The variety of reduced Rickart rings

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Outline

Motivation

- The class of reduced Rickart rings expanded with a suitably chosen derived operation is a variety.
- This talk is about some properties of this variety:
  - Congruence properties
  - Ideals and homomorphisms
  - Subdirectly irreducible reduced Rickart rings

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Reduced Rickart rings

Definition

A ring R is called right Rickart ring (or right PP ring) iff for every $a \in R$ there exists $e \in R$ such that

- e is idempotent,
- for all $x \in$,

$$ax = 0 \iff ex = x.$$

Definition

A ring R is called reduced iff

$$a^n = 0 \implies a = 0$$

for arbitrary $n \in \mathbb{N}$ and $a \in R$.

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Examples

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- Boolean rings
- Domains (i.e., rings without proper zero divisors)
- $\mathbb{Z}_n$ for any square-free n

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The focal operation

Lemma

In a reduced Rickart ring R, for every $a \in R$ there exists a unique idempotent a' such that, for all $x \in R$,

$$ax = 0 \iff a'x = x.$$

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The operation $'$ is called focal operation.

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Focal rings and π-rings

Definition

A reduced focal ring is an algebra $\langle R, *, +, -, ', 0, 1\rangle$ such that

- $\langle R, *, +, -, 0, 1\rangle$ is a reduced Rickart ring,
- $'$ is its focal operation.

Definition

A reduced π-ring is an algebra $\langle R, *, +, -, \pi, 0, 1\rangle$ such that

- $\langle R, *, +, -, 0, 1\rangle$ is a reduced Rickart ring,
- π is the unary operation defined by $\pi(x) := (x')'$.

- From any reduced π-ring R we obtain a reduced focal ring by defining $x' := 1 - \pi(x)$.

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Two varieties of reduced Rickart rings

Theorem

The class of reduced focal rings is a variety.

Corollary

The class of reduced π-rings is a variety.

- These varieties are isomorphic as categories.

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Some trivial consequences of being a ring

Definition

A variety is called

- congruence permutable iff the congruences of every algebra commute
  (i.e., $\theta \circ \sigma = \sigma \circ \theta$ for all congruences θ, σ, where $\circ$ is the composition of congruences).
- regular iff every algebra and any of its congruences θ, σ, if $[a]_\theta = [a]_\sigma$ then $\theta = \sigma$, where $[a] = \{x \in A \mid x\theta a\}$.

- The variety of rings is congruence permutable and regular.
- Hence, the same holds for the variety of reduced focal rings.

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Congruence distributivity

Definition

A variety is called congruence distributive iff the congruence lattices of all its algebras are distributive.

Theorem

The variety of reduced focal rings is congruence distributive.

- The majority term $t(x, y, z)$ is obtained as follows:
- $x \wedge y := (x - y)'x$,
- $x \triangleleft y := x + y - y''x$,
- $t(x, y, z) = ((x \wedge y) \triangleleft (y \wedge z)) \triangleleft (z \wedge x)$.

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Focal ideals and focal homomorphisms

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An ideal I of a reduced focal ring R is called focal ideal iff for all $a \in R$,

$$\text{if } a \in I, \text{ then } a'' \in I.$$

Definition

A ring homomorphism $\phi : R \to S$ between reduced focal rings is called focal homomorphism iff for all $a \in R$,

$$\phi(a') = (\phi(a))'.$$

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Quotients

Lemma

Let R be a reduced focal ring and P a focal ideal. Then

- *${}^{R}/_{I}$ is a reduced Rickart ring,*
- *the canonical epimorphism $R \to {}^{R}/_{I}$ is a focal homomorphism.*

Theorem

An ideal I of a reduced focal ring R is a focal ideal iff it is the kernel of a focal homomorphism.

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Minimal prime ideals

Definition

- Let R be a ring and $P \subset R$ a proper ideal.
- P is called prime iff for all ideals I, J of R,

$$\text{if } IJ \subseteq P\text{, then } I \subseteq P \text{ or } J \subseteq P.$$

Theorem

Any intersection of minimal prime ideals of a reduced Rickart ring is a focal ideal.

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Subdirect representations

Definition

- Let A and A_i, $i \in I$ be algebras from some variety $\mathcal{V}$.
- A subdirect representation of an algebra $A \in \mathcal{V}$ is a monomorphism

$$s : A \rightarrowtail \Pi_{i \in I} A_i$$

such that, for every projection $p_j : \Pi_{i \in I} A_i \to A_j$, the function

$$p_j \circ s : A \to A_j$$

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Subdirect irreducibility

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Let $\mathcal{V}$ be a variety.
An algebra $A \in \mathcal{V}$ is called subdirectly irreducible iff

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Subdirectly irreducible reduced focal rings

Theorem

- *Let R be a reduced focal ring.*
- *Then R is subdirectly irreducible iff its ring reduct is a domain.*
- *In this case, it is also simple.*

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Further research

- Free focal rings

Thank you for your attention

Questions?