Unbounded distributive lattices with SEKP as direct sums

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Outline

We shall study the notion of strong endomorphism property defined by Blyth and Silva in 2004. Let A be a universal algebra, $f : A \to A$ be an endomorphism, $\Theta \in Con(A)$ be a congruence on A. f is *compatible* with Θ if $a \equiv b(\Theta) \Rightarrow f(a) \equiv f(b)(\Theta)$.

Endomorphism f is *strong* (*congruence preserving*) (on A), if it is compatible with every congruence $\Theta \in Con(A)$.

Definition

An algebra A has the *strong endomorphism kernel property* (=SEKP) if and only if every congruence relation on A different from the universal congruence ι_A is the kernel of a strong endomorphism on A.

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Blyth and Silva considered the case of Ockham algebras and in particular of MS-algebras. They proved e.g. that

- a finite Boolean algebra has SEKP if and only if it is 2 element BA,
- a finite bounded distributive lattice possesses SEKP if and only if it is trivial or a 2 element chain and
- they provided full characterization of MS-algebras having SEKP.

Blyth, J. Fang and Wang in 2013 proved a full characterization of finite distributive double p-algebras and finite double Stone algebras having SEKP.

SEKP for distributive p-algebras and Stone algebras has been also studied and fully characterized by G. Fang and J. Fang in 2013.

J. Fang and Sun fully characterized semilattices with SEKP in 2013.

J. Guričan and M. Ploščica gave full characterization of unbounded distributive lattices with SEKP in 2016. We shall use this characterization in this talk.

The main approach in papers most of mentioned papers is done by using the duality theory of H. Priestley.

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Unbounded distributive lattices (UDL) with SEKP are fully described by the following theorem

Theorem

UDL L has SEKP if and only if L is locally finite and there exists $c \in L$ such that for every $x \in (c \uparrow \cup c \downarrow) \setminus \{c\}$ intervals $[x, c]$ (if $x < c$) and $[c, x]$ (if $x > c$) are (finite) Boolean.

We shall call elements c from the characterization above *boolean elements of L. There can be many boolean elements in a given UDL L.*
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Outline

We shall summarize most importat properties of Priestley spaces of unbounded distributive lattices with SEKP. Let us start with one property of the order relation $\subseteq$ of $\mathrm{Spec}(L)$ - the set of all prime ideals of L, including $\emptyset$ and L.

As we are considering only unbounded distributive lattices, we shall omit the "UDL" in the next part of the talk .

Lemma

If L has SEKP, then $X = \mathrm{Spec}(L) \setminus \{\emptyset, L\}$ is a disjoint union of three antichains $A_0 \cup A_1 \cup A_2$, where $A_1 = \{a \in X; (\exists b \in X)(a < b)\}$ ("bottom" elements), $A_2 = \{b \in X; (\exists a \in X)(a < b)\}$ ("top" elements) and $A_0 = X \setminus (A_1 \cup A_2)$ ("incomparable" elements).

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Here are most important topological properties of Priestey spaces of L (we shall use the notation $\mathcal{O}(\mathbf{D}(L)))$ with SEKP.

Lemma

Let L be any distributive lattice. Let L have SEKP, $P \in \operatorname{Spec}(L)$, $P \neq \emptyset, P \neq L$. Then P is a discrete point in the topology τ.

Using this lemma we see, that for any subset U of $\operatorname{Spec}(L)$ the closure $\overline{U}$ is a subset of $U \cup \{\emptyset, L\}$, it means that for a downset $U \in \mathcal{O}(\mathbf{D}(L))$ we have that $\overline{U} \subseteq U \cup \{L\}$.

Therefore for $A \in \mathcal{O}(\mathbf{D}(L))$ and $U \subseteq \operatorname{Spec}(L)$ we have that $A \cap \overline{U} = A \cap U$. It means that for L with SEKP we can ommit a word "closed" in the description of the congruence θ_U on $\mathcal{O}(\mathbf{D}(L))$, which is a set of pairs (A, B) such that $A \cap U = B \cap U$ for clopen downsets A, B.

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Let L have SEKP. Then for every $A \in \mathcal{O}(\mathbf{D}(L))$ the sets $A \cap A_2$ and $A_1 \setminus A$ are finite.

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Let L have SEKP. Then there exists clopen down set $C \in \mathcal{O}(\mathbf{D}(L))$ such that $A_1 \subseteq C$ and $C \cap A_2 = \emptyset$. Moreover, for any such C and for $A \in \mathcal{O}(\mathbf{D}(L))$ such that $A \subseteq C$ the interval $[A, C]$ of $\mathcal{O}(\mathbf{D}(L))$ is (finite) Boolean and also for $A \in \mathcal{O}(\mathbf{D}(L))$ such that $C \subseteq A$ the interval $[C, A]$ of $\mathcal{O}(\mathbf{D}(L))$ is (finite) Boolean.

The element $C \in \mathcal{O}(\mathbf{D}(L))$ with this property can be obtained from any $A \in \mathcal{O}(\mathbf{D}(L))$ as $C = A \cup (A_1 \setminus A) \setminus (A \cap A_2)$ and in $\mathcal{O}(\mathbf{D}(L))$ corresponds to a boolean element of L, we shall also call it a boolean element.

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Every boolean element (its $\mathcal{O}(\mathbf{D}(L))$ "form") is therefore uniquely determined by its intersection with the set A_0.

Here is the first information on the structure of boolean elements of L

Lemma

Let L be a distributive lattice with SEKP. Then the set of all boolean elements form a convex sublattice of L.

Next lemma can be used to describe the structure of the sublattice of boolean elements

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Let us take one boolean element C of $\mathcal{O}(\mathbf{D}(L))$ (it means a clopen set such that $A_1 \subseteq C$ and $C \cap A_2$ is empty). We know that by adding or by removing finitely many elements of A_0 to (or from) C we obtain a boolean element.

If the intersection $C \cap A_0$ is finite, we can (and shall) use $C = A_1 \cup \{\emptyset\}$, if the intersection $C \cap A_0$ is cofinite, we shall use $C = A_1 \cup A_0 \cup \{\emptyset\}$.

So, in interesting cases we have that $C \cap A_0$ is infinite and also its complement in A_0 is infinite.

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Large boolean elements

Now we are going to "separate" boolean elements larger than C.
Let L be a lattice with SEKP. Let $C \in \mathcal{O}(\mathbf{D}(L))$ be a boolean element of $\mathcal{O}(\mathbf{D}(L))$. Let us consider congruences α_C, β_C on $\mathcal{O}(\mathbf{D}(L))$ given by

$$\begin{aligned}(A, B) \in \alpha_C \quad &\Leftrightarrow \quad A \cap (C \cup A_2) = B \cap (C \cup A_2)\\ (A, B) \in \beta_C \quad &\Leftrightarrow \quad A \cap (A_0 \setminus C) = B \cap (A_0 \setminus C)\end{aligned}$$

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For congruences α_C, β_C on $\mathcal{O}(\mathbf{D}(L))$ defined above we have

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2. $\alpha_C \circ \beta_C = 1$

Therefore L is isomorphic to $\mathcal{O}(\mathbf{D}(L))/\alpha_C \times \mathcal{O}(\mathbf{D}(L))/\beta_C$ and one can prove that

$$\mathcal{O}(\mathbf{D}(L))/\alpha_C \cong \{A \cap (A_2 \cup C); A \in \mathcal{O}(\mathbf{D}(L))\}$$

and

$$\mathcal{O}(\mathbf{D}(L))/\beta_C \cong \{(A \cap (A_0 \setminus C)) \cup C; A \in \mathcal{O}(\mathbf{D}(L))\}$$

Clearly, $A \cap (A_0 \setminus C) \subseteq (A \cap A_0)\Delta(C \cap A_0)$, which is finite. Moreover, by the fact that all elements of A_0 are discrete, any finite subset of $A_0 \setminus C$ is of the form $A \cap (A_0 \setminus C)$ for appropriate $A \in \mathcal{O}(\mathbf{D}(L))$. Therefore $\mathcal{O}(\mathbf{D}(L))/\beta_C$ is isomorphic to the lattice of all finite subsets of $A_0 \setminus C$ which is isomorphic to the direct sum $\Sigma(\{0,1\}; i \in A_0 \setminus C)$ with distinguished elements 0.

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$$\mathcal{O}(\mathbf{D}(L))/\beta_C \cong \{(A \cap (A_0 \setminus C)) \cup C; A \in \mathcal{O}(\mathbf{D}(L))\}$$

Clearly, $A \cap (A_0 \setminus C) \subseteq (A \cap A_0)\Delta(C \cap A_0)$, which is finite. Moreover, by the fact that all elements of A_0 are discrete, any finite subset of $A_0 \setminus C$ is of the form $A \cap (A_0 \setminus C)$ for appropriate $A \in \mathcal{O}(\mathbf{D}(L))$. Therefore $\mathcal{O}(\mathbf{D}(L))/\beta_C$ is isomorphic to the lattice of all finite subsets of $A_0 \setminus C$ which is isomorphic to the direct sum $\Sigma(\{0,1\}; i \in A_0 \setminus C)$ with distinguished elements 0.

Small boolean elements

Now we are going to decompose the first part, i.e. $\mathcal{O}(\mathbf{D}(L))/\alpha_C$. It means that we shall work with a lattice L such that there is the boolean element $C \in \mathcal{O}(\mathbf{D}(L))$ with $C = A_1 \cup A_0 \cup \{\emptyset\}$.

Using congruences γ_C, δ_C on $\mathcal{O}(\mathbf{D}(L))$ given by

$$(A, B) \in \gamma_C \quad \Leftrightarrow \quad A \cap (A_1 \cup A_2) = B \cap (A_1 \cup A_2)$$
$$(A, B) \in \delta_C \quad \Leftrightarrow \quad A \cap A_0) = B \cap A_0)$$

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1. $\gamma_C \wedge \delta_C = 0$
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1. $\gamma_C \wedge \delta_C = 0$
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Therefore L is isomorphic to $\mathcal{O}(\mathbf{D}(L))/\gamma_C \times \mathcal{O}(\mathbf{D}(L))/\delta_C$ and one can prove that

$$\mathcal{O}(\mathbf{D}(L))/\gamma_C \cong \{A \cap (A_1 \cup A_2) \cup \{\emptyset\}; A \in \mathcal{O}(\mathbf{D}(L))\}$$

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We know that $A \cap A_0$ is a cofinite set in A_0. Moreover, by the fact that all elements of A_0 are discrete, any cofinite subset of A_0 is of the form $A \cap A_0$ for appropriate $A \in \mathcal{O}(\mathbf{D}(L))$. Therefore $\mathcal{O}(\mathbf{D}(L))/\delta_C$ is isomorphic to the lattice of all cofinite subsets of A_0 which is isomorphic to the direct sum $\Sigma(\{0,1\}; i \in A_0)$ with distinguished elements 1. It is also clear that $\{(A \cap A_0) \cup A_1 \cup \{\emptyset\}; A \in \mathcal{O}(\mathbf{D}(L))\}$ are all boolean elements of $\mathcal{O}(\mathbf{D}(L))$ (we must keep in mind that $\mathcal{O}(\mathbf{D}(L))$ is such that $C = A_1 \cup A_0 \cup \{\emptyset\}$ is a boolean element).

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Combining these result we can see that original lattice L with SEKP can be written as a direct product

$$L \cong A \times B \times C$$

where

- A is the lattice with SEKP having exactly one boolean element, there are no incomparable elements in $\operatorname{Spec} A$, i.e. A_0 is empty
- B is the direct sum $\Sigma(\{0,1\}; i \in X)$ with distinguished elements 1 for some set X
- C is the direct sum $\Sigma(\{0,1\}; i \in Y)$ with distinguished elements 0 for some set Y

Also, $B \times C$ is isomorphic to the subalgebra of all boolean elements of L

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Thank you for the attention.

I would like to thank the organizers for organizing this beautiful conference.

Outline

R. Balbes and P. Dwinger, *Distributive lattices.* University of Missouri Press, Columbia, Missouri, 1975.

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