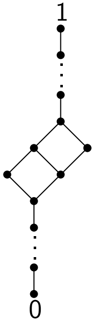
$\mathrm{Con}(L_{5,1}(n,k)) \cong \mathrm{Con}(L_{5,2}(n,k)) \cong (\mathcal{L}_2^2 \oplus \mathcal{L}_2^2) \times \mathcal{L}_2^{n-6}$

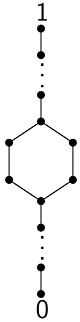
Let: $n \in 2(\mathbb{N} \setminus \{0, 1, 2, 3\})$ and $s \in 2\mathbb{N}^*$. Then:

$L_4(n, (n-s)/2 - 2, s) \in \mathbb{KL}$ $\quad$ $L_{5,2}(n, n/2 - 2) \cong_{\mathbb{I}} \mathcal{L}_{n/2-2} \oplus B_6 \oplus \mathcal{L}_{n/2-2} \in \mathbb{PKA}$

$L_4(n, n/2 - 2) \in \mathbb{KL}$







$M_{5,2}(n, n/2 - 2) \cong_{\mathbb{I}} \mathcal{L}_{n/2-2} \oplus (\mathcal{L}_4 \boxplus \mathcal{L}_4) \oplus \mathcal{L}_{n/2-2} \in \mathbb{BI} \setminus \mathbb{PKA}$

$|\mathrm{Con}_{\mathbb{I}}(L_4(n, (n-s)/2 - 2, s))| = |\mathrm{Con}_{\mathbb{I}}(L_4(n, n/2 - 2))| = 2^{n/2-1} < 5 \cdot 2^{n/2-3} =$
$|\mathrm{Con}_{\mathbb{I}}(L_{5,2}(n, n/2 - 2))| = |\mathrm{Con}_{\mathbb{I}}(M_{5,2}(n, n/2 - 2))|$

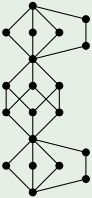
$\mathrm{Con}_{\mathbb{I}}(L_4(n, (n-s)/2 - 2, s)) \cong \mathrm{Con}_{\mathbb{I}}(L_4(n, n/2 - 2)) \cong \mathcal{L}_2^{n/2-1}$

$\mathrm{Con}_{\mathbb{I}}(L_{5,2}(n, n/2 - 2)) \cong (\mathcal{L}_2 \oplus \mathcal{L}_2^2) \times \mathcal{L}_2^{n/2-3}$

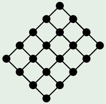
$\mathrm{Con}_{\mathbb{I}}(M_{5,2}(n, n/2 - 2)) \cong (\mathcal{L}_2^2 \oplus \mathcal{L}_2) \times \mathcal{L}_2^{n/2-3}$

Example (other finite involution lattices with the same number of elements, more lattice congruences and less full congruences)

$L \cong_{\mathbb{I}} (M_3 \boxplus \mathcal{L}_4) \oplus \mathcal{L}_2^3 \oplus (M_3 \boxplus \mathcal{L}_4) \in \mathbb{PKA}$

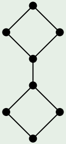


$\mathcal{L}_4 \times \mathcal{L}_5 \in \mathbb{KL}$

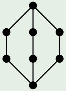


$|\mathcal{L}_4 \times \mathcal{L}_5| = |L| = 20$
$|\mathrm{Con}(\mathcal{L}_4 \times \mathcal{L}_5)| = 128 > 72 = |\mathrm{Con}(L)|$
$|\mathrm{Con}_{\mathbb{I}}(\mathcal{L}_4 \times \mathcal{L}_5)| = 16 < 24 = |\mathrm{Con}_{\mathbb{I}}(L)|$

$M \cong_{\mathbb{I}} \mathcal{L}_2^2 \oplus \mathcal{L}_2 \oplus \mathcal{L}_2^2 \in \mathbb{KL}$



$H \cong_{\mathbb{I}} \mathcal{L}_4 \boxplus \mathcal{L}_4 \boxplus \mathcal{L}_4 \in \mathbb{BI} \setminus \mathbb{PKA}$



$|M| = |H| = 8$
$|\mathrm{Con}(M)| = 32 > 9 = |\mathrm{Con}(H)|$
$|\mathrm{Con}_{\mathbb{I}}(M)| = 8 < 9 = |\mathrm{Con}_{\mathbb{I}}(H)|$

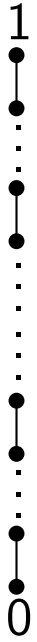
Next: the n–element i–lattices,
respectively BZ–lattices with the 0 meet–irreducible,
with the largest number of full congruences: $2^{\lfloor n/2 \rfloor}$, respectively $2^{\lfloor n/2 \rfloor - 1} + 1$.

$L_1(n) = \mathcal{L}_n$: antiortholattice
$L_2(n, n/2 - 1)$: antiortholattice if $n \in 2(\mathbb{N} \setminus \{0, 1, 2\})$

for $n \in 2(\mathbb{N} \setminus \{0, 1\})$:

$L_1(n) = \mathcal{L}_n \in \mathbb{KL}$ $\quad$ $L_2(n, n/2 - 1) = \mathcal{L}_{n/2-1} \oplus \mathcal{L}_2^2 \oplus \mathcal{L}_{n/2-1} \in \mathbb{KL}$

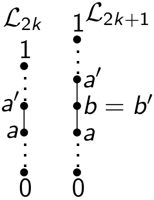
$L_2(n, n/2 - 1) \ncong_{\mathbb{I}} \mathcal{L}_{n/2-1} \oplus (\mathcal{L}_3 \boxplus \mathcal{L}_3) \oplus \mathcal{L}_{n/2-1}$

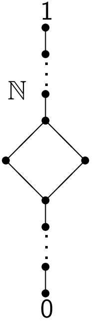


1

$2^{\lfloor n/2 \rfloor}$

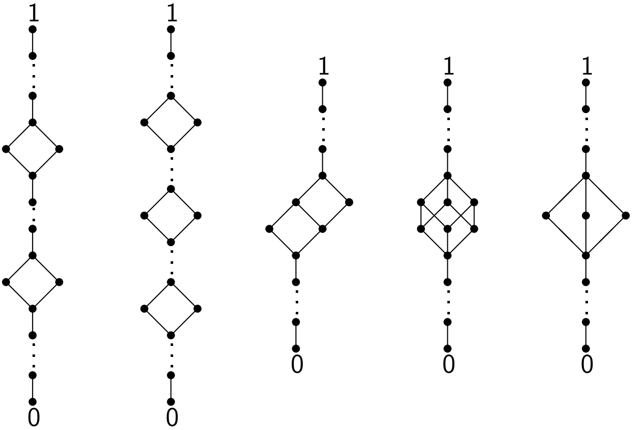
$= 2^k$, $n \in \{2k, 2k+1\}$, $k \in \mathbb{N}^*$ / $\mathbb{N}$



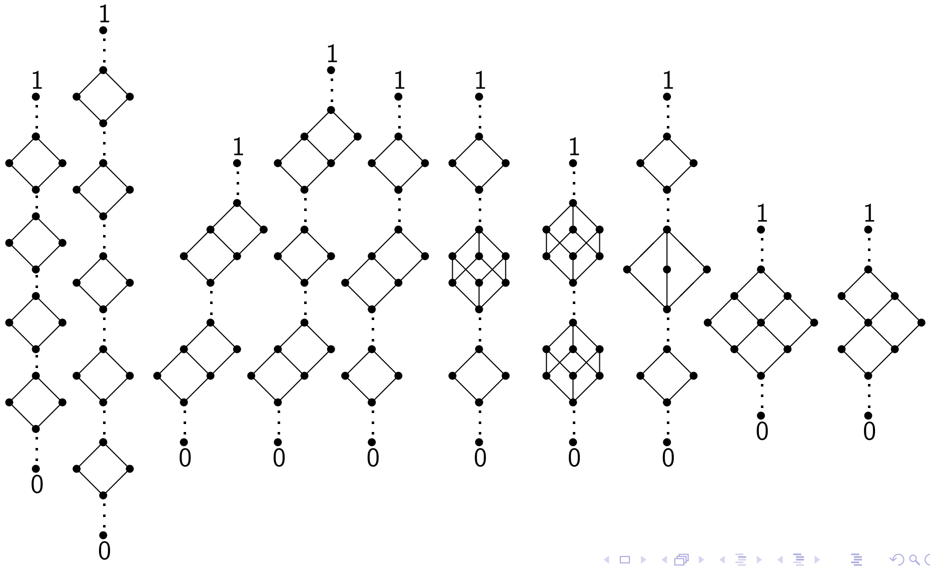


With the second equalities/isomorphisms for $n \in 2(\mathbb{N} \setminus \{0, 1\})$:
$|\mathrm{Con}_{\mathbb{I}}(L_1(n))| = 2^{\lfloor n/2 \rfloor} = |\mathrm{Con}_{\mathbb{I}}(L_2(n, n/2 - 1))|$
$\mathrm{Con}_{\mathbb{I}}(L_1(n)) \cong \mathcal{L}_2^{\lfloor n/2 \rfloor} \cong \mathrm{Con}_{\mathbb{I}}(L_2(n, n/2 - 1))$
For $n \geq 2$, respectively $n \in 2(\mathbb{N} \setminus \{0, 1, 2\})$:
$|\mathrm{Con}_{\mathbb{BZL}}(L_1(n))| = 2^{\lfloor n/2 \rfloor - 1} + 1 = |\mathrm{Con}_{\mathbb{BZL}}(L_2(n, n/2 - 1))|$
$\mathrm{Con}_{\mathbb{BZL}}(L_1(n)) \cong \mathcal{L}_2^{\lfloor n/2 \rfloor - 1} \oplus \mathcal{L}_2 \cong \mathrm{Con}_{\mathbb{BZL}}(L_2(n, n/2 - 1))$

Shapes of Some of the n–element i–lattices with $2^{\lfloor n/2 \rfloor - 1}$ Full Congruences (Triple Dots Represent Chains)



Shapes of Some of the n–element i–lattices with $2^{\lfloor n/2 \rfloor - 2}$ Full Congruences (Triple Dots Represent Chains)



The new results above can be found in:

C. Mureșan, Some Properties of Lattice Congruences Preserving Involutions and Their Largest Numbers in the Finite Case, arXiv:1802.05344v3 *(v4 coming up)* [math.RA].

THANK YOU FOR YOUR ATTENTION !