Homomorphism order of connected monounary algebras

D. Jakubíková-Studenovská
P. J. Šafárik University, Košice, Slovakia

Summer School on Algebra and Ordered Sets
Špindlerův Mlýn September 3–8, 2018

Inspiration

- Summer School on Algebra and Ordered Sets, Nový Smokovec, 2017

Inspiration

- Summer School on Algebra and Ordered Sets, Nový Smokovec, 2017
- Brian Davey presented a series of plenary lectures with a characterizing title

  The homomorphism order: from graphs to algebras

Introduction

- P. Hell, J. Nešetřil, *Graphs and Homomorphisms*, Oxford Lecture Series in Mathematics and Its Applications 28, Oxford University Press, 2004.

Introduction

- P. Hell, J. Nešetřil, *Graphs and Homomorphisms*, Oxford Lecture Series in Mathematics and Its Applications 28, Oxford University Press, 2004.

- type $\tau = (F, R)$

Introduction

- P. Hell, J. Nešetřil, *Graphs and Homomorphisms*, Oxford Lecture Series in Mathematics and Its Applications 28, Oxford University Press, 2004.

- type $\tau = (F, R)$
- A *homomorphism* of a structure $\mathcal{A} = (A, F, R)$ to $\mathcal{B} = (B, F, R)$ is a mapping $\varphi : A \to B$ commuting with all operations $f \in F$ and such that each relation $\rho \in R$ is compatible with φ.

Introduction

- P. Hell, J. Nešetřil, *Graphs and Homomorphisms*, Oxford Lecture Series in Mathematics and Its Applications 28, Oxford University Press, 2004.

- type $\tau = (F, R)$
- A *homomorphism* of a structure $\mathcal{A} = (A, F, R)$ to $\mathcal{B} = (B, F, R)$ is a mapping $\varphi : A \to B$ commuting with all operations $f \in F$ and such that each relation $\rho \in R$ is compatible with φ.
- Define $\mathcal{A} \leq \mathcal{B}$ if there is a homomorphism of $\mathcal{A}$ to $\mathcal{B}$.
  $\leq$ is a quasi-order.

Introduction

- P. Hell, J. Nešetřil, *Graphs and Homomorphisms*, Oxford Lecture Series in Mathematics and Its Applications 28, Oxford University Press, 2004.

- type $\tau = (F, R)$
- A *homomorphism* of a structure $\mathcal{A} = (A, F, R)$ to $\mathcal{B} = (B, F, R)$ is a mapping $\varphi : A \to B$ commuting with all operations $f \in F$ and such that each relation $\rho \in R$ is compatible with φ.
- Define $\mathcal{A} \leq \mathcal{B}$ if there is a homomorphism of $\mathcal{A}$ to $\mathcal{B}$.
  $\leq$ is a quasi-order.

- Set $\mathcal{A} \sim \mathcal{B}$ if $\mathcal{A} \leq \mathcal{B}$ and $\mathcal{B} \leq \mathcal{A}$.
  The system of algebraic structures of type τ factorized by the equivalence $\sim$ is partially ordered.

Introduction

Properties of the homomorphism order investigated for relational structures, mostly for graphs, in algebraic category theory, random and combinatorial context

Introduction

Properties of the homomorphism order investigated for relational structures, mostly for graphs, in algebraic category theory, random and combinatorial context

- existence of suprema and infima (supremum is a disjoint sum, infimum is a categorical or relational product)

Introduction

Properties of the homomorphism order investigated for relational structures, mostly for graphs, in algebraic category theory, random and combinatorial context

- existence of suprema and infima (supremum is a disjoint sum, infimum is a categorical or relational product)
- maximal chains and antichains

Basic notions

Basic notions

- **monounary algebra** (A, f)

Basic notions

- **monounary algebra** (A, f)
- well vizualisable: corresponding directed graph (A, E) such that

$$(x, y) \in E \;\Leftrightarrow\; f(x) = y$$

Basic notions

- **monounary algebra** (A, f)
- well vizualisable: corresponding directed graph (A, E) such that
$$(x, y) \in E \Leftrightarrow f(x) = y$$
- **connected**: $\forall\ x, y \in A\ \exists\ n, m \in \mathbb{N} \cup \{0\}$ such that $f^n(x) = f^m(y)$

Basic notions

- **monounary algebra** (A, f)
- well vizualisable: corresponding directed graph (A, E) such that
$$(x, y) \in E \;\Leftrightarrow\; f(x) = y$$
- **connected**: $\forall\ x, y \in A\ \exists\ n, m \in \mathbb{N} \cup \{0\}$ such that $f^n(x) = f^m(y)$
- **connected component** of (A, f): maximal connected subalgebra

Basic notions

- **monounary algebra** (A, f)
- well vizualisable: corresponding directed graph (A, E) such that
$$(x, y) \in E \;\Leftrightarrow\; f(x) = y$$
- **connected**: $\forall\ x, y \in A\ \exists\ n, m \in \mathbb{N} \cup \{0\}$ such that $f^n(x) = f^m(y)$
- **connected component** of (A, f): maximal connected subalgebra
- $c \in A$ is **cyclic** if $f^k(c) = c$ for some $k \in \mathbb{N}$

Basic notions

- **monounary algebra** (A, f)
- well vizualisable: corresponding directed graph (A, E) such that
$$(x, y) \in E \iff f(x) = y$$
- **connected**: $\forall\ x, y \in A\ \exists\ n, m \in \mathbb{N} \cup \{0\}$ such that $f^n(x) = f^m(y)$
- **connected component** of (A, f): maximal connected subalgebra
- $c \in A$ is **cyclic** if $f^k(c) = c$ for some $k \in \mathbb{N}$
- the set of all cyclic elements of some connected component of A is **cycle** of (A, f)

Notation and aim

- $\mathcal{L}$ - the class of all connected monounary algebras

Notation and aim

- $\mathcal{L}$ - the class of all connected monounary algebras
- $\mathbb{L} = \mathcal{L}/\sim$ - the class $\mathcal{L}$ factorized by $\sim$

Notation and aim

- $\mathcal{L}$ - the class of all connected monounary algebras
- $\mathbb{L} = \mathcal{L}/\sim$ - the class $\mathcal{L}$ factorized by $\sim$
- **Note!** in the class of all (not necessarily connected) monounary algebras

Notation and aim

- $\mathcal{L}$ - the class of all connected monounary algebras
- $\mathbb{L} = \mathcal{L}/\sim$ - the class $\mathcal{L}$ factorized by $\sim$
- **Note!** in the class of all (not necessarily connected) monounary algebras
  - supremum = disjoint union

Notation and aim

- $\mathcal{L}$ - the class of all connected monounary algebras
- $\mathbb{L} = \mathcal{L}/\sim$ - the class $\mathcal{L}$ factorized by $\sim$
- **Note!** in the class of all (not necessarily connected) monounary algebras
  - supremum $=$ disjoint union
  - infimum $=$ direct product

Notation and aim

- $\mathcal{L}$ - the class of all connected monounary algebras
- $\mathbb{L} = \mathcal{L}/\sim$ - the class $\mathcal{L}$ factorized by $\sim$
- **Note!** in the class of all (not necessarily connected) monounary algebras
  - supremum = disjoint union
  - infimum = direct product
  - disjoint union of connected monounary algebras is never connected, direct product of connected monounary algebras fails to be connected in general

Notation and aim

- $\mathcal{L}$ - the class of all connected monounary algebras
- $\mathbb{L} = \mathcal{L}/\sim$ - the class $\mathcal{L}$ factorized by $\sim$
- **Note!** in the class of all (not necessarily connected) monounary algebras
    - supremum $=$ disjoint union
    - infimum $=$ direct product
    - disjoint union of connected monounary algebras is never connected, direct product of connected monounary algebras fails to be connected in general
- **Aim:** $\mathbb{L}$ with the homomorphism order is a lattice.

Notation and aim

- $\mathcal{L}$ - the class of all connected monounary algebras
- $\mathbb{L} = \mathcal{L}/\sim$ - the class $\mathcal{L}$ factorized by $\sim$
- **Note!** in the class of all (not necessarily connected) monounary algebras
  - supremum $=$ disjoint union
  - infimum $=$ direct product
  - disjoint union of connected monounary algebras is never connected, direct product of connected monounary algebras fails to be connected in general
- **Aim:** $\mathbb{L}$ with the homomorphism order is a lattice.
  - some properties of $\mathbb{L}$

Basic notions

Basic notions

- (A, f) is **unbounded** if

Basic notions

- (A, f) is **unbounded** if
  (i) there exists no sequence $\{a_n\}_{n \in \mathbb{N}_0}$ of elements in A such that $f(a_n) = a_{n-1}$ for each $n \in \mathbb{N}$

Basic notions

- (A, f) is **unbounded** if
  - (i) there exists no sequence $\{a_n\}_{n\in\mathbb{N}_0}$ of elements in A such that $f(a_n) = a_{n-1}$ for each $n \in \mathbb{N}$
  - (ii) if $x \in A$, $n \in \mathbb{N}$, then there is $m \in \mathbb{N}$ such that $f^{-(m+n)}(f^m(x)) \neq \emptyset$.

Basic notions

- (A, f) is **unbounded** if
  (i) there exists no sequence $\{a_n\}_{n \in \mathbb{N}_0}$ of elements in A such that $f(a_n) = a_{n-1}$ for each $n \in \mathbb{N}$
  (ii) if $x \in A$, $n \in \mathbb{N}$, then there is $m \in \mathbb{N}$ such that $f^{-(m+n)}(f^m(x)) \neq \emptyset$.
- (A, f) is **bounded** if (A, f) satisfies (i) and does not fulfill (ii).

Basic notions

- (A, f) is **unbounded** if
  (i) there exists no sequence $\{a_n\}_{n\in\mathbb{N}_0}$ of elements in A such that $f(a_n) = a_{n-1}$ for each $n \in \mathbb{N}$
  (ii) if $x \in A$, $n \in \mathbb{N}$, then there is $m \in \mathbb{N}$ such that $f^{-(m+n)}(f^m(x)) \neq \emptyset$.
- (A, f) is **bounded** if (A, f) satisfies (i) and does not fulfill (ii).
- $(\mathbb{Z}, suc)$, $(\mathbb{N}, suc)$ and $(\mathbb{Z}_m, suc)$ - the algebras such that *suc* is the operation of the successor
  (i.e., $(\mathbb{Z}_m, suc)$ is an m-element cycle)

Lemma

Let $(A, f) \in \mathcal{L}$*.*

(i) $(\mathbb{N}, suc) \leq (A, f) \leq (\mathbb{Z}_1, suc)$.

(ii) *If* $m \in \mathbb{N}$*, then* $(\mathbb{Z}, suc) \leq (\mathbb{Z}_m, suc)$.

(iii) $(\mathbb{Z}_n, suc) \leq (\mathbb{Z}_m, suc)$ *if and only if* m *divides* n

(iv) *If* (A, f) *contains no cycle, then* $(A, f) \leq (\mathbb{Z}, suc)$.

Lemma

Let $(A, f) \in \mathcal{L}$*.*

(a) $(A, f) \sim (\mathbb{Z}, suc)$ *if and only if* (A, f) *contains a subalgebra isomorphic to* $(\mathbb{Z}, suc)$.

(b) $(A, f) \sim (\mathbb{Z}_m, suc)$ $(m \in \mathbb{N})$ *if and only if* (A, f) *contains an* m*-element cycle.*

(c) $(A, f) \sim (\mathbb{N}, suc)$ *if and only if* (A, f) *is bounded.*

Shape of the partially ordered class $\mathbb{L}$

Shape of the partially ordered class $\mathbb{L}$

Lemma

The system $\mathbb{L}$ is a proper class.

Proposition

The partially ordered class $\mathbb{L}$ is bounded, with the upper bound $[(\mathbb{Z}_1, suc)]_\sim$ and with the lower bound $[(\mathbb{N}, suc)]_\sim$. Further, $\mathbb{L}$ is a disjoint union of a one element set $\{[(\mathbb{Z}, suc)]_\sim\}$ and of two parts, the upper one $\mathbb{L}_1$ (over $[(\mathbb{Z}, suc)]_\sim$) and the lower one $\mathbb{L}_2$ (below $[(\mathbb{Z}, suc)]_\sim$). Next, $\mathbb{L}_1$ is dually isomorphic to $\mathbb{N}$ ordered by divisibility.

The lattice $\mathbb{L}$

The lattice $\mathbb{L}$

The following theorem is proved by a constructive way:.

Theorem

The partially ordered class $\mathbb{L}$ *forms a lattice.*

The lattice $\mathbb{L}$

The following theorem is proved by a constructive way:.

Theorem

The partially ordered class $\mathbb{L}$ forms a lattice.

Theorem

The lattice $\mathbb{L}$ is distributive.

Idea of proof:

- class of all sequences of ordinals ($\wedge$, $\vee$ coordinate-wise) is a distributive lattice
- class S of all increasing sequences of ordinals is a distributive lattice
- S factorized by some congruence is a distributive lattice
- this lattice is isomorphic to $\mathbb{L}$

Antichains in $\mathbb{L}$

Antichains in $\mathbb{L}$

The following theorem was proved in , in that time formulated in the language of category theory.

Antichains in $\mathbb{L}$

The following theorem was proved in , in that time formulated in the language of category theory.

Theorem

(i) *If M is an antichain in $\mathcal{L}$ then M is a set with at most continuum elements.*

Antichains in $\mathbb{L}$

The following theorem was proved in , in that time formulated in the language of category theory.

Theorem

(i) *If M is an antichain in $\mathcal{L}$ then M is a set with at most continuum elements.*

(ii) *There exists an antichain M in $\mathcal{L}$ possessing continuum elements.*

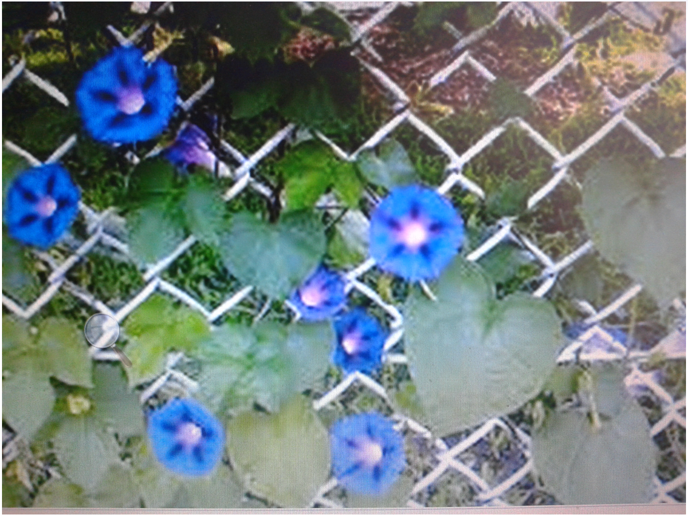
Antichains in $\mathbb{L}$

The following theorem was proved in 1978 , in that time formulated in the language of category theory.

Theorem

(i) *If M is an antichain in $\mathcal{L}$ then M is a set with at most continuum elements.*

(ii) *There exists an antichain M in $\mathcal{L}$ possessing continuum elements.*



Thank You for Your Attention !