The sum of observables on a σ-frame effect algebra

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56th Summer School on General Algebra and Ordered Sets

Špindlerův Mlýn

2-7 September, 2018

Effect algebras

Definition (Foulis, Bennett, 1994)

A partial algebra $(E; \oplus, 0, 1)$ is called an *effect algebra* if 0, 1 are two distinct elements and $\oplus$ is a partially defined binary operation on E which satisfy the following conditions for any $x, y, z \in E$:

(Ei) $x \oplus y = y \oplus x$ if $x \oplus y$ is defined,

(Eii) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ if one side is defined,

(Eiii) for every $x \in E$ there exists a unique $y \in E$ such that $x \oplus y = 1$ (we put $x' = y$),

(Eiv) if $1 \oplus x$ is defined then $x = 0$.

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A partial order $\leq$ on E can be introduced by:

$$x \leq y \quad \text{iff } x \oplus z \quad \text{is defined and} \quad x \oplus z = y.$$

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- σ-*frame* effect algebra – if $(E, \leq)$ is a σ-*frame*, i.e., σ-complete lattice which for countable I satisfies

$$a \wedge (\bigvee_{i \in I} a_i) = \bigvee_{i \in I} (a \wedge a_i).$$

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An effect algebra E is σ-*frame* if and only E is σ-*coframe*.

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- Interval effect algebras – let $(G; +, \leq)$ be a partially ordered commutative group, $a \in G, 0 < a$. Then $[0, a] \subseteq G$ with $x \oplus y = x + y$ iff $x + y \leq a$ is an effect algebra.

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- Interval effect algebras – let $(G; +, \leq)$ be a partially ordered commutative group, $a \in G, 0 < a$. Then $[0, a] \subseteq G$ with $x \oplus y = x + y$ iff $x + y \leq a$ is an effect algebra.
- $\mathcal{E}(\mathcal{H}) := [\mathbf{0}, I] \subseteq \mathcal{B}(\mathcal{H})$ – an interval on bounded self-adjoint linear operators on a complex Hilbert space $\mathcal{H}$, with the usual addition, $A \leq B$ if $(Ax, x) \leq (Bx, x)$ for all $x \in \mathcal{H}$.

Observables

Definition

Let E be a monotone σ-complete effect algebra. An *observable* is a map $x : \mathcal{B}(\mathbb{R}) \to E$ such that

(i) $x(\mathbb{R}) = 1$,

(ii) if $A \cap B = \emptyset$, then $x(A \cup B) = x(A) \oplus x(B)$,

(iii) if $\{A_i\}_{i \in \mathbb{N}}$, $A_i \subseteq A_{i+1}$, then $x(\bigcup_i A_i) = \bigvee_i x(A_i)$.

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Measurable functions (random variables) $f : \Omega \to \mathbb{R}$ on a measure space $(\Omega, \mathcal{A}, p)$ induce σ-homomorphisms $x : \mathcal{B}(\mathbb{R}) \to \mathcal{A}$ by $x(B) = f^{-1}(B)$.

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Example

Observables on the prototype effect algebra $\mathcal{E}(\mathcal{H})$ bounded positive self-adjoint linear operators on a complex Hilbert space $\mathcal{H}$ between $\mathbf{0}$ and I are (normalized) positive-operator valued measures (POVM).

Theorem (Dvurečenskij, Kuková, 2014)

Let x be an observable on a σ-lattice effect algebra E. Let us set

(1) $B_x(t) := x((-\infty, t))$.

Spectral resolutions

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We call $\{B_x(t)\}_{t\in\mathbb{R}}$ a *spectral resolution* of x.

Olson order

Definition (Dvurečenskij, 2016)

Let $x, y \in \mathcal{BO}(E)$ be bounded observables on a monotone σ-complete effect algebra E. A relation $\leq$ on $\mathcal{BO}(E)$ given by

$$x \leq y \text{ iff } B_y(t) \leq_E B_x(t)$$

for every $t \in \mathbb{R}$ is a partial order, so called *Olson order*.

The sum of observables

Theorem (Dvurečenskij, 2016)

Let E be a σ-frame effect algebra and let $x, y \in \mathcal{BO}(E)$. Let:

$$B_{x+y}(t) := \bigvee_{r \in \mathbb{Q}} (B_x(r) \wedge B_y(t-r))$$

for all $t \in \mathbb{R}$. Then there is a unique bounded observable z on E such that $B_z(t) = B_{x+y}(t)$ for every $t \in \mathbb{R}$. We call z the sum of x, y.

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Theorem (Dvurečenskij, 2016)

Let E be a σ-frame effect algebra and $\mathcal{BO}(E)$ the set of all bounded observables on E. Then $\mathcal{BO}(E)$ is a distributive lattice and a lattice ordered commutative semigroup w.r.t. Olson order and the sum of observables.

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In the rest of talk, E will denote a σ-frame effect algebra.

Expressions of the sum of observables

Proposition

Let $x, y \in \mathcal{BO}(E)$, $\sigma(x) \subseteq [a_x, b_x), \sigma(y) \subseteq [a_y, b_y)$ *for some* $a_x, b_x, a_y, b_y \in \mathbb{R}$ *and let* $K := \mathbb{Q} \cap (a_x \vee t - c_y, t - a_y \wedge c_x)$. *Then*

$$B_{x+y}(t) = \bigvee_{r \in K} (B_x(r) \wedge B_y(t-r)).$$

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Theorem

Let $x, y \in \mathcal{B}(\mathbb{R})$. Let $M \subseteq \mathbb{R}$ be a subset of $\mathbb{R}$ such that for every $p \in \sigma(x)$ there exists $M_p \subseteq M$ such that $p = \bigwedge M_p$. Then

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$$B_{x+y}(t) = \bigvee_{m \in M} (B_x(m] \wedge B_y(t-m)).$$

We can take $M = \sigma(x)$, i.e., the expression becomes finite when x is a simple observable.

Expressions of the sum of observables

Corollary

Let x, y be bounded observables on a σ-distributive lattice effect algebra E. Then there exists at most countable set $M \subseteq \sigma(x)$ such that

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Theorem

Let $x, y \in \mathcal{BO}(E)$. Then

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Continuity of the spectral resolution

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Theorem

Let $x, y \in \mathcal{BO}(E)$ such as x has a continuous spectral resolution and y is simple. Then $x + y$ has a continuous spectral resolution.

Continuity of the spectral resolution

Let I be a set and $\{J_i \mid i \in I\}$ a family of non-empty sets. A complete lattice L is *completely distributive* if for any subset $\{x_{ij}\}_{i\in I, j\in J}$ of L we have $\bigwedge_{i\in I}(\bigvee_{j\in J} x_{ij}) = \bigvee_{f:I\to J}(\bigwedge_{i\in I} x_{if(j)})$.

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Theorem

Let $x, y \in \mathcal{BO}(E)$ be bounded observables on a completely distributive lattice effect algebra E. If x has a continuous spectral resolution, then $x + y$ has a continuous spectral resolution.

Spectra of the sum

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Corollary

Let $x, y \in \mathcal{BO}(E)$ *such that* $|\sigma(x)| = m$ *and* $|\sigma(y)| = n$. *Then* $\sigma(x+y) \subseteq \{r + s \mid r \in \sigma(x), s \in \sigma(y)\}$, *that is,* $|\sigma(x+y)| \leq m \cdot n$.

There exists an example of $x, y \in \mathcal{BO}(E)$ such that $|\sigma(x+y)| = m \cdot n$.

Extremal points of the spectrum of the sum

Theorem

Let $x, y \in \mathcal{BO}(E)$. *TFAE:*

1.) *if* $p, q \in \mathbb{Q}$ *and* $B_x(p) > 0, B_y(q) > 0$, *then* $B_x(p) \wedge B_y(q) > 0$,

2.) $\bigwedge \sigma(x) + \bigwedge \sigma(y) \in \sigma(x+y)$,

3.) $\bigwedge \sigma(x) + \bigwedge \sigma(y) = \bigwedge \sigma(x+y)$.

Moreover, TFAE:

1.) *if* $p, q \in \mathbb{Q}$, $p < \bigvee \sigma(x), q < \bigvee \sigma(y)$, *then* $B_x(p) \vee B_y(q) < 1$,

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Inverse elements

- Let $f : \mathbb{R} \to \mathbb{R}$ be a mapping defined by $f(t) := -t$. Define $-x : \mathcal{B}(\mathbb{R}) \to E$ by $-x(A) := x(f(A))$,

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Theorem (Dvurečenskij 2016)

Let E be a σ-frame effect algebra. The set of sharp bounded observables $\mathcal{SBO}(E) \subseteq \mathcal{BO}(E)$ is with respect to Olson order and the sum of observables a lattice-ordered group in which $-x$ is the inverse element of x and the neutral element q_0 is given by $\sigma(q_0) = \{0\}$. Moreover, $\mathcal{SBO}(E)$ is a subsemigroup and a sublattice of the lattice-ordered semigroup $\mathcal{BO}(E)$.

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Theorem

Let $x, y \in \mathcal{BO}(E)$. Then $(-x) + (-y) = -(x + y)$.

Decomposition to a sharp and a meager part

By $\hat{x}$ ($\tilde{x}$ respectively) we denote the least (greatest) sharp observable greater (less) than x. We say that x is a *meager* observable if $\sigma(\tilde{x}) = \{a\}$ and a *dense* observable if $\sigma(\hat{x}) = \{b\}$.

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Let $x \in \mathcal{BO}(E)$. Then $x = \tilde{x} + x_m = \hat{x} + x_d$ where x_m is a meager observable and x_d a dense observable. Moreover, we have $x_m = -(-x)_d$ and $\bigwedge \sigma(x_m) = \bigvee \sigma(x_d) = 0$.

Decomposition to a sharp and a meager part

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Theorem

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Theorem

Subsets $\mathcal{M}_0\mathcal{BO}(E), \mathcal{D}_0\mathcal{BO}(E) \subseteq \mathcal{BO}(E)$ of meager, resp. dense, observables such that $\bigwedge \sigma(x_m) = 0$, resp. $\bigvee \sigma(x_d) = 0$, forms subsemigroups and sublattices of the lattice-ordered semigroup $\mathcal{BO}(E)$. Moreover,

$$\mathcal{BO}(E) \cong \mathcal{SBO}(E) \oplus \mathcal{M}_0\mathcal{BO}(E) \cong \mathcal{SBO}(E) \oplus \mathcal{D}_0\mathcal{BO}(E).$$

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Thank you for your attention!