ON SPECIAL ELEMENTS OF LATTICE OF EPIGROUP VARIETIES

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An *epigroup* is a semigroup S with the following property: for any $x \in S$ there is n such that x^n lies in some subgroup of S.

All periodic semigroups as well as all completely regular semigroups are epigroups.

Epigroups may be considered as *unary semigroups*, that is semigroups with an additional unary operation.

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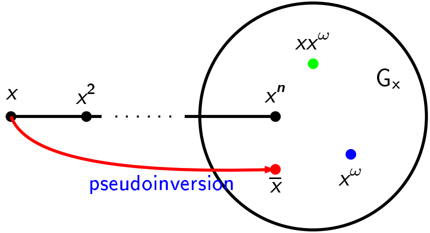
Let S be an epigroup, $x \in S$, G_x is the maximum subgroup of S containing x.

Let x^ω be a unit element of G_x. Then $xx^\omega = x^\omega x \in G_x$. Put

$$\overline{x} = (xx^\omega)^{-1} \text{ in } G_x.$$

$\overline{x}$ is called *pseudoinverse* to x



Epigroups were first studied by Douglas Munn in 1961, who called them *pseudoinvertible*.

Shevrin L N. *On the theory of epigroups*. I, II. Russian Academy of Sciences. Sbornik. Mathematics, 1995, Vol.82 №2, P. 485–512, Vol.83 №1, P.133–154.

Shevrin L.N. *Epigroups* in *Structural Theory of Automata, Semigroups, and Universal Algebra*, Kudryavtsev V.B., Rosenberg I.G. (eds.), 2005, P.331–380.

Shevrin L.N., Vernikov B.M., Volkov M.V. *Lattices of semigroup varieties*, Russian Math. Izv. VUZ, 2009, Vol.53, №3, P.1–28.

Every periodic semigroup variety can be considered as a variety of epigroups.

If an epigroup variety **V** consists of periodic semigroups then the operation of pseudoinversion may be defined by multiplication. Namely, if **V** satisfies the identity $x^m = x^{m+n}$ then $\overline{x} = x^{(m+1)n-1}$. Thus a variety of periodic epigroups can be considered as epigroup variety.

Periodic varieties of epigroups may be identified with periodic varieties of semigroups.

Periodic case

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Periodic varieties of epigroups may be identified with periodic varieties of semigroups.

If S is completely regular then $\overline{x} = x^{-1}$.

Varieties of completely regular semigroups are varieties of epigroups.

Three equivalent formulations of the distributive law:

1) $\forall x, y, z: \quad x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$
2) $\forall x, y, z: \quad x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
3) $\forall x, y, z: \quad (x \vee y) \wedge (y \vee z) \wedge (z \vee x) = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$

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   (ii) $\forall x, z: \quad x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ — y *costandard* element

2) $\forall x, y, z: \quad x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

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(v) $\forall y, z:\quad (x \vee y) \wedge (y \vee z) \wedge (z \vee x) = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$ — x *neutral* element

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The modularity law: $\forall x, y, z: \quad y \leq z \to (x \vee y) \wedge z = (x \wedge z) \vee y$

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(vi) $\forall y, z: \quad y \leq z \to (x \vee y) \wedge z = (x \wedge z) \vee y$ — x *modular*

(vii) $\forall x, z: \quad y \leq z \to (x \vee y) \wedge z = (x \wedge z) \vee y$ — y *lower-modular*

(viii) $\forall x, y: \quad y \leq z \to (x \vee y) \wedge z = (x \wedge z) \vee y$ — z *upper-modular*

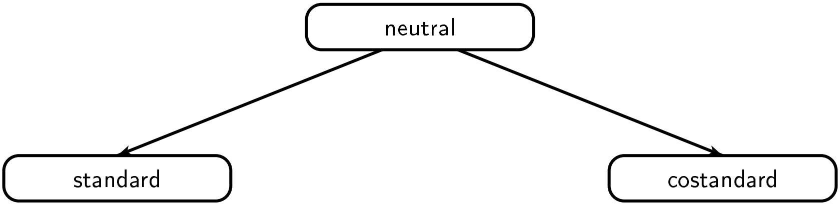
Cancellable elements

$$\forall y, z \quad (x \vee y = x \vee z \ \& \ x \wedge y = x \wedge z \to y = z)$$

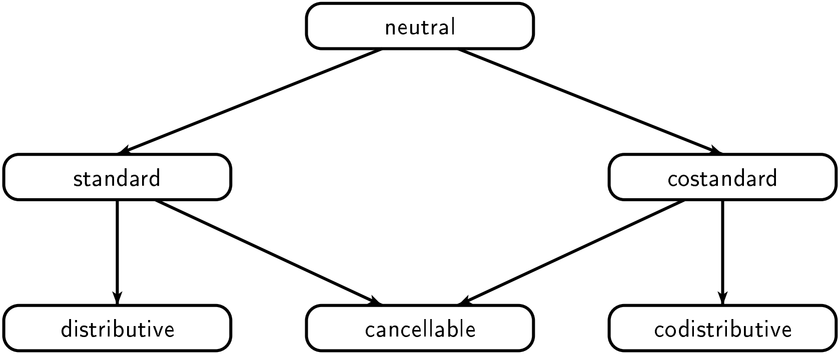
x — *cancellable element*

neutral

Interrelations between types of elements in abstract lattices



Interrelations between types of elements in abstract lattices



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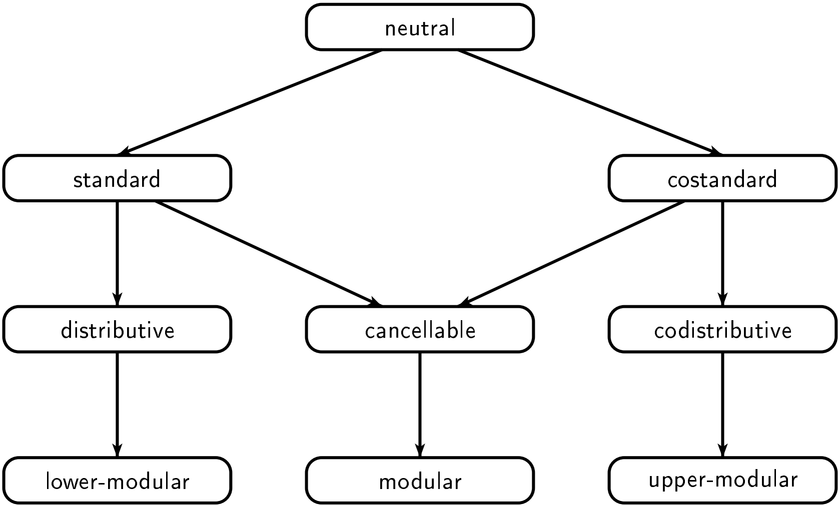

Theorem (Shaprynskii, Skokov, Vernikov - 2016)

Neutral elements of the lattice of epigroup varieties are

1. *the trivial variety* $\mathbf{T}$,
2. *the variety of all semilattices* $\mathbf{SL}$,
3. *the variety of all semigroups with zero multiplication* $\mathbf{ZM}$,
4. *the variety* $\mathbf{SL} \vee \mathbf{ZM}$

and only they.

The lattice of completely regular varieties has infinitely many neutral elements including

- all varieties of bands;
- the variety of all groups;
- the variety of all completely simple semigroups

and some others (Trotter, 1989).

Theorem (Shaprynskii, Skokov, Vernikov - 2016)

For an epigroup variety $\mathbf{V}$, *the following are equivalent:*

1. $\mathbf{V}$ *is a neutral element of the lattice* $\mathbf{EPI}$;
2. $\mathbf{V}$ *is a costandard element of the lattice* $\mathbf{EPI}$;
3. $\mathbf{V}$ *coincides with one of the varieties* $\mathbf{T}$, $\mathbf{SL}$, $\mathbf{ZM}$ *or* $\mathbf{SL} \vee \mathbf{ZM}$.

permutational identity: $x_1x_2\cdots x_m \approx x_{\pi(1)}x_{\pi(2)}\cdots x_{\pi(m)}$ where $\pi \in S_m$ — $p_m[\pi]$

$\mathbf{X}_{\infty,\infty} = \mathrm{var}\{x^2y \approx xyx \approx yx^2 \approx 0\}$,
$\mathbf{X}_{m,\infty} = \mathbf{X}_{\infty,\infty} \wedge \mathrm{var}\{p_m[\pi] \mid \pi \in S_m\}$ where $2 \le m < \infty$,
$\mathbf{X}_{m,n} = \mathbf{X}_{m,\infty} \wedge \mathrm{var}\{x_1x_2\cdots x_n \approx 0\}$ where $2 \le m \le n < \infty$,
$\mathbf{Y}_{m,n} = \mathbf{X}_{m,n} \wedge \mathrm{var}\{x^2 \approx 0\}$ where $2 \le m \le n \le \infty$.

Theorem (Shaprynskii, Skokov, Vernikov - 2016)

For an epigroup variety $\mathbf{V}$, *the following are equivalent:*

1. $\mathbf{V}$ *is a distributive element of the lattice* **EPI**;
2. $\mathbf{V}$ *is a standard element of the lattice* **EPI**;
3. $\mathbf{V} = \mathbf{M} \vee \mathbf{N}$ *where* $\mathbf{M}$ *is one of the varieties* $\mathbf{T}$ *or* $\mathbf{SL}$, *and* $\mathbf{N}$ *is one of the varieties* $\mathbf{X}_{\infty,\infty}$, $\mathbf{X}_{n,n}$, $\mathbf{Y}_{\infty,\infty}$, $\mathbf{Y}_{n,n}$.

0-reduced identity: $w \approx 0$, that is $wx \approx xw \approx w$ where x is a letter that does not occur in the word w.

Substitutive identity: $v \approx w$ where w is obtained from v by renaiming of letters

Examples: $xy \approx yx$, $xyz \approx yxz$, $x^2y \approx y^2x$, $xyx \approx yxy$ etc.

Theorem (Shaprynskii, Skokov, Vernikov - 2016)

If an epigroup variety $\mathbf{V}$ *is a modular element of the lattice of epigroup varieties then* $\mathbf{V} = \mathbf{X} \vee \mathbf{N}$ *where* $\mathbf{X}$ *is one of the varieties* $\mathbf{T}$ *or* $\mathbf{SL}$, *and* $\mathbf{N}$ *is a nil-variety given by 0-reduced and substitutive identities only.*

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0-reduced identity: $w \approx 0$, that is $wx \approx xw \approx w$ where x is a letter that does not occur in the word w.

Theorem (Skokov - 2016)

An epigroup variety $\mathbf{V}$ *is a lower-modular element of the lattice* $\mathbf{EPI}$ *if and only if* $\mathbf{V} = \mathbf{M} \vee \mathbf{N}$ *where* $\mathbf{M}$ *is one of the varieties* $\mathbf{T}$ *or* $\mathbf{SL}$, *while* $\mathbf{N}$ *is a 0-reduced variety.*

permutational identity: $x_1x_2\cdots x_m \approx x_{\pi(1)}x_{\pi(2)}\cdots x_{\pi(m)}$ where $\pi \in S_m$ — $p_m[\pi]$

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Theorem (Shaprynskii, Skokov, Vernikov - 2018)

An epigroup variety $\mathbf{V}$ *is a cancellable element of the lattice* **EPI** *if and only if* $\mathbf{V} = \mathbf{M} \vee \mathbf{N}$ *where* $\mathbf{M}$ *is one of the varieties* $\mathbf{T}$ *or* $\mathbf{SL}$, *while* $\mathbf{N}$ *is one of the varieties* $\mathbf{T}$, $\mathbf{X}_{m,n}$ *or* $\mathbf{Y}_{m,n}$ *with* $2 \le m \le n \le \infty$.

Theorem (Shaprynskii, Skokov, Vernikov - 2016)

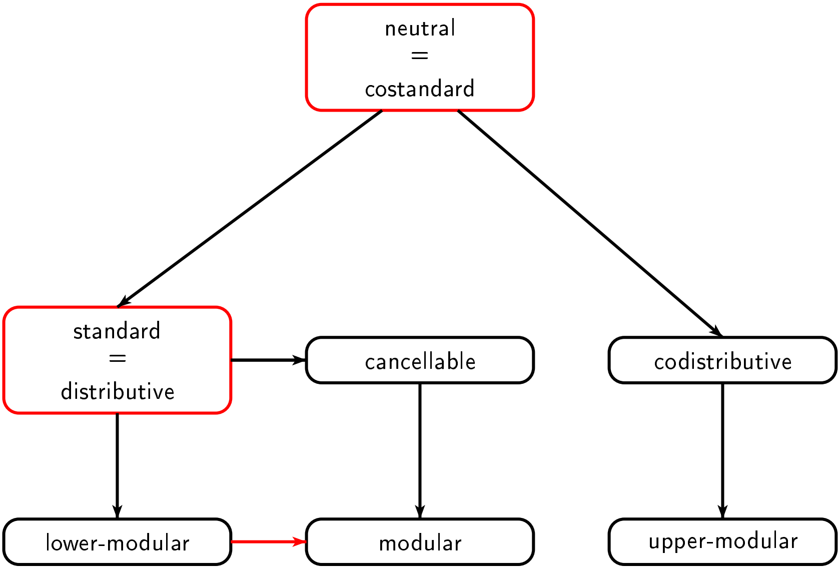
A commutative epigroup variety $\mathbf{V}$ *is a codistributive element of the lattice* $\mathbf{EPI}$ *if and only if* $\mathbf{V} = \mathbf{G} \vee \mathbf{X}$ *where* $\mathbf{G}$ *is an Abelian group variety, while* $\mathbf{X}$ *is one of the varieties* $\mathbf{T}$, $\mathbf{SL}$, $\mathbf{ZM}$ *or* $\mathbf{SL} \vee \mathbf{ZM}$.

$\mathbf{C_m} = \operatorname{var}\{x^m \approx x^{m+1}, xy \approx yx\}$

Theorem (Shaprynskii, Skokov, Vernikov - 2016)

A commutative epigroup variety $\mathbf{V}$ *is an upper-modular element of the lattice* **EPI** *if and only if one of the following holds:*

1. $\mathbf{V} = \mathbf{M} \vee \mathbf{N}$ *where* $\mathbf{M}$ *is one of the varieties* $\mathbf{T}$ *or* $\mathbf{SL}$*, and* $\mathbf{N}$ *is a nilvariety satisfying the commutative law and the identity* $x^2y \approx xy^2$;
2. $\mathbf{V} = \mathbf{G} \vee \mathbf{C}_m \vee \mathbf{N}$ *where* $\mathbf{G}$ *is an Abelian group variety,* $0 \le m \le 2$ *and* $\mathbf{N}$ *satisfies the commutative law and the identity* $x^2y \approx 0$.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costandard
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