TWO SMALL MONOID VARIETIES WITH THE LARGE JOIN

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The lattices $\mathbb{MON}$ and $\mathbb{SEM}$

We consider monoids as algebras of type (2,0).
The collection of all monoid varieties forms a lattice under class-theoretical inclusion. This lattice is denoted by $\mathbb{MON}$.

The lattice $\mathbb{MON}$ is embedded in the lattice $\mathbb{SEM}$ of all semigroup varieties. The lattice $\mathbb{SEM}$ is studied since the first half of 1960s. There are about 250 articles on this subject. Among them, there are several surveys.The last survey is the following

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Analog for $\mathbb{SEM}$: *none.*

The following two questions still remain open:

Does the join of two semigroup varieties with the ascending chain condition for subvarieties satisfy this condition for subvarieties?
Does a cover of a semigroup variety with the ascending chain condition for subvarieties satisfy this condition for subvarieties?

This concerns the ascending chain condition.
What about *the descending chain condition*?

Two semigroup varieties $\mathbf{U}$ and $\mathbf{V}$ with the descending chain condition for subvarieties was found by Mark Sapir such that the join $\mathbf{U} \vee \mathbf{V}$ does not satisfy this condition for subvarieties and covers $\mathbf{U}$.

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This concerns the descending chain condition in the lattice $\mathbb{SEM}$.

As for the lattice $\mathbb{MON}$ then the following two questions were remain open so far:

1. Are there monoid varieties **U** and **V** with the descending chain condition for subvarieties such that the join $\mathbf{U} \vee \mathbf{V}$ does not satisfy the same condition for subvarieties?
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We give an affirmative answer to these two questions!

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The main result

Put

$$\mathbf{X} = \operatorname{var}\{x^2yz \approx xyxzx \approx yzx^2, xzxyty \approx xzyxty, xyzxy \approx yxzxy, xzytxy \approx xzytyx\},$$

$$\mathbf{Y} = \operatorname{var}\{x^2yz \approx xyxzx \approx yzx^2, xzxyty \approx xzyxty, xyzxty \approx yxzxty\}.$$

The subvariety lattice of $\mathbf{X}$ is the 6-element chain (Jackson, 2005).
The subvariety lattice of $\mathbf{Y}$ is the 7-element chain (Gusev and Vernikov, 2018).

$\overleftarrow{\mathbf{X}}$ – the variety dual to $\mathbf{X}$

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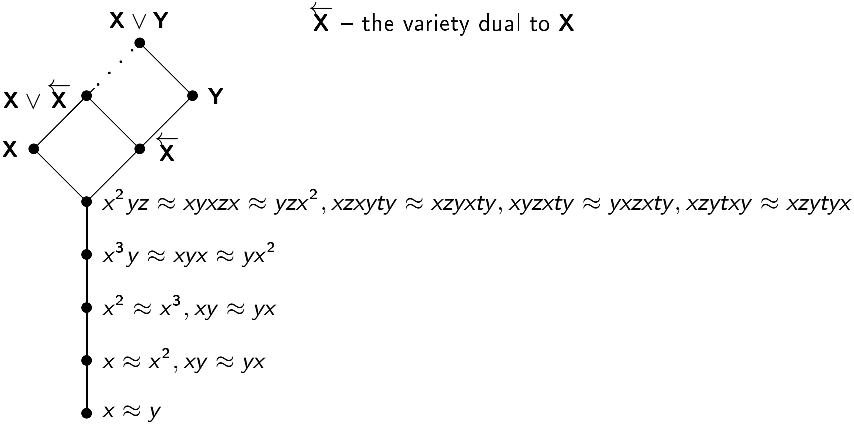

Theorem

The variety $\mathbf{X} \vee \mathbf{Y}$ *covers the variety* $\mathbf{Y}$*, while the subvariety lattice of* $\mathbf{X} \vee \mathbf{Y}$ *does not satisfy the descending chain condition.*

So, the class of monoid varieties whose subvariety lattices satisfy the descending chain condition is not closed with respect to the join of the varieties and to coverings.

Also the class of monoid varieties with finite subvariety lattice is not closed with respect to coverings.

Theorem

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Thank you very much!