Meet-irreducibility of congruence lattices of connected algebras

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Introduction

- **congruence of $\mathcal{A} = (A, F)$**: a binary relation on A, which is
  - relation of equivalence (i.e. reflexive, transitive and symmetric)
  - compatible with all fundamental operations of $\mathcal{A}$

- the lattice $(\mathrm{Con}(\mathcal{A}), \subseteq)$ of all congruences of an algebra $\mathcal{A}$

- $\mathcal{E}_A$ - system of all $\mathrm{Con}(A, F)$ - **the lattice of congruence lattices** of all algebras $\mathcal{A} = (A, F)$

- $\mathcal{E}_A$ forms a lattice (with respect to $\subseteq$)

Introduction

L - lattice, $x, a, b \in L$

- x is $\wedge$-irreducible if $x = a \wedge b$ implies $x \in \{a, b\}$
- x is $\vee$-irreducible if $x = a \vee b$ implies $x \in \{a, b\}$

$F \subseteq G$ implies $\mathrm{Con}(A, G) \subseteq \mathrm{Con}(A, F)$, hence we get $\mathrm{Con}(A, \{f_1, f_2, \dots\}) = \bigwedge \mathrm{Con}(A, f_i)$, $i \in 1, 2, \dots$

$\Rightarrow$ All $\wedge$-irreducible elements in $\mathcal{E}_A$ are of the form $\mathrm{Con}(A, f)$ for a single mapping f i.e. it is sufficient to consider monounary algebras.

Preliminary

(A, f) - monounary algebra

- operation $f \in A^A$ is called **trivial**, if it is identity of constant
- $f \in A^A$ is called **acyclic**, if each its cycle has length 1
- if $f(x)$ belongs to a cycle for $\forall x \in A$, then (A, f) is called **an algebra with short tails**
- if each cycle of (A, f) contains at most 2 elements, then (A, f) is called **an algebra with small cycles**

Meet-irreducibility in $\mathcal{E}_A$

Open problem: characterization of $\wedge$-irreducible elements in $\mathcal{E}_A$

Partial answers: characterization of $\wedge$-irreducible elements in $\mathcal{E}_A$ in the case that:

- (A, f) is algebra with short tails, or
- (A, f) is algebra with small cycles

Our aim

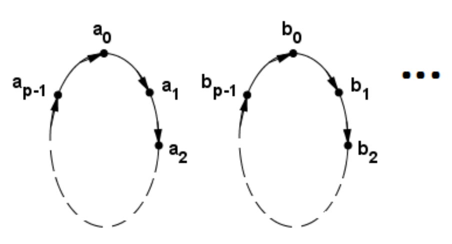
Characterization of $\wedge$-irreducible elements in $\mathcal{E}_A$ in the case that (A, f) is **connected algebra**.

Lemma (JPR)

Let $f \in A^A$ be a transposition $(|A| \geq 3)$. Then $\mathrm{Con}(A, f)$ is not $\wedge$-irreducible

Theorem (JPR)

A congruence lattice $Con(A, f)$ with a nontrivial permutation f is $\wedge$-irreducible in $\mathcal{E}$ is and only if f is of prime power order p^m with at least two cycles of length p^m.



Meet-irreducibility for permutation-algebras with short tails

Theorem

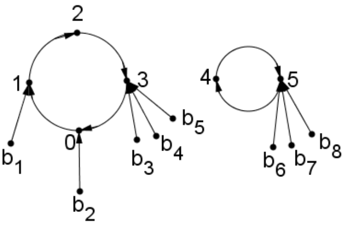
Let $A, B \neq \emptyset$, $f \in A^A$ a nontrivial permutation. Let $\bar{A} = A \cup B$ and $\bar{f}(x) = \begin{cases} f(x), & x \in A \\ \text{some element of } A, & x \in B. \end{cases}$

Then $\mathrm{Con}(\bar{A}, \bar{f})$ is $\wedge$-irreducible in $\mathcal{E}_{\bar{A}} \Leftrightarrow \mathrm{Con}(A, f)$ is $\wedge$-irreducible in $\mathcal{E}_A$.

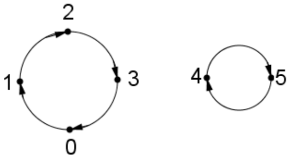
example:

$B = \{b_1, b_2, \ldots, b_8\}$

$(\bar{A}, \bar{f})$:



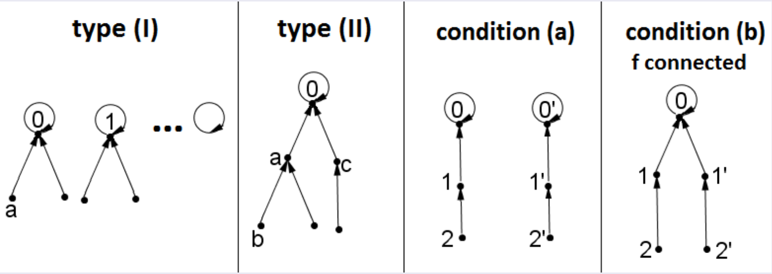
(A, f):



Meet-irreducibility for acyclic case

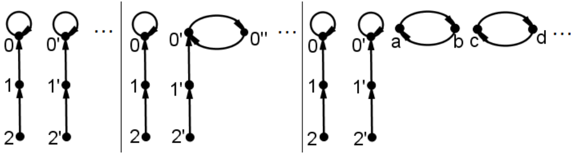
Theorem (JPR)

A congruence lattice $Con(A, f)$ with an acyclic $f \in A^A$ is $\wedge$-irreducible in $\mathcal{E}$ if and only if f is of type (I) or (II) or satisfies condition (a) or (b).

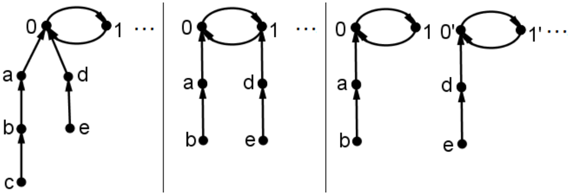


(All labeled elements are mandatory, all others are optional.)

Meet-irreducibility for small cycles



Obr. : Operations satisfying the condition(c)



Obr. : Operations satisfying the condition(d)

Theorem

Let (A, f) be a monounary algebra with small cycles. Then $\mathrm{Con}(A, f)$ is $\wedge$-irreducible iff one of the following holds:

(1) *(A, f) is connected and f is of type (II) or satisfies condition (b), or*

(2) *(A, f) is a permutation with short tails such that the corresponding permutation is either identity or contains at least two nontrivial cycles, or*

(3) *f satisfies condition (c) or (d).*

Connected algebras

Lemma

Let $f, g \in A^A$ be nontrivial and $\mathrm{Con}(A, f) \subseteq \mathrm{Con}(A, g)$. Then we have

(i) $\forall x, y \in A : (x, y) \in \varkappa \in \mathrm{Con}(A, f) \implies (g(x), g(y)) \in \varkappa$, *in particular we have* $(g(x), g(y)) \in \theta_f(x, y)$ *and* $\theta_g(x, y) \subseteq \theta_f(x, y)$.

(ii) *Let B be a subalgebra of (A, f). Then either B is also a subalgebra of (A, g) or g is constant on B, where the constant does not belong to B.*

Lemma

Let $f, g \in A^A$ be nontrivial. Then
$\mathrm{Con}(A, f) \subseteq \mathrm{Con}(A, g) \Leftrightarrow \forall x, y \in A$: $(g(x), g(y)) \in \theta_f(x, y)$.

Let (A, f) be a monounary algebra.

- We denote $f^{-1}(a) := \{x \in A, f(x) = a\}$.
- Let $t_f(a) := \min\{n \in \mathbb{N}_0, f^n(a)\text{is cyclic}\}$.
- For every $x \in A$ acyclic, such that $t_f(x) = t$, there exists unique cyclic element $x' \in A$ such that $f^t(x) = f^t(x')$. We call x' a colleague of x.
- For a cyclic element $x \in A$ and for $n \in \mathbb{N}$, we denote the set of all acyclic $y \in A$ such that $f^n(y) = x$ by $C_n(x)$.

Connected algebras

Lemma

Let (A, f) be a monounary algebra such that $\mathrm{Con}(A, f)$ is not $\wedge$-irreducible. Let a, b be noncyclic elements of (A, f) with $f(b) = a$. Then there exists $g \in A^A$ nontrivial such that $\mathrm{Con}(A, f) \subsetneq \mathrm{Con}(A, g)$ and one of the following holds:

(1) *$g(a) = a'$, $g(b) = a = g(a')$ and g is constant on the set of all cyclic elements of (A, f), or*

(2) *$g(b) = a' = g(a)$, $g(a') = a$ and g is constant on the set of all cyclic elements of (A, f), or*

(3) *g is constant on $\{a, b, a', b'\}$ and the constant is a, or*

(4) *$g(b) = a, g(b') = a'$, and there exists $i \in \mathbb{N}$ such that $g^j(a) = g^j(a')$ for every $j \in \mathbb{N}, j \geq i$, or*

(5) *$g(b) = a, g(b') = a'$ and there exists $i \in \mathbb{N}_0$ such that $g^j(a) = g^i(a), g^j(a') = g^i(a')$, $g^j(a) \neq g^j(a')$ for every $j \in \mathbb{N}_0, j \geq i$.*

Connected algebras

Lemma

Let (A, f) be a connected monounary algebra with short tails and with one simple tail of the length $k \geq 2$. Then $\mathrm{Con}(A, f)$ is not $\wedge$-irreducible.

Proof

If $|A| = 3$ then it is trivial to see that $\mathrm{Con}(A, f)$ fails to be $\wedge$-irreducible. Hence we can assume that $|A| \geq 4$.
Denote the elements of simple tail as $a_1, a_2, \ldots, a_k$ where $t_f(a_i) = i$ for every i. Let us define mappings g_1, g_2 on A as follows:

$$g_1(x) = \begin{cases} f(a'_k), & \text{if } x = a_k, \\ f(x), & \text{otherwise}, \end{cases}$$

$$g_2(x) = \begin{cases} a'_{k-1}, & \text{if } x \in \{a_k, a_{k-1}\}, \\ a_{k-1}, & \text{otherwise}. \end{cases}$$

Connected algebras

Proposition

Let (A, f) be a connected monounary algebra with cycle of the length ≥ 2, such that for every cyclic element $x \in A$ holds $C_2(x) \neq \emptyset$. Then $\mathrm{Con}(A, f)$ is $\wedge$-irreducible.

By way of contradiction, assume that $\mathrm{Con}(A, f)$ is $\wedge$-reducible i.e. that there exist nontrivial operations $g_i \in A^A, i \in I, |I| \geq 2$ such that for every $i \in I$ is $\mathrm{Con}(A, f) \subsetneq \mathrm{Con}(A, g_i)$ and

$$\mathrm{Con}(A, f) = \bigcap_{i \in I} \mathrm{Con}(A, g_i).$$

Therefore for every $x, y \in A$ holds

$$\theta_f(x, y) = \bigvee_{i \in I} \theta_{g_i}(x, y)...$$

Connected algebras

Lemma

Let (A, f) be a connected monounary algebra with $p \geq 3$ cyclic elements, where p is a prime number. Let there be $x, y \in A$ cyclic such that $C_2(x) \neq \emptyset$ and $C_2(y) = \emptyset$, and let $C_3(a) = \emptyset$ for every cyclic element $a \in A$. Then $\mathrm{Con}(A, f)$ is $\wedge$-reducible.

Proof

We denote the cyclic elements of (A, f) by $0, 1, 2, \ldots, p-1$ with $f(i) = i + 1$ for every $i \in \{0, 1, \ldots, p-2\}$, $f(p-1) = 0$. Consider cyclic element $k \in A$ such that $C_2(f(k)) = \emptyset$. Then we define:

$$g_1(x) = \begin{cases} f(x'), & \text{if } t_f(x) = 2, \\ f(x), & \text{otherwise}, \end{cases}$$

$$g_2(x) = \begin{cases} l, & l \in f^{-2}(k) \cap C(k), \text{if } x \in f^{-1}(k), \\ f(x), & \text{otherwise}. \end{cases}$$

Ďakujem za Vašu pozornosť.

Bibliography

1. G. Grätzer, E. T. Schmidt, *Characterizations of congruence lattices of abstract algebras*. Acta Sci. Math. (Szeged) 24, (1963), 34–59.
2. D. Jakubíková-Studenovská, L. Janičková, *Meet-irreducible congruence lattices*. Algebra Universalis (to appear).
3. D. Jakubíková-Studenovská, R. Pöschel, S. Radeleczki, *The lattice of congruence lattices of algebra on a finite set*. Algebra Universalis (to appear). arXiv:1612.07648 [math.GM]. (accessed April 20, 2017).
4. D. Jakubíková-Studenovská, R. Pöschel, S. Radeleczki, *The lattice of quasiorder lattices of algebras on a finite set*. Algebra Universalis 75(2), (2016), 197–220.
5. L. Janičková, *Zlúčiteľné relácie na algebrách*. Košice: P.J. Šafárik Univ., 2017.