The balanced {2, 3}-hyperidentities of length four in invertible algebras and {3}-hyperidentities of associativity in semigroups

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INTRODUCTION

Definition of hyperidentity.

$$\forall X_1,\ldots, X_m ; \forall x_1,\ldots, x_n \ (W_1 = W_2) \qquad (1)$$

or

$$W_1 = W_2$$

According to the definition, the hyperidentity $W_1=W_2$ is said to be satisfied in the algebra (Q,Σ) if this equality holds when every functional variable X_i is replaced by any arbitrary operation of the corresponding arity from Σ and every object variable x_j is replaced by any arbitrary element from Q .

If the arities of the functional variables are:

$$|X_1|=n_1,\ldots, |X_m|=n_m$$

then the hyperidentity $W_1=W_2$ is called $\{n_1,\ldots,n_m\}$-hyperidentity.

- A hyperidentity is balanced if each object variable of the hyperidentity occurs in both parts of the equality $W_1 = W_2$ only once.

- A balanced hyperidentity is called first sort hyperidentity, if the object variables on the left and right parts of the equality are ordered identically.

- The number of the object variables in a balanced hyperidentity is called length of   this hyperidentity.

- The algebra (Q,Σ) with the binary and ternary operations is called {2, 3}-algebra.

- A {2, 3}-algebra is called non-trivial, if the sets of its binary and ternary operations are not singleton.

Classification of {2,3}-hyperidentities

The balanced first sort {2, 3}-hyperidentities of length four in non-trivial invertible algebras are classified.

$$X(Y(x,y,z),u) = X(x,Y(y,z,u)) \qquad (1)$$

$$Y(X(x,y),u,v) = Y(x,X(y,u),v) \qquad (2)$$

$$Y(X(x,y),u,v) = Y(x,y,X,(u,v)) \qquad (3)$$

$$X(Y(x,y,z),u) = Y(X(x,y),z,u) \qquad (4)$$

$$X(Y(x,y,z),u) = Y(x,X(y,z),u) \qquad (5)$$

$$X(Y(x,y,z),u) = Y(x,y,X(z,u)) \qquad (6)$$

The invertible {2, 3}-algebras with a binary group operation, which satisfy the balanced first sort {2, 3}-hyperidentities of the length four are described

Theorem 1. Suppose that an invertible {2,3}–algebra $Q(\Sigma)$ has a binary operation $(\cdot)\in\Sigma$, such that $Q(\cdot)$ is a group. Then the hyperidentity

$$\mathbf{X(Y(x,y,z),u)= X(x,Y(y,z,u))} \qquad \mathbf{(1)}$$

is satisfied in the algebra $Q(\Sigma)$, if each ternary operation $A_i \in\Sigma$ is defined by the rule

$$\mathbf{A_i(x,y,z)=x\cdot y\cdot z\cdot t_i}$$

where

$$\mathbf{t_i \in Z(Q)}$$

(which is the center of the group $Q(\cdot)$),

and each binary operation $B_j \in \Sigma$ is defined by the rule

$$\mathbf{B_j\,(x,y)=\alpha_j(x \cdot y)}$$

where

$$\mathbf{\alpha_j : Q \longrightarrow Q}$$

is a bijection.

The invertible {2, 3}-algebras with a ternary group operation,which satisfy the balanced first sort {2, 3}-hyperidentities of the length four are described

Theorem 2. Suppose that an invertible {2,3} -algebra $Q(\Sigma)$ has a ternary operation $A \in \Sigma$, such that $Q(A)$ is a ternary group. Then the hyperidentity

$$\mathbf{X(Y(x,y,z),u)= X(x,Y(y,z,u)} \qquad \mathbf{(1)}$$

is satisfied in the algebra $Q(\Sigma)$, if each ternary quasigroup operation $A_i \in \Sigma$ is defined by the rule

$$\mathbf{A_i(x,y,z)=x\cdot\theta y \cdot c \cdot z \cdot t_i}$$

and each binary quasigroup operation is defined by the rule

$$\mathbf{B_j\ (x,y)=\ \alpha_j\,(x\cdot\theta y)}$$

where Q (·) is group, θ is an automophysics of Q (·) group,

$$\mathbf{t_i,\ c\in Q\ ,\ \ \theta c=c,\ \theta(t_i)=t_i,\ \theta^2(x)=c\cdot x\cdot c^{-1}\ ,\ t_i\in Z(Q)}$$

(which is the center of the Q (·) group),

$$\mathbf{\alpha_j : Q \longrightarrow Q}$$

is a bijection.

The description of semigroups, which polynomially satisfy associative {3}- hyperidentities

Theorem 3. In order that

$$F(F(x,x,y),x,x)=F(x,x,F(y,x,x))$$

hyperidentity takes place in polynomiallian way in $Q(\cdot)$ semigroup, its necessary and satisfactary that the following 4 identities should take place in $Q(\cdot)$ semigroup.

$$\begin{cases} x^3 = x^2, \\ xyx^2yx = (xy)^2x, \\ x^2(yx)^2 = x^2y^2x, \\ (xy)^2x^2 = xy^2x^2 : \end{cases}$$

Theorem 4

The semigroup $Q(\cdot)$ polynomially satisfies the following hyperidentity

$$\mathbf{F(F(x,x,x),x,x)=F(x,x,F(x,x,x))} \qquad \mathbf{(7)}$$

if $Q(\cdot)$ is a semigroup with the identity:

$$x^3=x^2$$

Theorem 5

The semigroup $Q(\cdot)$ polynomially satisfies the following hyperidentity

$$\mathbf{F(F(x,x,x),x,x)=F(x,x,F(x,y,x))} \qquad \mathbf{(8)}$$

if $Q(\cdot)$ is a semigroup with the identities:

$$\begin{cases} \mathbf{x^2y = xy,} \\ \mathbf{xy^2 = xy} \end{cases}$$

Theorem 6

The semigroup $Q(\cdot)$ polynomially satisfies the following hyperidentity

$$\mathbf{F(F(x,x,x),x,y)=F(x,x,F(x,x,y))} \qquad \mathbf{(9)}$$

if $Q(\cdot)$ is a semigroup with the identities:

$$\begin{cases} \mathbf{x^2y = xy,} \\ \mathbf{xy^2 = xy} \end{cases}$$

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