Structures with small orbit growth

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Main result

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1. *The class of all structures $\mathfrak{A}$ for which there exist constants c and $d < 1$ so that the number of injective n-orbits of $\mathrm{Aut}(\mathfrak{A})$ is at most cn^{dn}. ($\mathcal{K}_{exp+}$)*
2. *The class of finite coverings of reducts of unary ω-categorical structures. ($F(R(\mathcal{U}))$)*

Reducts

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Notation

If $\mathcal{C}$ is a class of structures, then $R(\mathcal{C})$ denotes the class of reducts of structures in $\mathcal{C}$.

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Reducts and automorphism groups

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- *the closed supergroups of $\mathrm{Aut}(\mathfrak{A})$.*

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- In terms of automorphism groups: $\prod_{i=1}^{n} \mathrm{Sym}(X_i)$.

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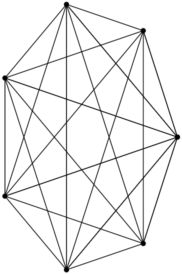
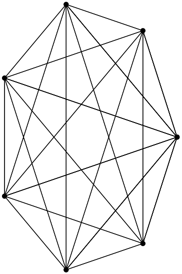


K_ω



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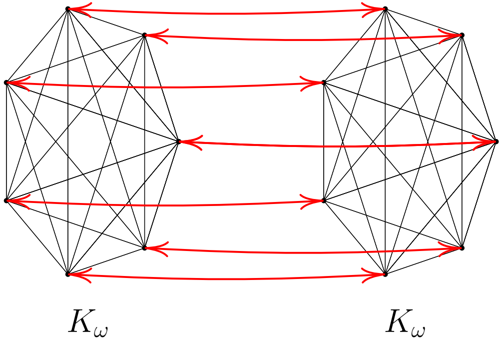

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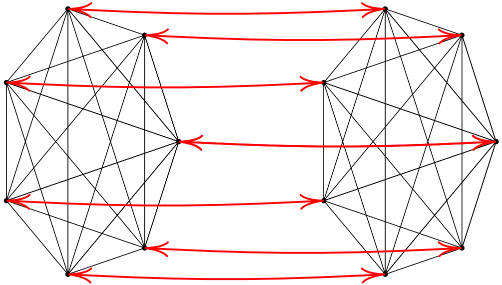
That is all. (...)



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4. The action of $\mathrm{Aut}(\mathfrak{A})$ on $\mathfrak{B}$ is $\mathrm{Aut}(\mathfrak{B})$.

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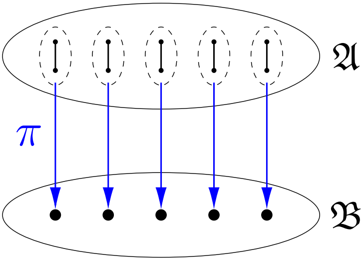

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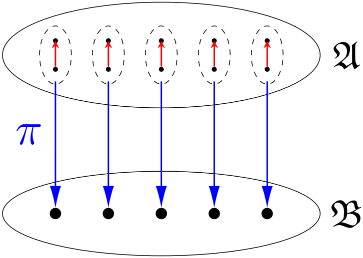

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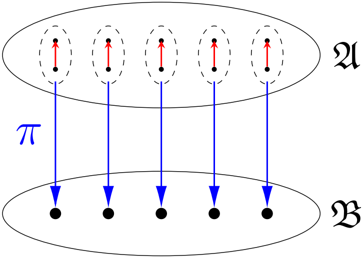

Notation

$F(\mathcal{C})$ is the class of finite coverings of structures in $\mathcal{C}$.

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Is there anything else?

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Definition

- $\mathcal{K}_{exp} := \{\mathfrak{A} : \exists c(\mathcal{O}_n(\mathfrak{A}) \leq c^n)\}$,
- $\mathcal{K}_{exp+} := \{\mathfrak{A} : \exists c, d < 1(\mathcal{O}_n(\mathfrak{A}) \leq cn^{dn})\}$.

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The class $F(R(\mathcal{U}))$ is closed under taking reducts, and thus $F(R(\mathcal{U})) = R(F(\mathcal{U}))$.

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Corollary

$\mathcal{K}_{exp+} = F(R(\mathcal{U}))$ is the smallest nontrivial class of countable structures which is closed under R, F, and C (adding finitely many constants).

Primitive case:

Theorem (Macpherson, 1984)

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Corollary

We are done in the primitive case. That is if $\mathfrak{A} \in \mathcal{K}_{exp}$ (or in $\mathcal{K}_{exp+}$), and $\mathrm{Aut}(\mathfrak{A})$ is primitive, then $\mathrm{Aut}(\mathfrak{A}) = \mathrm{Sym}(\mathfrak{A})$.

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- We show that $G|_{X_i}$ is primitive and closed $\rightarrow G|_{X_i} = \mathrm{Sym}(X_i)$.
- Then if $G \not\geq \mathrm{Sym}(X_1) \times \mathrm{Sym}(X_2)$, then there is bijection $e : X_1 \rightarrow X_2$ such that $\{(x, e(x)\}$ is an invariant partition.
  Not possible because of orbit growth.

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We need:

- $\mathcal{K}_{exp+} = F(R(\mathcal{U}))$,
- A quasi-characterization of structures in $F(R(\mathcal{U}))$.

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Theorem (Bodirsky, Mottet, 2016)

The dichotomy conjecture is true for structures in $R(\mathcal{U})$. (Using the finite dichotomy conjecture.)

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Conjecture

$\mathfrak{A} \in \mathcal{K}_{exp+}$.
If Pol($\mathfrak{A}$) contains a (canonical) pseudo-cyclic term, then CSP($\mathfrak{A}$) is in **P**, otherwise it is **NP**-complete.

Future work

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- Describe all structures in $\mathcal{K}_{exp+} = F(R(U))$.
- Decidability problems in $\mathcal{K}_{exp+}$.

Thank you for your attention!

European Research Council

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