A note on homomorphisms between products of algebras

Ivan Chajda, Martin Goldstern and Helmut Länger

Acknowledgement.The first and the third author gratefully acknowledge support of this research by ÖAD, project CZ 04/2017, as well as by IGA, project PřF 2018 012. The second author gratefully acknowledges support by the Austrian Science Fund (FWF), project I 3081-N35. The third author gratefully acknowledges support of this research by the Austrian Science Fund (FWF), project I 1923-N25.

Palacký
University
Olomouc

2018

Let $\mathbf{A}_i, i \in I$, and $\mathbf{B}_j, j \in J$, be algebras of the same type and f a homomorphism from $\prod_{i \in I} \mathbf{A}_i$ to $\prod_{j \in J} \mathbf{B}_j$. For every $k \in J$ let p_k denote the projection from $\prod_{j \in J} \mathbf{B}_j$ onto $\mathbf{B}_k$ and $f_k := p_k \circ f$. More generally, for any $J_0 \subseteq J$ we let $p_{J_0} \colon \prod_{j \in J} \mathbf{B}_j \to \prod_{j \in J_0} \mathbf{B}_j$ be the canonical projection map. It is evident that $f = (f_j : j \in J)$. Hence, the task of describing f is reduced to the task of describing the homomorphisms f_k from $\prod_{i \in I} \mathbf{A}_i$ to $\mathbf{B}_k$.

In [3] the authors solve this problem for the case that the algebras $\mathbf{A}_i$ and $\mathbf{B}_j$ are conservative median algebras and the index sets are finite. More generally, [2] considers the case that the $\mathbf{A}_i$ are median algebras and the $\mathbf{B}_j$ are tree-median algebras. It turns out that the method developed in [2] and [3] can be further generalized to lattices. Let us note that every distributive lattice is a median algebra (but not conversely). We are even able to extend this result to arbitrary lattices $\mathbf{A}_i$ provided the $\mathbf{B}_j$ are chains. For lattice concepts used in the rest of the paper the reader is referred to the monographs [1] and [5].

We call a mapping $f\colon \prod_{i\in I} A_i \to C$ *essentially unary* if there exists a $i_0 \in I$ and a mapping $g\colon A_{i_0} \to C$ with $g \circ p_{i_0} = f$. In this case we say that "f depends only on the i_0-th coordinate", or that "f factors through p_{i_0}".
From $f = g \circ p_{i_0}$ it easily follows that g is a homomorphism if and only if f is.

Definition 1

A class $\mathcal{K}$ of algebras has the Fraser-Horn property if there are no skew congruences on any product $\mathbf{A}_1 \times \mathbf{A}_2$ with $\mathbf{A}_1, \mathbf{A}_2 \in \mathcal{K}$, or more explicity:

> *For all $\mathbf{A}_1, \mathbf{A}_2 \in \mathcal{K}$, for every congruence $\theta \in \mathrm{Con}(\mathbf{A}_1 \times \mathbf{A}_2)$ there are congruences $\theta_1 \in \mathrm{Con}(\mathbf{A}_1)$, $\theta_2 \in \mathrm{Con}(\mathbf{A}_2)$ such that $\theta = \theta_1 \times \theta_2$, i.e. $\theta = \{((x_1, x_2), (y_1, y_2)) \mid x_1\theta_1 y_1, x_2\theta_2 y_2\}$.*

The following lemma is known from [4].

Lemma 1

Let $\mathcal{K}$ be a congruence distributive (CD) variety. Then $\mathcal{K}$ has the Fraser-Horn property.

For the rest of the paper we fix a variety $\mathcal{K}$ with the Fraser-Horn property.
We call an algebra non-trivial if its universe contains at least two elements.

Definition 2

We call an algebra $\mathbf{A}$ *hereditarily directly irreducible (HDI) if every subalgebra* $\mathbf{B} \leq \mathbf{A}$ *is directly irreducible, i.e., is not isomorphic to a direct product of two non-trivial factors.*

Fact 1

1. *The variety of lattices is congruence distributive.*
2. *A lattice is HDI if and only if it is a chain.*

Theorem 2

Let $\mathcal{K}$ be a variety with the Fraser-Horn property. If n is a positive integer, $\mathbf{A}_1, \ldots, \mathbf{A}_n$ are in $\mathcal{K}$ and $\mathbf{C} \in \mathcal{K}$ is HDI, then every homomorphism f from $\mathbf{A}_1 \times \cdots \times \mathbf{A}_n$ to $\mathbf{C}$ is essentially unary, i.e., factors through one of the projections p_i.

Remark 1

If $f: \mathbf{A}_1 \times \mathbf{A}_2 \to \mathbf{C}$ is not constant, then there is at most one $i \in \{1, 2\}$ such that f factors through p_i.

Theorem 3

If n is a positive integer, $\mathbf{A}_1, \ldots, \mathbf{A}_n \in \mathcal{K}$, $(\mathbf{C}_j; j \in J)$ is a non-empty family of HDI algebras in $\mathcal{K}$, and f is a homomorphism from $\mathbf{A}_1 \times \cdots \times \mathbf{A}_n$ to $\prod_{j \in J} \mathbf{C}_j$ then there exists a mapping $\sigma\colon J \to \{1, \ldots, n\}$ and for every $j \in J$ a homomorphism g_j from $\mathbf{A}_{\sigma(j)}$ to $\mathbf{C}_j$ such that

$$f(x_1, \ldots, x_n) = (g_j(x_{\sigma(j)}); j \in J)$$

for all $(x_1, \ldots, x_n) \in A_1 \times \cdots \times A_n$.

Corollary 1

If an algebra in $\mathcal{K}$ is isomorphic to a finite product of non-trivial HDI algebras, then these factors are uniquely determined up to order and isomorphisms.

We can generalize this to infinite direct products as follows. Recall that an ultrafilter on a set I is a family U of subsets of I which is upwards closed and also closed under intersections such that for all $I_0 \subseteq I$ exactly one of I_0, $I \setminus I_0$ is in U. For any family $(A_i : i \in I)$ of sets and any ultrafilter U on I we define the equivalence relation $\sim_U$ on $\prod_i A_i$ by

$$(x_i : i \in I) \sim_U (y_i : i \in I) \;\Leftrightarrow\; \{i \in I \mid x_i = y_i\} \in U,$$

and we write $\prod_i A_i/U$ for the set of equivalence classes, the "*ultraproduct* of the A_i modulo U". The canonical map from $\prod_i A_i$ to $\prod A_i/U$ is denoted by κ_U. If $(\mathbf{A}_i)_{i\in I}$ is a family of algebras of the same type, then the relation $\sim_U$ is a congruence relation on the product $\prod_i \mathbf{A}_i$.

Theorem 4

Let $\mathcal{K}$ be a variety with the Fraser-Horn property. Let I be a non-empty set, and for each $i \in I$ let $\mathbf{A}_i$ be an algebra in $\mathcal{K}$. Let $\mathbf{C}$ be an HDI algebra in $\mathcal{K}$, and let $h\colon \prod_{i\in I} \mathbf{A}_i \to \mathbf{C}$ be a homomorphism which is not constant. Then there is a unique ultrafilter U on I such that h factors through κ_U, i.e.,

> *There is a homomorphism $h'\colon \prod_{i\in I} \mathbf{A}_i/{\sim_U} \to \mathbf{C}$ such that $h = h' \circ \kappa_U$.*

In particular: If there is no $i \in I$ such that h factors through p_i, then U will be a non-principal ultrafilter.

Remark 2

Theorem 2 was used in the proof of Theorem 4; but we can also view Theorem 2 as a special case of Theorem 4, as any ultrafilter on a finite index set must be principal.

Theorem 4 is in some sense best possible, in the sense that homomorphisms from an infinite product $\prod_i \mathbf{A}_i$ into an HDI algebra will in general not factor through any single projection p_j, as the following example shows.

Example 1

Let U be an ultrafilter on the infinite index set I, and for each $i \in I$ let $\mathbf{A}_i$ be the 2-element lattice $\{0,1\}$. Then the ultraproduct $\prod_{i\in I} \mathbf{A}_i/U$ is again the 2-element lattice.
Identifying $\prod_i \mathbf{A}_i$ with the power set lattice $(P(I),\cup,\cap)$, the canonical map $\kappa_U\colon P(I) \to \{0,1\}$ maps each element of U to 1 and everything else to 0. If U is a non-principal ultrafilter, then h_U does not factor through any projection.

This example can be generalized to any Fraser-Horn variety where the class of HDI algebras is described by a set of first order formulas: If $\prod_i \mathbf{A}_i$ is a product of algebras, and $(h_i : i \in I)$ is a family of homomorphisms $h_i \colon \mathbf{A}_i \to \mathbf{C}_i$, where each $\mathbf{C}_i$ is HDI, then the family $(h_i : i \in I)$ naturally defines a homomorphism $h \colon \prod_i \mathbf{A}_i \to \prod_i \mathbf{C}_i$.

If U is an ultrafilter on I, then the algebra $\mathbf{C} := \prod_i \mathbf{C}_i/U$ is again HDI (as $\mathbf{C}$ satisfies all first order statements that are true in each $\mathbf{C}_i$). Let $\kappa_U^C \colon \prod_i \mathbf{C}_i \to \prod_i \mathbf{C}_i/U$ and $\kappa_U^A \colon \prod_i \mathbf{A}_i \to \prod_i \mathbf{A}_i/U$ be the canonical maps. Then the map $\bar{h} := \kappa_U^C \circ h \colon \prod_i \mathbf{A}_i \to \mathbf{C}$ trivially factors through κ_U^A, i.e., there is $h' \colon \prod_i \mathbf{A}_i/U \to \mathbf{C}$ with $\bar{h} = h' \circ \kappa_U^A$. By the uniqueness claim in Theorem 4, we see that U is the set of all $M \subseteq I$ such that $\bar{h}$ factors through p_M. So if U is non-principal, then $\bar{h}$ does not factor through any p_i.

Fact 5

Let **A** *be a lattice. Then the following are equivalent:*

- *There is a non-constant homomorphism from* **A** *into a chain.*
- *There is a non-constant homomorphism from* **A** *into the 2-element chain.*
- *The lattice* **A** *has a prime ideal.*

The following corollary can be seen as a weak version of Theorem 2.

Corollary 2

The class of lattices without a prime ideal is closed under finite direct products.

The following example shows that even this weak version cannot be generalized to infinite products, not even if all factors are equal.

Example 2

(a) *There are non-trivial lattices* $\mathbf{M}$ *such that no (finite or infinite) direct power of* $\mathbf{M}$ *has a prime ideal.*

(b) *On the other hand, there are lattices* $\mathbf{A}$ *without a prime ideal such that any infinite direct power* $\mathbf{A}^I$ *will contain a prime ideal.*

The end!

Thanks for your attention!!

- Birkhoff, G.: Lattice Theory. Corr. repr. of the 1967 3rd edn. American Mathematical Society Colloquium Publications, vol. 25. American Mathematical Society (AMS), Providence, R.I. (1979)
- Couceiro, M., Foldes, S., Meletiou, G.C.: On homomorphisms between products of median algebras. Algebra Universalis **78**, 545–553 (2017)
- Couceiro, M., Marichal, J.L., Teheux, B.: Conservative median algebras and semilattices. Order **33**, 121–132 (2016)
- Fraser, G.A., Horn, A.: Congruence relations in direct products. Proc. Amer. Math. Soc. **26**, 390–394 (1970)
- Grätzer, G.: Lattice Theory: Foundation. Birkhäuser, Basel (2011)