# On minimality of generating sets of aggregation clones

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# 1. Generating of an aggregation clone

# Aggregation function

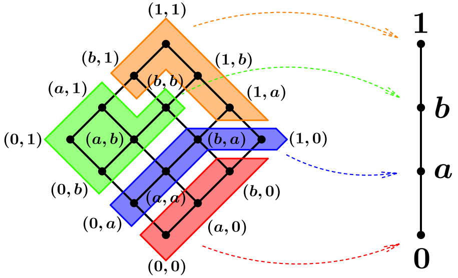
## Definition

Let $(L, \leq)$ be a finite lattice and $k \in \mathbb{N}$. A mapping $A : L^k \to L$ is called a *k-ary aggregation function* on lattice $L$, if it is non-decreasing, i.e., for any $\mathbf{x}, \mathbf{y} \in L^n$:

$$\mathbf{x} \leq \mathbf{y} \Rightarrow A(\mathbf{x}) \leq A(\mathbf{y})$$

and it fulfills boundary conditions

$$A(0, \dots, 0) = 0 \quad \text{and} \quad A(1, \dots, 1) = 1.$$

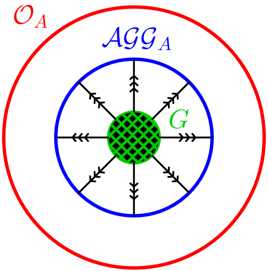


# Aggregation clone

**Definition**

Let $\mathcal{O}_A$ be a set of all (finitary) operations defined on a set $A$. Then subset $\mathcal{C} \subseteq \mathcal{O}_A$ is called a *clone* on $A$ if it contains all projection operations on $A$ and is closed under the composition of functions.

- all aggregation functions form a clone ( *aggregation clone* $\mathcal{AGG}_A$ )



# The generating set of an aggregation clone using functions $\chi$ and $\oplus$

**Definition**

Let $L$ be a finite lattice. Then functions $\chi_a : L \to L$ and $\oplus_a : L^2 \to L$ for all $a \in L$ are defined as follows

$$\chi_a(x) = \begin{cases} 1 & ; x \geq a, x \neq 0 \\ 0 & ; \text{otherwise}, \end{cases} \qquad x \oplus_a y = \begin{cases} 1 & ; x = 1, y = 1 \\ 0 & ; x = 0, y = 0 \\ a & ; \text{otherwise}. \end{cases}$$

- the number of functions is $2n + 2$ for $n$-element lattice $L$

# The generating set of an aggregation clone using functions $\chi$ and $\oplus$

**Theorem**

*Let L be a finite lattice. Then the aggregation clone $\mathcal{AGG}_L$ on L is generated by the lattice operations $\vee$, $\wedge$ and by functions $\chi_a$, $\oplus_a$, $a \in L$.*

- Let $A : L^k \to L$ be an aggregation function on lattice $L$. Then

$$A(\mathbf{x}) = \bigvee_{\mathbf{a} \in L_*^k} \left( \bigwedge_{i \in J_{\mathbf{a}}} \chi_{a_i}(x_i) \wedge \bigoplus_{i=1}^{k}{}_{A(\mathbf{a})} \, x_i \right)$$

where $\mathbf{x} = (x_1, \ldots, x_k)$, $\mathbf{a} = (a_1, \ldots, a_k)$,

$L_*^k = L^k \setminus \{(0, \ldots, 0), (1, \ldots, 1)\}$, $J_{\mathbf{a}} = \{1 \leq i \leq k \mid a_i \neq 0\}$,

$\bigoplus_{i=1}^{k}{}_{A(\mathbf{a})} \, x_i = x_1 \oplus_{A(\mathbf{a})} x_2 \oplus_{A(\mathbf{a})} \cdots \oplus_{A(\mathbf{a})} x_k$.

# 2. Properties of functions $\chi$ and $\oplus$

# Properties of functions $\oplus$

**Lemma**

*Let $L$ be a finite lattice. Then the equalities are valid:*

i) $x \oplus_0 y = \chi_1(x) \wedge \chi_1(y)$,

ii) $x \oplus_1 y = \chi_0(x) \vee \chi_0(y)$.

*Moreover:*

i) $x \oplus_0 x = \chi_1(x)$,

ii) $x \oplus_1 x = \chi_0(x)$.

**Lemma**

*Let $L$ be a finite lattice. Then the following equalities are valid for all $a, b \in L$:*

i) $(x \oplus_a y) \wedge (x \oplus_b y) = x \oplus_{a \wedge b} y$,

ii) $(x \oplus_a y) \vee (x \oplus_b y) = x \oplus_{a \vee b} y$.

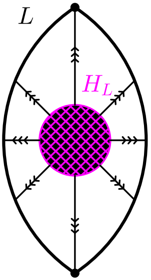
# Properties of functions $\oplus$

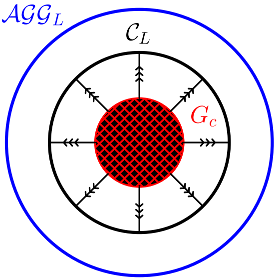
## Proposition

*Let $L$ be a finite lattice and let $H_L$ be a minimal generating set of lattice $L$. Denote*

$$G = \Big\{\chi_a \mid a \in L\Big\} \cup \Big\{\oplus_b \mid b \in H_L \setminus \{0,1\}\Big\} \cup \Big\{\vee, \wedge\Big\}.$$

*Then the set $G_c = G \setminus \{\oplus_c\}$ does not generate the aggregation clone $\mathcal{AGG}_L$ for any $c \in H_L \setminus \{0,1\}$.*



# Properties of functions $\chi$

**Lemma**

*Let $L$ be an $n$-element lattice, $n \geq 4$. Then for any $a, b, c \in L$ such that $a, b \neq 1$, $a \not\leq b$, $c \neq 0$ we have*

i) $\chi_0(x) = \chi_c\,(x \oplus_c x)$,

ii) $\chi_1(x) = \chi_a\,(x \oplus_b x)$,

iii) $\chi_0(x) = \chi_\alpha(x)$ *if $\alpha$ is a unique atom of $L$.*

**Lemma**

*Let $L$ be a finite lattice. Then for any $a_1, \ldots, a_p \in L$ we have*

i) $\chi_{a_1}(x) \wedge \ldots \wedge \chi_{a_p}(x) = \chi_{a_1 \vee \ldots \vee a_p}(x)$,

ii) $\chi_{a_1}(x) \vee \ldots \vee \chi_{a_p}(x) = \chi_{a_1 \wedge \ldots \wedge a_p}(x) = \chi_0(x)$ *assuming $a_1, \ldots, a_p$ are just all the atoms of $L$.*

# Properties of atomary functions $\chi$

- An atom $\alpha \Rightarrow \chi_\alpha \ldots$ an *atomary function*
- $$\underbrace{\overbrace{\chi_{\alpha_1}(x)}^{0} \vee \ldots \vee \chi_{\alpha_i}(x) \vee \ldots \vee \overbrace{\chi_{\alpha_q}(x)}^{0} = \chi_0(x), \quad x \mapsto \boxed{x \wedge (x \oplus_{\alpha_i} x)}}_{\Downarrow}$$

## Lemma

*Let L be a finite lattice with atoms $\alpha_1, \ldots, \alpha_q$. Then all atomary functions $\chi_{\alpha_i}$ can be generated according to the following formula:*

$$\chi_{\alpha_i}(x) = \chi_0\Big(x \wedge (x \oplus_{\alpha_i} x)\Big).$$

# 3. The forms of a generating set

# The cardinality of generating sets

- $\chi_{a_1}(x) \wedge \ldots \wedge \chi_{a_p}(x) = \chi_{a_1 \vee \ldots \vee a_p}(x) \Rightarrow J_L \ldots$ *join-irreducible elements of L*
  $\chi_{\alpha_1}(x) \vee \ldots \vee \chi_{\alpha_q}(x) = \chi_{\alpha_1 \wedge \ldots \wedge \alpha_q}(x) \Rightarrow At_L \ldots$ *atoms of L*
- Duality:
  $\mu_{a_1}(x) \vee \ldots \vee \mu_{a_p}(x) = \mu_{a_1 \wedge \ldots \wedge a_p}(x) \Rightarrow M_L \ldots$ *meet-irreducible elements of L*
  $\mu_{\omega_1}(x) \wedge \ldots \wedge \mu_{\omega_r}(x) = \mu_{\omega_1 \vee \ldots \vee \omega_r}(x) \Rightarrow CoAt_L \ldots$ *coatoms of L*
- $\left.\begin{array}{l} (x \oplus_a y) \wedge (x \oplus_b y) = x \oplus_{a \wedge b} y \\ (x \oplus_a y) \vee (x \oplus_b y) = x \oplus_{a \vee b} y \end{array}\right\} \Rightarrow H_L \ldots$ a *generating set of lattice L*

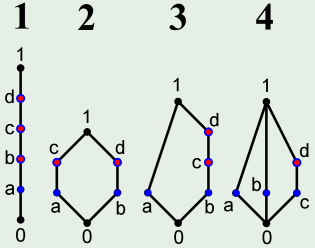
**Theorem**

*Let L be a finite lattice with q atoms and r coatoms. Let $H_L \subset L$ be a set of minimal cardinality generating the lattice L. Let $G \subset \mathcal{O}_A$ be a set of minimal cardinality generating the aggregation clone $\mathcal{AGG}_L$. Then*

$$|G| \leq \min\left(|J_L| - q, |M_L| - r\right) + |H_L| + 2.$$

# The cardinality of $(\chi,\oplus)$-generating sets

## Example

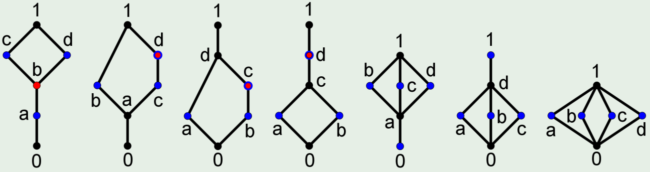



- $G_1 = \{\chi_b, \chi_c, \chi_d\} \cup \{\oplus_a, \oplus_b, \oplus_c, \oplus_d\} \cup \{\vee, \wedge\}$ . . . *greatest cardinality*
- $G_2 = \{\chi_c, \chi_d\} \cup \{\oplus_a, \oplus_b, \oplus_c, \oplus_d\} \cup \{\vee, \wedge\}$
- $G_3 = \{\chi_c, \chi_d\} \cup \{\oplus_a, \oplus_b, \oplus_c, \oplus_d\} \cup \{\vee, \wedge\}$
- $G_4 = \{\chi_d\} \cup \{\oplus_a, \oplus_b, \oplus_c, \oplus_d\} \cup \{\vee, \wedge\}$

# The cardinality of $(\chi,\oplus)$-generating sets

## Example

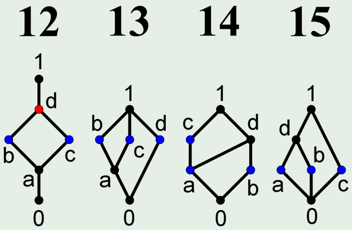
**5 6 7 8 9 10 11**




- $G_5 = \{\chi_b\} \cup \{\oplus_a, \oplus_c, \oplus_d\} \cup \{\vee, \wedge\}$
- $G_6 = \{\chi_d\} \cup \{\oplus_a, \oplus_c, \oplus_d\} \cup \{\vee, \wedge\}$
- $G_7 = \{\chi_c\} \cup \{\oplus_a, \oplus_b, \oplus_c\} \cup \{\vee, \wedge\}$
- $G_8 = \{\chi_d\} \cup \{\oplus_a, \oplus_b, \oplus_d\} \cup \{\vee, \wedge\}$
- $G_9 = \{\oplus_0, \oplus_a, \oplus_b, \oplus_c\} \cup \{\vee, \wedge\}$
- $G_{10} = \{\oplus_a, \oplus_b, \oplus_c, \oplus_1\} \cup \{\vee, \wedge\}$
- $G_{11} = \{\oplus_a, \oplus_b, \oplus_c, \oplus_d\} \cup \{\vee, \wedge\}$

# The cardinality of $(\chi,\oplus)$-generating sets

## Example




- $G_{12} = \{\chi_d\} \cup \{\oplus_b, \oplus_c\} \cup \{\vee, \wedge\}$ . . . *smallest cardinality*
- $G_{13} = \{\oplus_b, \oplus_c, \oplus_d\} \cup \{\vee, \wedge\}$ . . . *smallest cardinality*
- $G_{14} = \{\oplus_a, \oplus_b, \oplus_c\} \cup \{\vee, \wedge\}$ . . . *smallest cardinality*
- $G_{15} = \{\oplus_a, \oplus_b, \oplus_c\} \cup \{\vee, \wedge\}$ . . . *smallest cardinality*

# The greatest cardinality of a $(\chi,\oplus)$-generating set

## Proposition

*Let $n \geq 4$ be a positive integer and $L$ be $n$-element lattice. Let $\alpha \in L$ be an atom of lattice $L$. Consider the set*

$$G = \Big\{\chi_b \mid b \in L \setminus \{0, \alpha, 1\}\Big\} \cup \Big\{\oplus_c \mid c \in L \setminus \{0, 1\}\Big\} \cup \Big\{\vee, \wedge\Big\}.$$

*Then the lattice $L$ is a chain if and only if the set $G$ is a minimal $(\chi, \oplus)$-generating set of the aggregation clone $\mathcal{AGG}_L$.*

- $|G| = 2n - 3$
- $n = 2 \quad \ldots \quad G = \{\vee, \wedge\}$
- $n = 3 \quad \ldots \quad G = \{\oplus_0, \oplus_a, \oplus_1\} \cup \{\vee, \wedge\}$

# A generating set of an aggregation clone on an atomary lattice

**Definition**

Let $L$ be a lattice. Let $J_L$ be the set of join-irreducible elements of $L$ and $At_L$ be the set of atoms of $L$. Then $L$ is called an *atomary lattice* if $J_L = At_L$.

**Proposition**

*Let L be a finite atomary lattice. Let $H_L$ be a minimal generating set of lattice L. Then the set $\{\oplus_b \mid b \in H_L\} \cup \{\vee, \wedge\}$ is a minimal ($\chi$,$\oplus$)-generating set of the aggregation clone $\mathcal{AGG}_L$.*

# The smallest cardinality of a $(\chi,\oplus)$-generating set

## Hypothesis

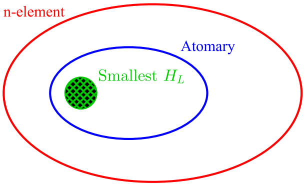
*Let L be an n-element atomary (resp. coatomary) lattice such that for another n-element atomary (resp. coatomary) lattice M we have*

$$|H_L| \leq |H_M|,$$

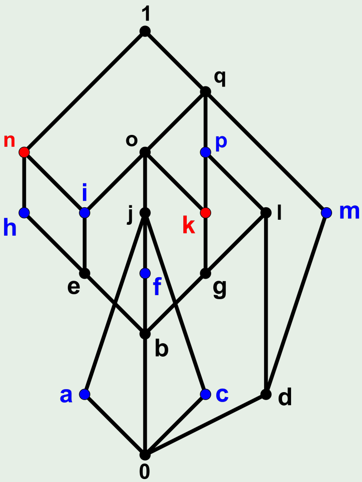
*where $H_L$ and $H_M$ are sets of minimal cardinality generating lattices L and M, respectively. Then the sets $G_L$ and $G_M$ of minimal cardinality generating aggregation clones $\mathcal{AGG}_L$ and $\mathcal{AGG}_M$ fulfill*

$$|G_L| \leq |G_M|.$$

- $|G_L| = |H_L| + 2$



# Example




- $G = \{\chi_n, \chi_k\} \cup \{\oplus_a, \oplus_c, \oplus_f, \oplus_h, \oplus_i, \oplus_m, \oplus_p\} \cup \{\vee, \wedge\}$

# References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Thank you for your attention