On Monounary Algebras with EKP

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56th SSAOS
Špindlerův Mlýn, 2. – 7. 9. 2018

Outline

Introduction

- Definition of EKP
- Terminology

$\mathcal{A}$ ..... algebra
$End(\mathcal{A})$ ..... set of all endomorphisms of $\mathcal{A}$
$Con(\mathcal{A})$ ..... set of all congruences of $\mathcal{A}$

$Con(\mathcal{A}) = \{ker(\varphi), \varphi \in End(\mathcal{A})\}$ ..... $\mathcal{A}$ has EKP

Lemma

$\mathcal{A}$ has EKP *iff* *every homomorphic image of $\mathcal{A}$ is isomorphic to a subalgebra of $\mathcal{A}$.*

J. Chvalina, O. Kopeček, M. Novotný:
Homomorphic transformations - why and possible ways to how,
Brno, 2012.

$\mathcal{A} = (A, h), \ h : A \to A$

Terminology

1. D. Jakubíková-Studenovská, J. Pócs: *Monounary algebras*, Košice, 2009.
2. J. Chvalina, O. Kopeček, M. Novotný: *Homomorphic transformations - why and possible ways to how*, Brno, 2012.
3. R. McKenzie, G. McNulty, W. Taylor: *Algebras, Lattices, Varieties, vol.1*, Wadsworth, 1987.
4. J. Berman, P. M. Idziak: *Generative complexity in algebra*, vol.175 of Memoirs of the AMS, AMS, 2005.

General Properties

- Cycles
- Components

$\mathcal{A} = (A, h), \;\; h : A \to A$

If $h \in \{id_A, const\}$ then $\mathcal{A}$ has EKP.

Suppose that $\mathcal{A} = (A, h)$ has EKP.

If $\mathcal{A}$ consists of finitely many components, then every component of $\mathcal{A}$ has an 1-element cycle.

The number of fixed points of h is equal to the number of components of $\mathcal{A}$.

Let $k, l \in \mathbb{N}$ be such that l divides k, $l < k$.

The set of all cycles of $\mathcal{A}$ forms an algebra with EKP.

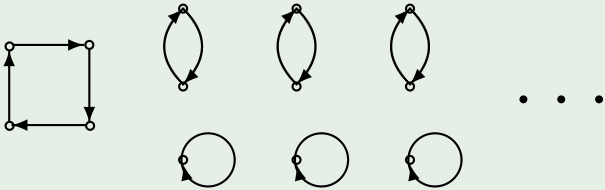
If κ is a cardinal number and $\mathcal{A}$ has κ cycles of length k,
then $\mathcal{A}$ has at least $\kappa \cdot \aleph_0$ cycles of length l.

.... the condition (γ)

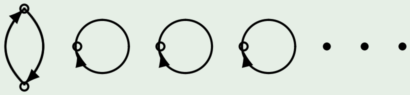
Let $k, l \in \mathbb{N}$ be such that l divides k, $l < k$.

(γ): If κ is a cardinal number and $\mathcal{A}$ has κ cycles of length k, then $\mathcal{A}$ has at least $\kappa \cdot \aleph_0$ cycles of length l.

the condition (γ) satisfied



Suppose that $\mathcal{A} = (A, h)$ has EKP and it is not connected.



components removed

has not EKP

EKP remains

one component of $\mathcal{A}$ removed	all components of $\mathcal{A}$ with a cycle	all components of $\mathcal{A}$ with 1-element cycle	. . .

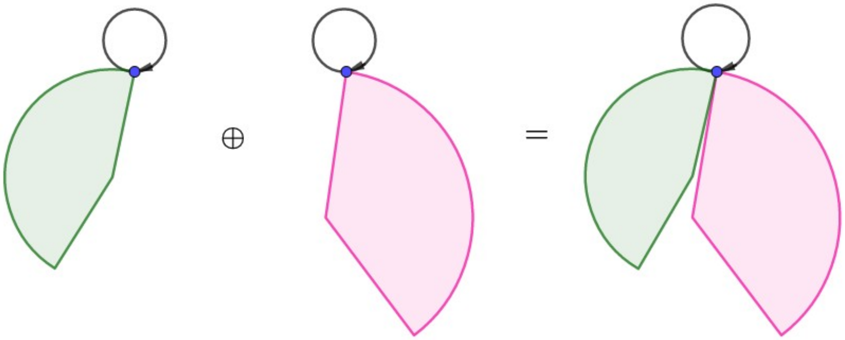
Equivalent Conditions

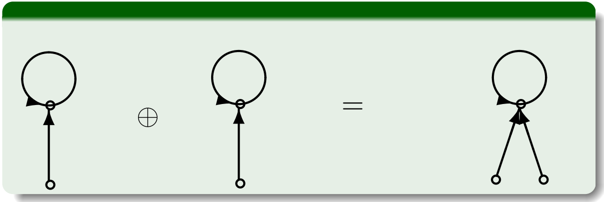
- Finitely Many Components
- Finitely Many Non-Injective Components

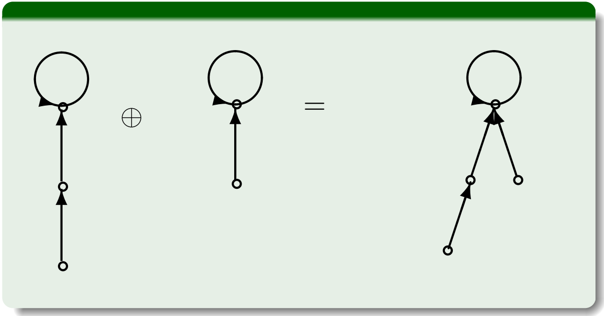
Let $\mathcal{A} = (A, h)$, $\mathcal{B} = (B, h)$

- be connected,
- have 1-element cycles,
- $A \cap B = \emptyset$.

Then the algebra $\mathcal{A} \oplus \mathcal{B}$ is defined as:







Suppose that $\mathcal{A} = (A, h)$
consists of finitely many components.

Theorem

Let I be a finite set and $\{A_i, i \in I\}$ be a component partition of the algebra $\mathcal{A}$. The algebra $\mathcal{A}$ has EKP if and only if

1. *the algebra $\mathcal{A}_i = (A_i, h)$ has EKP for every $i \in I$,*
2. *for each $J \subseteq I$, $J = \{j_1, \ldots, j_m\}$ there exists $j \in J$ such that $\mathcal{A}_{j_1} \oplus \cdots \oplus \mathcal{A}_{j_m}$ is isomorphic to a subalgebra of $\mathcal{A}_j$.*

Let $\mathcal{A} = (A, h)$, $\mathcal{B} = (B, h)$ and $A \cap B = \emptyset$.

$$\text{Denote } \mathcal{A} + \mathcal{B} = (A \cup B, h)$$

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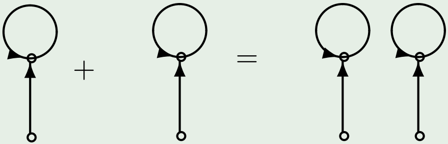
the operation + does not keep EKP



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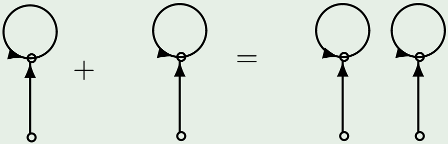

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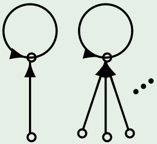


Theorem

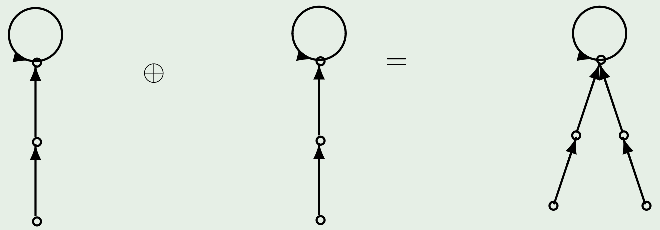
Let algebras $\mathcal{A} = (A, h), \mathcal{B} = (B, h)$ be such that

a) $A \cap B = \emptyset$,
b) *the function h is injective on A,*
c) *$\mathcal{B}$ consists of finitely many components,*
d) *for every component D of $\mathcal{B}$*
the function h is not injective on D.

Then the following statements are equivalent

1. *The algebra $\mathcal{A} + \mathcal{B}$ has EKP.*
2. *Algebras $\mathcal{A}$, $\mathcal{B}$ have EKP.*

the operation $\oplus$ does not keep EKP



Full Descriptions

- Basic Monounary Algebras
- Injective Monounary Algebras
- Class $\mathcal{F}$

A monounary algebra $\mathcal{A} = (A, h)$ is basic if for every $a, b \in A$ there is either $f^n(a) = b$ or $f^n(b) = a$ for some natural number n.

Theorem

Let $\mathcal{A} = (A, h)$ be a basic algebra.
The algebra $\mathcal{A}$ has EKP if and only if the function h has a fixed point.

A monounary algebra $\mathcal{A} = (A, h)$ is injective if the function h is injective on A.

Theorem

Let $\mathcal{A}$ be injective. The algebra $\mathcal{A}$ has EKP if and only if

- *every component of $\mathcal{A}$ is a cycle and*
- *the condition (γ) is satisfied for $\mathcal{A}$.*

$\mathcal{F}$ .... the class of all monounary algebras $\mathcal{A}$ such that the number of arrows that enter to every cyclic point of $\mathcal{A}$ is finite

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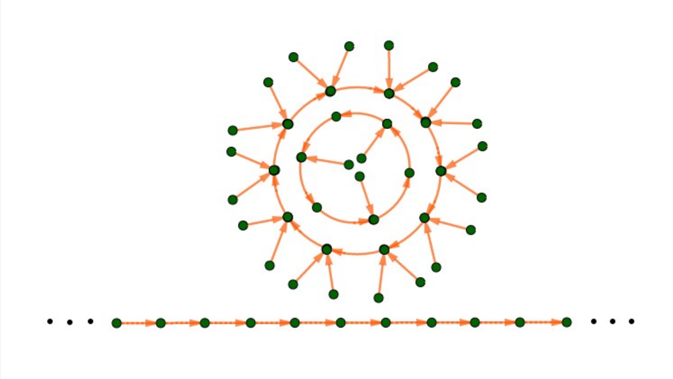
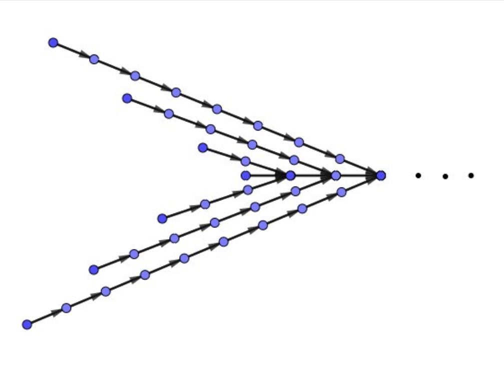


$\mathcal{F}$ .... the class of all monounary algebras $\mathcal{A}$ such that at most finitely many arrows enter into every cyclic point of $\mathcal{A}$.



Theorem

Let $\mathcal{A} \in \mathcal{F}$. The algebra $\mathcal{A}$ has EKP if and only if there exist algebras $\mathcal{B}, \mathcal{D}_1, \mathcal{D}_2$ such that

1. *$\mathcal{B}$ is injective with EKP,*
2. *$\mathcal{D}_1$ is basic with EKP,*
3. *$\mathcal{D}_2$ has a constant operation.*

and $\mathcal{A} \in \{\mathcal{B}, \mathcal{D}_1, \mathcal{D}_2\}$ or $\mathcal{A} = \mathcal{B} + \mathcal{D}_1$ or $\mathcal{A} = \mathcal{B} + \mathcal{D}_2$ or $\mathcal{A} = \mathcal{D}_1 \oplus \mathcal{D}_2$ or $\mathcal{A} = \mathcal{B} + (\mathcal{D}_1 \oplus \mathcal{D}_2)$.

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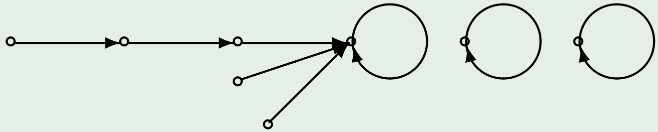

Conclusion

- Summary
- Published EKP Papers

1. The full characterization of algebras with EKP is done in several proper classes of monounary algebras.
2. The number of elements that enter cyclic points is important in these classes.
3. Monounary algebras with EKP are not closed with respect to operations $+$ and $\oplus$.
4. Some monounary algebras with EKP can be decomposed to smaller ones with this property.

1 T.S. Blyth, J. Fang and H. J. Silva: *EKP in finite distributive lattices and de Morgan algebras*, Communications in Algebra, 32 (6), (2004), 2225-2242.

2 T.S. Blyth, H. J. Silva: *SEKP in Ockham algebras*, Communications in Algebra, 36 (5), (2004), 1682-1694.

3 T.S. Blyth, J. Fang and L.-B. Wang: *SEKP in distributive double p-algebras*, Sci. Math. Jpn. 76 (2), (2013), 227-234.

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5 J. Fang, Z.-J. Sun: *Semilattices with the SEKP*, Algebra Univ. 70 (4), (2013), 393-401.

6 H. Gaitán, Y.J. Cortés: *EKP in finite Stone algebras*, JP J. of Algebra, Number Theory and Appl., 14(1), (2009), 51-64.

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8 J. Guričan: *A note on EKP*, JP J. of Algebra, Number Theory and Appl., 33 (2), (2014), 133-139.

9 J. Guričan: *SEKP for Brouwerian algebras*, JP J. of Algebra, Number Theory and Appl., 36(3), (2015), 241-258.

10 J. Guričan, M. Ploščica: *SEKP for modular p-algebras and for distributive lattices*, Algebra Univ., 75 (2), (2016), 243-255.

11 E. Halušková: *SEKP for monounary algebras*, Mathematica Bohemica 143-2 (2018), 161-171.

SEKP

$\varphi \in End\mathcal{A}$
φ is strong called, if φ preserves all congruences of $\mathcal{A}$

${}^{s}End\mathcal{A}$ ...... set of all strong endomorphisms of $\mathcal{A}$

Definition

$\mathcal{A}$ has SEKP if

$$\{Ker\varphi, \varphi \in {}^{s}End\mathcal{A}\} = Con\mathcal{A}$$