A simple identity forcing a lattice to be Boolean

Léonard Kwuida – Helmut Länger

Bern University of Applied Sciences, Bern, Swizerland,
email: leonard.kwuida@bfh.ch
TU Wien, Vienna, Austria, and
Palacký University Olomouc, Olomouc, Czech Republic,
email: helmut.laenger@tuwien.ac.at

Support of the research of the second author by the Austrian Science Fund (FWF), project I 1923-N25, by ÖAD, project CZ 04/2017, and by IGA, project PřF 2018 012, is gratefully acknowledged.

56th Summer School on Algebra and Ordered Sets, Špindlerův Mlýn, Czech Republic, September 6, 2018

Result

Result

Theorem 1

A non-empty lattice $\mathbf{L} = (L, \vee, \wedge, ')$ *with a unary operation is Boolean if and only if it satisfies the identity*

Result

Theorem 1

A non-empty lattice $\mathbf{L} = (L, \vee, \wedge, ')$ *with a unary operation is Boolean if and only if it satisfies the identity*

$$(x \wedge y) \vee (x \wedge y') \approx (x \vee y) \wedge (x \vee y'). \tag{1}$$

Outline of the proof

Outline of the proof

Let $\mathbf{L} = (L, \vee, \wedge, ')$ be a non-empty lattice with a unary operation satisfying (1).

Outline of the proof

Let $\mathbf{L} = (L, \vee, \wedge, ')$ be a non-empty lattice with a unary operation satisfying (1).
Step 1: $(x \wedge y) \vee (x \wedge y') \approx x \approx (x \vee y) \wedge (x \vee y')$,

Outline of the proof

Let $\mathbf{L} = (L, \vee, \wedge, ')$ be a non-empty lattice with a unary operation satisfying (1).
Step 1: $(x \wedge y) \vee (x \wedge y') \approx x \approx (x \vee y) \wedge (x \vee y')$,
Step 2: $\mathbf{L}$ is bounded and $'$ a complementation,

Outline of the proof

Let $\mathbf{L} = (L, \vee, \wedge, ')$ be a non-empty lattice with a unary operation satisfying (1).
Step 1: $(x \wedge y) \vee (x \wedge y') \approx x \approx (x \vee y) \wedge (x \vee y')$,
Step 2: $\mathbf{L}$ is bounded and $'$ a complementation,
Step 3: $'$ is antitone,

Outline of the proof

Let $\mathbf{L} = (L, \vee, \wedge, ')$ be a non-empty lattice with a unary operation satisfying (1).

Step 1: $(x \wedge y) \vee (x \wedge y') \approx x \approx (x \vee y) \wedge (x \vee y')$,

Step 2: $\mathbf{L}$ is bounded and $'$ a complementation,

Step 3: $'$ is antitone,

Step 4: $'$ is an involution,

Outline of the proof

Let $\mathbf{L} = (L, \vee, \wedge, ')$ be a non-empty lattice with a unary operation satisfying (1).

Step 1: $(x \wedge y) \vee (x \wedge y') \approx x \approx (x \vee y) \wedge (x \vee y')$,
Step 2: $\mathbf{L}$ is bounded and $'$ a complementation,
Step 3: $'$ is antitone,
Step 4: $'$ is an involution,
Step 5: $\mathbf{L}$ satisfies the de Morgan laws,

Outline of the proof

Let $\mathbf{L} = (L, \vee, \wedge, ')$ be a non-empty lattice with a unary operation satisfying (1).

Step 1: $(x \wedge y) \vee (x \wedge y') \approx x \approx (x \vee y) \wedge (x \vee y')$,

Step 2: $\mathbf{L}$ is bounded and $'$ a complementation,

Step 3: $'$ is antitone,

Step 4: $'$ is an involution,

Step 5: $\mathbf{L}$ satisfies the de Morgan laws,

Step 6: $\mathbf{L}$ is orthomodular,

Outline of the proof

Let $\mathbf{L} = (L, \vee, \wedge, ')$ be a non-empty lattice with a unary operation satisfying (1).

Step 1: $(x \wedge y) \vee (x \wedge y') \approx x \approx (x \vee y) \wedge (x \vee y')$,

Step 2: $\mathbf{L}$ is bounded and $'$ a complementation,

Step 3: $'$ is antitone,

Step 4: $'$ is an involution,

Step 5: $\mathbf{L}$ satisfies the de Morgan laws,

Step 6: $\mathbf{L}$ is orthomodular,

Step 7: $\mathbf{L}$ is distributive.

Proof

Proof

Proof.

Proof

Proof. Let $\mathbf{L} = (L, \vee, \wedge, ')$ be a non-empty lattice with a unary operation satisfying (1).

Step 1: $(x \wedge y) \vee (x \wedge y') \approx x \approx (x \vee y) \wedge (x \vee y')$

Step 1: $(x \wedge y) \vee (x \wedge y') \approx x \approx (x \vee y) \wedge (x \vee y')$

This follows from

Step 1: $(x \wedge y) \vee (x \wedge y') \approx x \approx (x \vee y) \wedge (x \vee y')$

This follows from

$$(x \wedge y) \vee (x \wedge y') \leq x \leq (x \vee y) \wedge (x \vee y')$$

Step 1: $(x \wedge y) \vee (x \wedge y') \approx x \approx (x \vee y) \wedge (x \vee y')$

This follows from

$$(x \wedge y) \vee (x \wedge y') \leq x \leq (x \vee y) \wedge (x \vee y')$$

and (1).

Step 2: $\mathbf{L}$ is bounded and $'$ a complementation

Step 2: $\mathbf{L}$ is bounded and $'$ a complementation

We have

Step 2: $\mathbf{L}$ is bounded and $'$ a complementation

We have

$$y \wedge y' \leq (x \wedge y) \vee (x \wedge y') \approx x \approx (x \vee y) \wedge (x \vee y') \leq y \vee y'$$

Step 2: $\mathbf{L}$ is bounded and $'$ a complementation

We have

$$y \wedge y' \leq (x \wedge y) \vee (x \wedge y') \approx x \approx (x \vee y) \wedge (x \vee y') \leq y \vee y'$$

and hence $y \wedge y' \approx 0$ and $y \vee y' \approx 1$.

Step 3: $'$ is antitone

Step 3: $'$ is antitone

If $x \leq y$ then

Step 3: $'$ is antitone

If $x \leq y$ then

$$y' \approx (y' \wedge x) \vee (y' \wedge x') \leq (y' \wedge y) \vee (y' \wedge x') \approx y' \wedge x' \leq x'.$$

Step 4: $'$ is an involution

Step 4: $'$ is an involution

This follows from

Step 4: $'$ is an involution

This follows from

$$x \approx (x \wedge x') \vee (x \wedge x'') \approx x \wedge x'' \leq x'' \approx (x'' \wedge x) \vee (x'' \wedge x') \approx x'' \wedge x \leq x.$$

Step 5: $\mathbf{L}$ satisfies the de Morgan laws

Step 5: $\mathbf{L}$ satisfies the de Morgan laws

This follows since $'$ is an antitone involution on the lattice $\mathbf{L}$.

Step 6: **L** is orthomodular

Step 6: $\mathbf{L}$ is orthomodular

If $x \leq y$ then

Step 6: **L** is orthomodular

If $x \leq y$ then

$$x \vee (y \wedge x') = (y \wedge x) \vee (y \wedge x') = y.$$

Step 6: $\mathbf{L}$ is orthomodular

If $x \leq y$ then

$$x \vee (y \wedge x') = (y \wedge x) \vee (y \wedge x') = y.$$

This means that $x \leq y$ and $y \wedge x' = 0$ together imply $x = y$.

Step 7: **L** is distributive

Step 7: $\mathbf{L}$ is distributive

We have

Step 7: $\mathbf{L}$ is distributive

We have

$$(x \wedge z) \vee (y \wedge z) \leq (x \vee y) \wedge z$$

Step 7: $\mathbf{L}$ is distributive

We have

$$(x \wedge z) \vee (y \wedge z) \leq (x \vee y) \wedge z$$

and

Step 7: $\mathbf{L}$ is distributive

We have

$$(x \wedge z) \vee (y \wedge z) \leq (x \vee y) \wedge z$$

and

$$\begin{aligned}
&(x \vee y) \wedge z \wedge ((x \wedge z) \vee (y \wedge z))' \approx \\
&\approx (x \vee y) \wedge z \wedge (x' \vee z') \wedge (y' \vee z') \approx \\
&\approx (x \vee y) \wedge z \wedge (x' \vee z) \wedge (x' \vee z') \wedge (y' \vee z) \wedge (y' \vee z') \approx \\
&\approx (x \vee y) \wedge z \wedge x' \wedge y' \approx \\
&\approx (x \vee y) \wedge z \wedge (x \vee y)' \approx 0
\end{aligned}$$

Step 7: $\mathbf{L}$ is distributive

We have

$$(x \wedge z) \vee (y \wedge z) \leq (x \vee y) \wedge z$$

and

$$\begin{aligned}
&(x \vee y) \wedge z \wedge ((x \wedge z) \vee (y \wedge z))' \approx \\
&\approx (x \vee y) \wedge z \wedge (x' \vee z') \wedge (y' \vee z') \approx \\
&\approx (x \vee y) \wedge z \wedge (x' \vee z) \wedge (x' \vee z') \wedge (y' \vee z) \wedge (y' \vee z') \approx \\
&\approx (x \vee y) \wedge z \wedge x' \wedge y' \approx \\
&\approx (x \vee y) \wedge z \wedge (x \vee y)' \approx 0
\end{aligned}$$

whence

Step 7: $\mathbf{L}$ is distributive

We have

$$(x \wedge z) \vee (y \wedge z) \leq (x \vee y) \wedge z$$

and

$$\begin{aligned}
&(x \vee y) \wedge z \wedge ((x \wedge z) \vee (y \wedge z))' \approx \\
&\approx (x \vee y) \wedge z \wedge (x' \vee z') \wedge (y' \vee z') \approx \\
&\approx (x \vee y) \wedge z \wedge (x' \vee z) \wedge (x' \vee z') \wedge (y' \vee z) \wedge (y' \vee z') \approx \\
&\approx (x \vee y) \wedge z \wedge x' \wedge y' \approx \\
&\approx (x \vee y) \wedge z \wedge (x \vee y)' \approx 0
\end{aligned}$$

whence

$$(x \wedge z) \vee (y \wedge z) \approx (x \vee y) \wedge z.$$

References

[1] I. Chajda, H. Länger and R. Padmanabhan, Single identities forcing lattices to be Boolean. Math. Slovaca **68** (2018), 713–716.

[2] I. Chajda and R. Padmanabhan, Lattices with unique complementation. Acta Sci. Math. (Szeged) **83** (2017), 31–34.

[3] L. Kwuida, Dicomplemented lattices. A contextual generalization of Boolean algebras. Dissertation, TU Dresden, Shaker Verlag, 2004.

[4] L. Kwuida, Axiomatization of Boolean algebras via weak dicomplementations. ArXiv: 0907.1279 (2009).

[5] L. Kwuida and H. Länger, A simple identity forcing a lattice to be Boolean. Algebra Universalis (submitted).

[6] R. Padmanabhan and S. Rudeanu, Axioms for Lattices and Boolean Algebras. World Scientific, Hackensack, NJ, 2008. ISBN: 978-981-283-454-6; 981-283-454-0.

Thank you for your attention!