Pairwise comparable or disjoint elemets in a poset

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SSAOS 2018. September 5.

CD-independent subsets in posets

Let $\mathbb{P} = (P, \leq)$ be a partially ordered set, and let $a, b \in P$.

The elements a and b are called *disjoint* and we write $a \perp b$ if

either $\mathbb{P}$ has least element $0 \in P$ and $\inf\{a, b\} = 0$,
or $\mathbb{P}$ is without 0 and the elements a and b have no common lowerbound.

A nonempty set $X \subseteq P$ is called *CD-independent* if for any $x, y \in X$, $x \leq y$ or $y \leq x$, or $x \perp y$ holds.

Maximal CD-independent sets (with respect to $\subseteq$) are called *CD-bases.*

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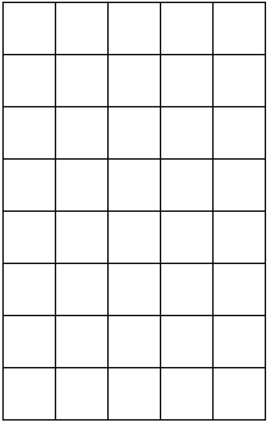
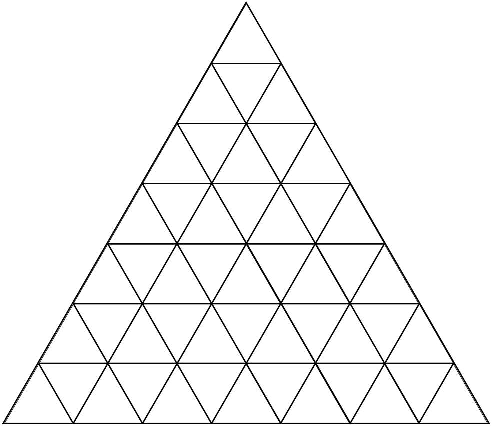
Island

Digital islands

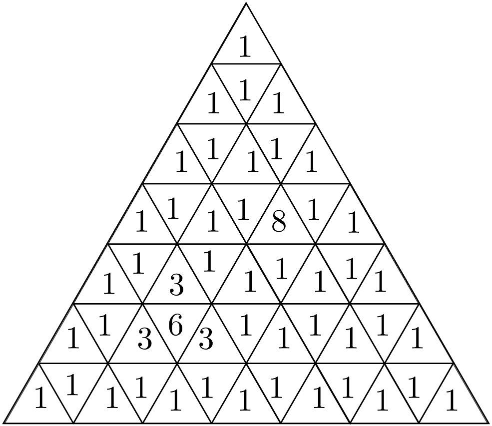




Grid

Digital islands

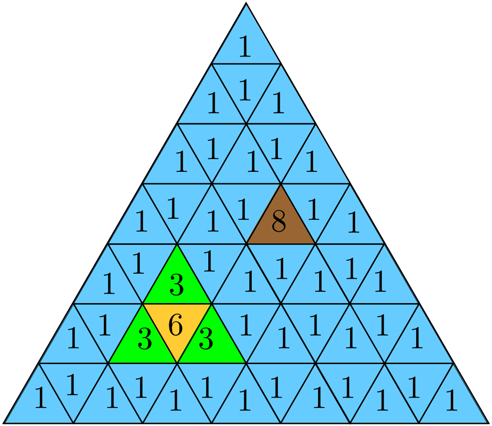
1	1	1	1	1
1	8	8	1	1
1	1	1	1	1
1	3	3	3	1
1	6	6	3	1
1	6	6	3	1
1	3	3	3	1
1	1	1	1	1



Grid, height function

Digital islands

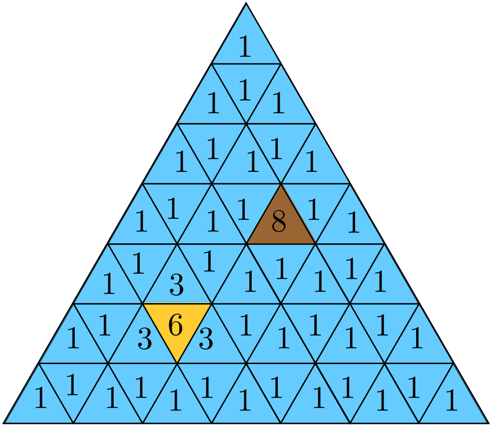
1	1	1	1	1
1	8	8	1	1
1	1	1	1	1
1	3	3	3	1
1	6	6	3	1
1	6	6	3	1
1	3	3	3	1
1	1	1	1	1



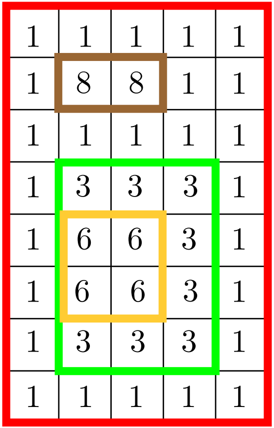
Grid, height function, water level: 2

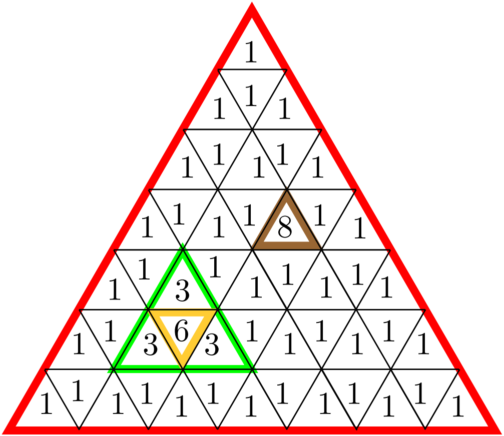
Digital islands

1	1	1	1	1
1	8	8	1	1
1	1	1	1	1
1	3	3	3	1
1	6	6	3	1
1	6	6	3	1
1	3	3	3	1
1	1	1	1	1



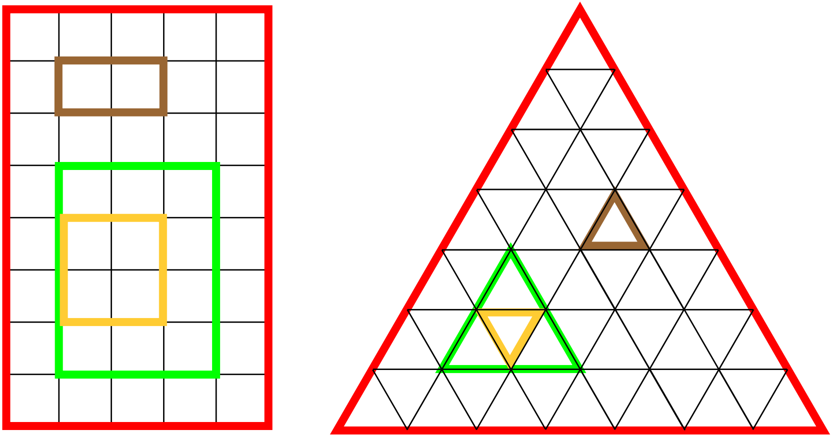
Grid, height function, water level: 4





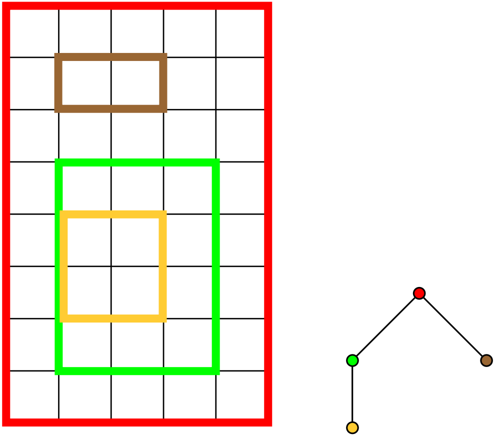
Grid, height function, island system

Digital islands



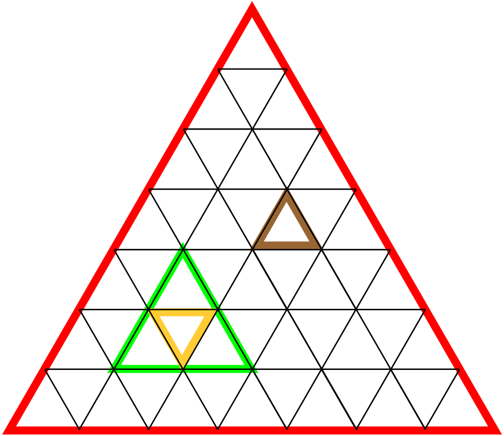
CD-independent: Comparable or Disjoint

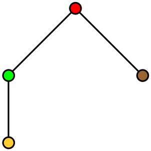
Digital islands



CD-independent: Comparable or Disjoint

Tree





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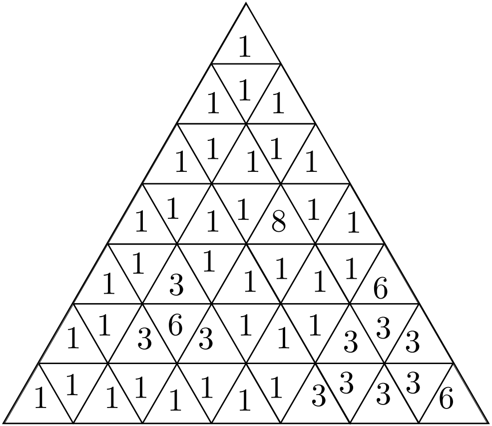

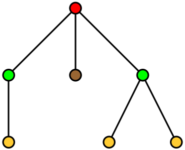

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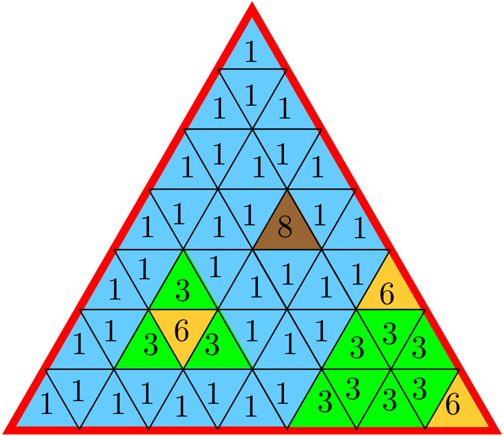

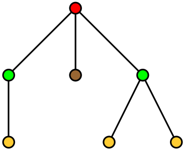

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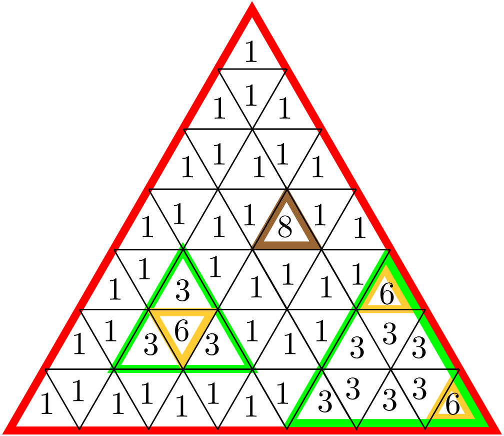

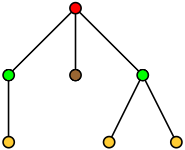

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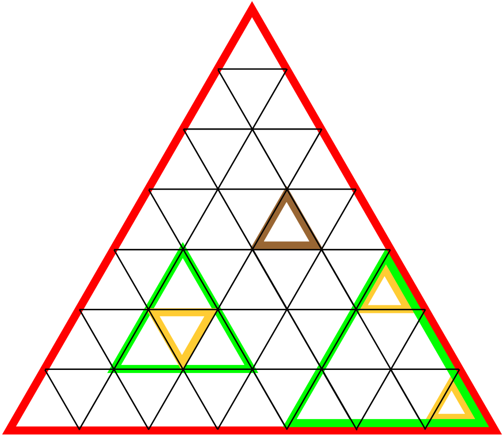

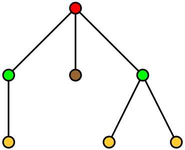

CD-independent: Comparable or Disjoint

CD-independent subsets in distributive lattices

G. Czédli, M. Hartmann and E. T. Schmidt: CD-independent subsets in distributive lattices, Publicationes Mathematicae Debrecen, 74/1-2 (2009).

Any two CD-bases of a finite distributive lattice have the same number of elements.

If all finite lattices in a lattice variety have this property, then the variety must coincide with the variety of distributive lattices.

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Sets of pairwise disjoint elements

Definition

A nonempty set D of nonzero elements of P is called a *set of pairwise disjoint element* in $\mathbb{P}$ if $x \perp y$ holds for all $x, y \in D$, $x \neq y$; if $\mathbb{P}$ has 0-element, then $\{0\}$ is considered to be a set of pairwise disjont elements, too.

D is a set of pairwise disjoint elements if and only if it is a CD-independent antichain in $\mathbb{P}$.

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Order ideals

Let $X \subseteq P$.

The order ideal $\{y \in P \mid y \leq x \text{ for some } x \in X\}$ is denoted by $\downarrow X$.

The order-ideals of any poset form a (distributive) lattice with respect to $\subseteq$.

So, the antichains of a poset can be ordered as follows:

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$\leqslant$ is a partial order.

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If A_1, A_2 are antichains in $\mathbb{P}$, then we say that A_1 *is dominated by* A_2, and we denote it by $A_1 \leqslant A_2$ if $\downarrow A_1 \subseteq \downarrow A_2$.

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$\mathcal{D}(\mathbb{P})$

Let $\mathcal{D}(\mathbb{P})$ denote the set of all sets of pairwise disjont elements of $\mathbb{P}$.

As sets of pairwise disjont elements of $\mathbb{P}$ are also antichains, restricting $\leqslant$ to $\mathcal{D}(\mathbb{P})$, we obtain a poset $(\mathcal{D}(\mathbb{P}), \leqslant)$.

The connection between CD-bases of a poset $\mathbb{P}$ and the poset $(\mathcal{D}(\mathbb{P}), \leqslant)$ is shown by the next theorem:

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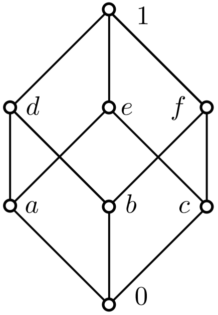

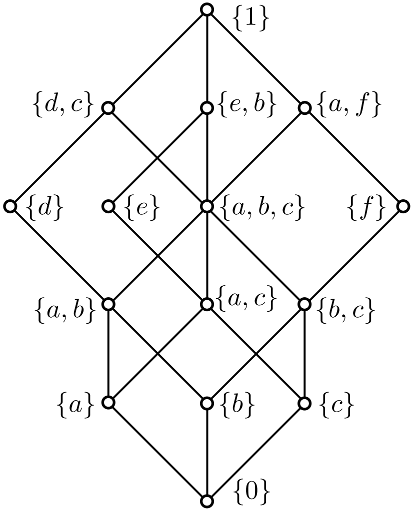

Theorem (E. K. H., S. Radeleczki)

Let B *be a CD-base of a finite poset* $(P, \leq)$, *and* let $|B| = n$.

Then there exists a maximal chain $\{D_i\}_{1 \leq i \leq n}$ *in* $\mathcal{D}(P)$ *such that* $B = \bigcup_{i=1}^{n} D_i$.

Moreover, for any maximal chain $\{D_i\}_{1 \leq i \leq m}$ *in* $\mathcal{D}(P)$ *the set* $D = \bigcup_{i=1}^{m} D_i$ *is a CD-base in* $(P, \leq)$ *with* $|D| = m$.

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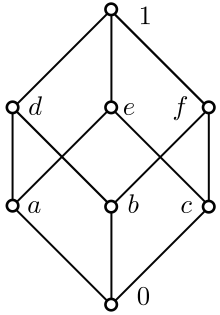

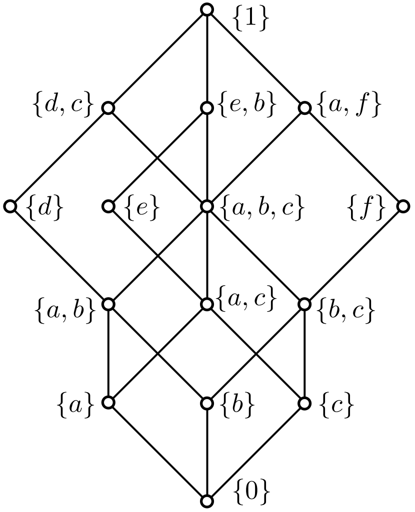

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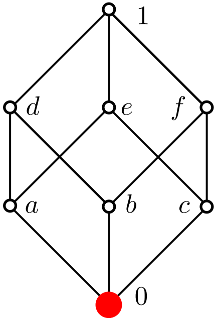

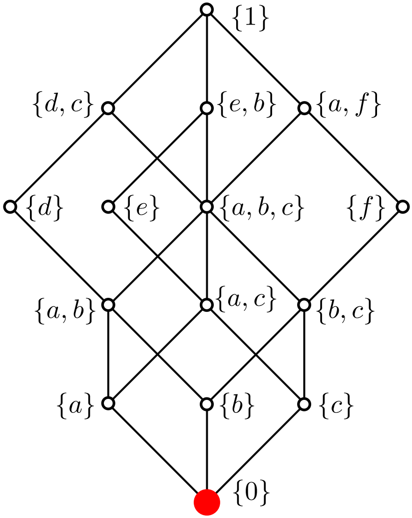

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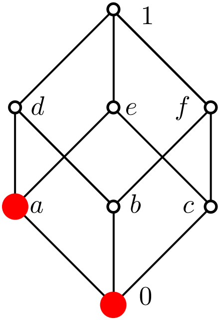

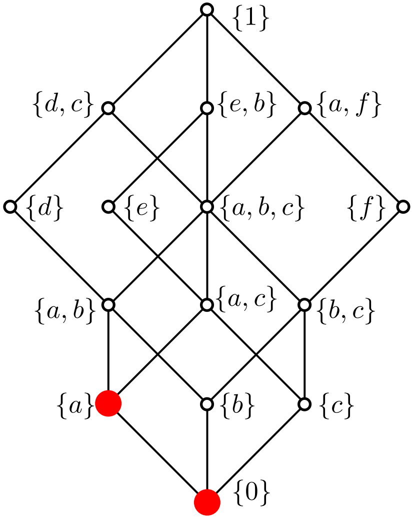

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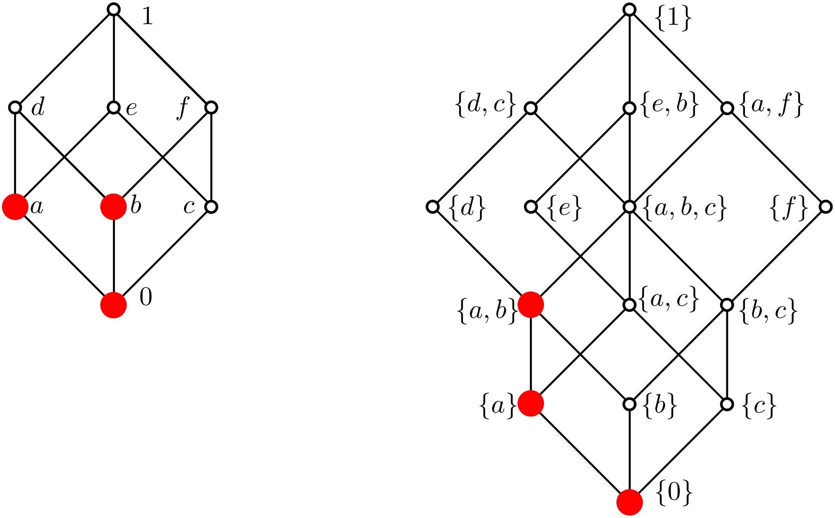

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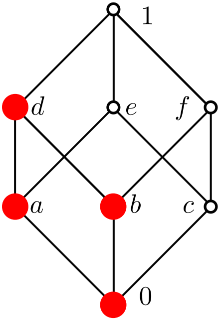

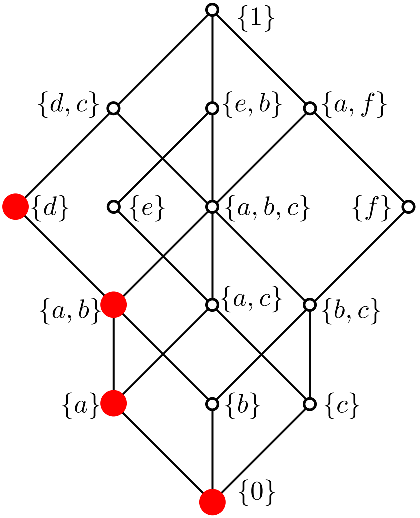

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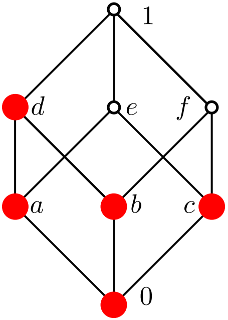

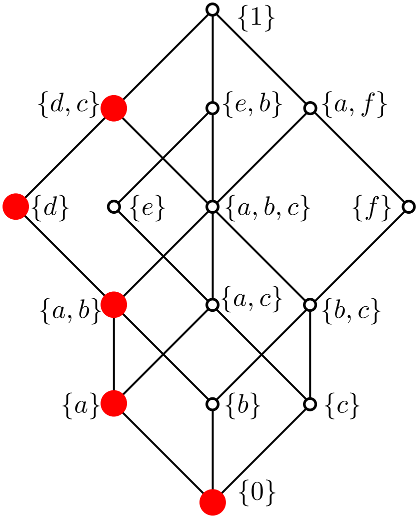

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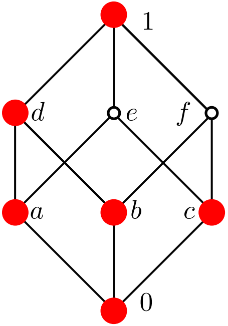
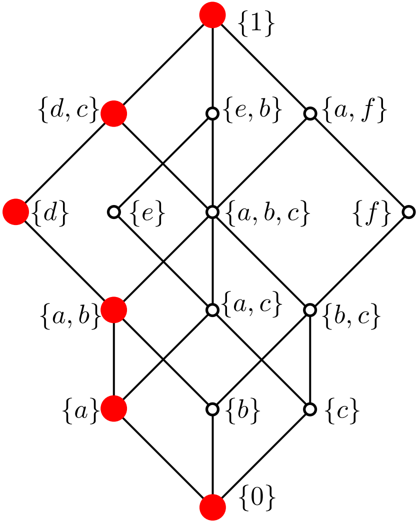
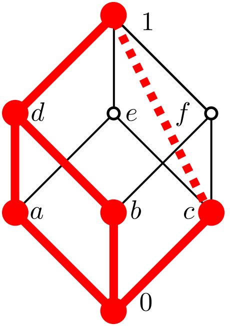
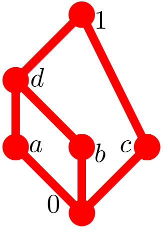
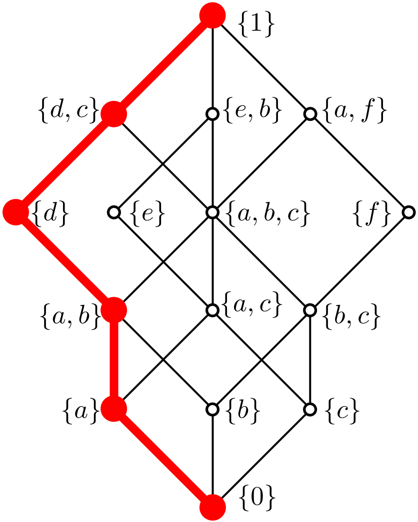




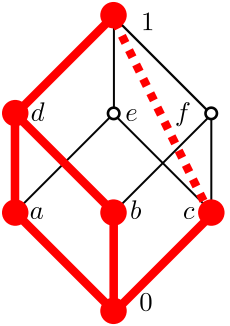
$\mathbb{P}_1$ és $\mathcal{D}(\mathbb{P}_1)$, maximális lánc $\mathcal{D}(\mathbb{P}_1)$-ben

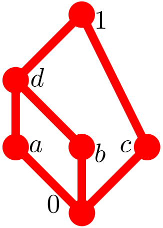


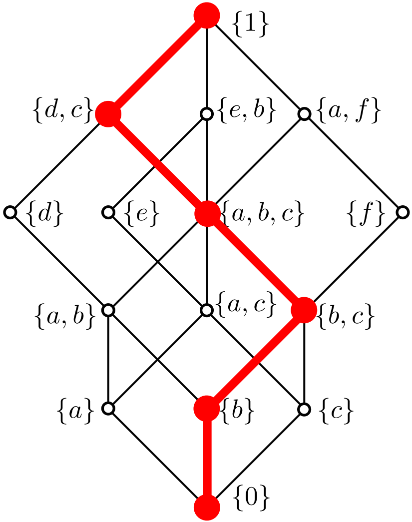




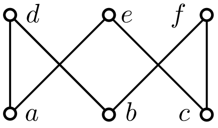
$\mathbb{P}_1$ és $\mathcal{D}(\mathbb{P}_1)$, maximális lánc $\mathcal{D}(\mathbb{P}_1)$-ben

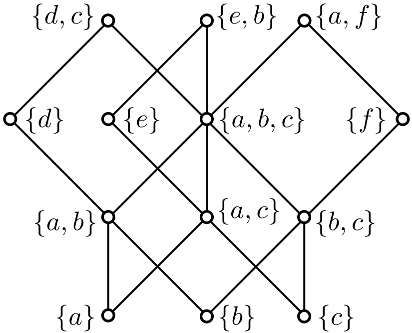




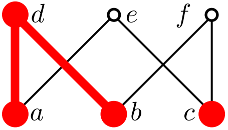


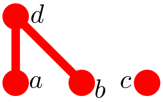
$\mathbb{P}_2$ and $\mathcal{D}(\mathbb{P}_2)$

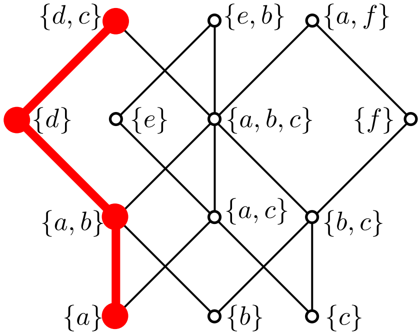




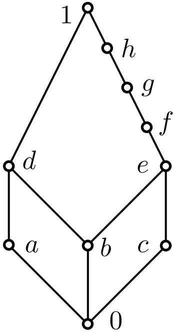
$\mathbb{P}_2$ és $\mathcal{D}(\mathbb{P}_2)$, maximális lánc $\mathcal{D}(\mathbb{P}_2)$-ben

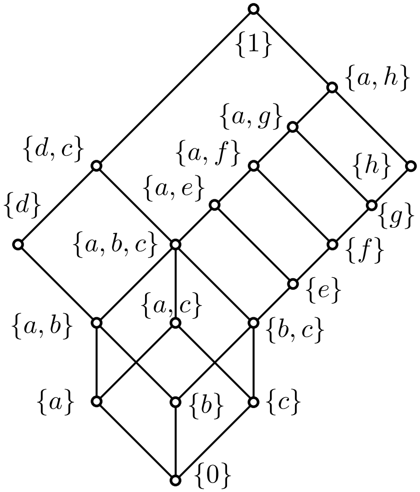




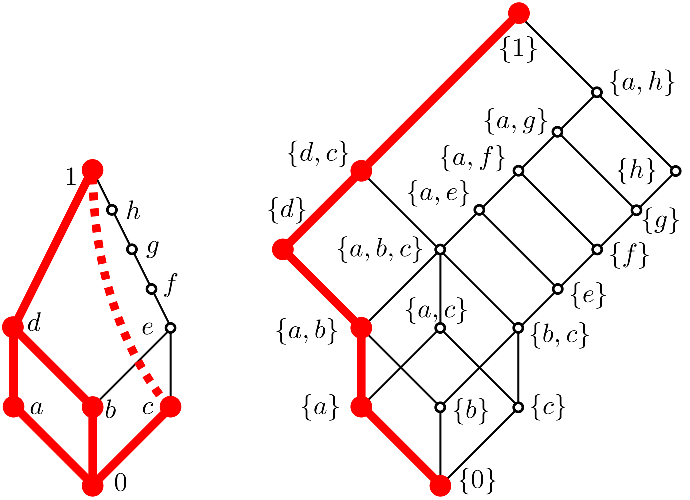


$\mathbb{P}_3$ and $\mathcal{D}(\mathbb{P}_3)$

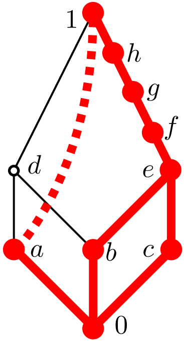


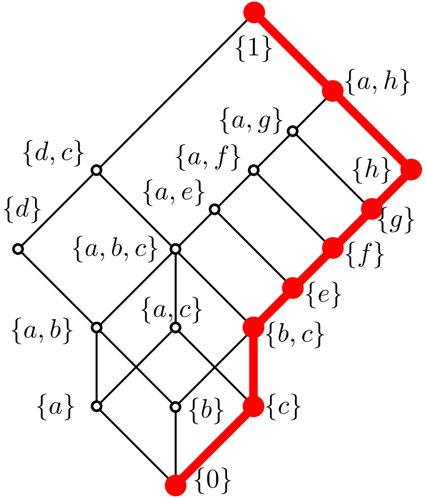


$\mathbb{P}_3$ és $\mathcal{D}(\mathbb{P}_3)$, maximális lánc $\mathcal{D}(\mathbb{P}_3)$-ban



$\mathbb{P}_3$ és $\mathcal{D}(\mathbb{P}_3)$, maximális lánc $\mathcal{D}(\mathbb{P}_3)$-ban





Proof

Any poset $(P, \leq)$ without least element becomes a poset with 0 by adding a new element 0 to P. In this way both the number of the elements in the CD-bases of $\mathbb{P}$ and the length of the maximal chains in $\mathcal{D}(P)$ are increased by one. Therefore, without loss of generality we may assume that $\mathbb{P}$ contains 0 and $|P| \geq 2$.

To prove the first part of Theorem 1.5, assume that B is a CD-base in $\mathbb{P}$. Then clearly $0 \in B$ and $|B| \geq 2$. Let $D_1 = \max(B)$. Take any $m_1 \in D_1$ and form $D_2 = \max(B \setminus \{m_1\})$. Then, in view of Lemma 1.7, $D_1, D_2 \in \mathcal{D}(P)$, $D_1 \succ D_2$, and D_1 is a maximal element in $\mathcal{D}(P)$. Further, suppose that we already have a sequence (D_i, m_i), $1 \leq i \leq k$ $(k \geq 2)$ such that $m_i \in D_i$, $D_1 \succ \ldots \succ D_k$ in $\mathcal{D}(P)$ and

$$D_k = \max(B \setminus \{m_1, \ldots, m_{k-1}\}).$$

We show that for all $i \in \{1, \ldots, k-1\}$ and $d \in D_k$ we have $d \not\geq m_i$. (5)

This is clear for $i = 1$ since $m_1 \in \max(B)$ and $d \in B$, $d \neq m_1$. If $2 \leq i \leq k-1$, then $m_i \in \max(B \setminus \{m_1, \ldots, m_{i-1}\})$, and since $d \in B \setminus \{m_1, \ldots, m_{i-1}\}$, $d \geq m_i$ would imply $m_i = d \in B \setminus \{m_1, \ldots, m_i, \ldots, m_{k-1}\}$, a contradiction.
Further, if $|B \setminus \{m_1, \ldots, m_{k-1}\}| \geq 2$, then form the next set $D_{k+1} := \max(B \setminus \{m_1, \ldots, m_{k-1}, m_k\})$ and let $m_{k+1} \in D_{k+1}$. Since D_{k+1} is an antichain in the CD-base B, it is a disjoint set, and clearly $D_{k+1} \neq D_k$.
In order to prove $D_k \succ D_{k+1}$, consider the subposet $(I(D_k), \leq)$. By Proposition 1.4, $B_k := B \cap I(D_k)$ is a CD-base in $(I(D_k), \leq)$. We claim that

$$B_k = B \setminus \{m_1, \ldots, m_{k-1}\}.$$

Indeed, $D_k = \max(B \setminus \{m_1, \ldots, m_{k-1}\})$ implies $B \setminus \{m_1, \ldots, m_{k-1}\} \subseteq B \cap I(D_k) = B_k$. On the other hand, (5) implies $\{m_1, \ldots, m_{k-1}\} \cap I(D_k) = \emptyset$, whence we get $B_k \subseteq B \setminus \{m_1, \ldots, m_{k-1}\}$, proving our claim. Hence $D_k = \max(B_k)$, and $D_{k+1} = \max\,(B \setminus \{m_1, \ldots, m_{k-1}, m_k\}) = \max(B_k \setminus \{m_k\})$.
Now, by applying Lemma 1.7, we obtain that $D_{k+1} \prec D_k$ holds in $\mathcal{D}(I(D_k))$. Finally, observe that any $S \in \mathcal{D}(P)$ with $S \leqslant D_k$ is also a disjoint set in $(I(D_k), \leq)$ according to (A). Moreover, since $D_{k+1} \prec D_k$ holds in $\mathcal{D}(I(D_k))$, $D_{k+1} \leqslant S \leqslant D_k$ implies either $S = D_k$ or $S = D_{k+1}$. This means that $D_{k+1} \prec D_k$ holds in $\mathcal{D}(P)$, too.
Thus we conclude by induction that the chain $D_1 \succ \ldots \succ D_k \succ \ldots$ can be continued as long as the condition $|B \setminus \{m_1, \ldots, m_{k-1}\}| \geq 2$ is still valid. Since P is finite, the process stops after finite - let say $n-1$ steps, when $|B \setminus \{m_1, \ldots, m_{n-1}\}| = 1$, and the last set is $D_n = B \setminus \{m_1, \ldots, m_{n-1}\}$. As $0 \in B$, and since $0 \notin \max(X)$ whenever $|X| \geq 2$, we get $\{0\} = B \setminus \{m_1, \ldots, m_{n-1}\} = D_n$. As D_1 is a maximal element and $D_n = \{0\}$ is the least element in $\mathcal{D}(P)$, $D_1 \succ \ldots \succ D_n$ is a maximal chain in $\mathcal{D}(P)$. Since $B = \{m_1, \ldots, m_{n-1}, 0\}$, we obtain $|B| = n$.

Proof

Any poset $(P, \leq)$ without least element becomes a poset with 0 by adding a new element 0 to P. In this way both the number of the elements in the CD-bases of $\mathbb{P}$ and the length of the maximal chains in $\mathcal{D}(P)$ are increased by one. Therefore, without loss of generality we may assume that $\mathbb{P}$ contains 0 and $|P| \geq 2$.

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We show that for all $i \in \{1, \ldots, k-1\}$ and $d \in D_k$ we have $d \ngeq m_i$. (5)

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Further, if $|B \setminus \{m_1, \ldots, m_{k-1}\}| \geq 2$, then form the next set $D_{k+1} := \max(B \setminus \{m_1, \ldots, m_{k-1}, m_k\})$ and let $m_{k+1} \in D_{k+1}$. Since D_{k+1} is an antichain in the CD-base B, it is a disjoint set, and clearly $D_{k+1} \neq D_k$.
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$$B_k = B \setminus \{m_1, \ldots, m_{k-1}\}.$$

Indeed, $D_k = \max(B \setminus \{m_1, \ldots, m_{k-1}\})$ implies $B \setminus \{m_1, \ldots, m_{k-1}\} \subseteq B \cap I(D_k) = B_k$. On the other hand, (5) implies $\{m_1, \ldots, m_{k-1}\} \cap I(D_k) = \emptyset$, whence we get $B_k \subseteq B \setminus \{m_1, \ldots, m_{k-1}\}$, proving our claim. Hence $D_k = \max(B_k)$, and $D_{k+1} = \max\,(B \setminus \{m_1, \ldots, m_{k-1}, m_k\}) = \max(B_k \setminus \{m_k\})$.
Now, by applying Lemma 1.7, we obtain that $D_{k+1} \prec D_k$ holds in $\mathcal{D}(I(D_k))$. Finally, observe that any $S \in \mathcal{D}(P)$ with $S \leqslant D_k$ is also a disjoint set in $(I(D_k), \leq)$ according to (A). Moreover, since $D_{k+1} \prec D_k$ holds in $\mathcal{D}(I(D_k))$, $D_{k+1} \leqslant S \leqslant D_k$ implies either $S = D_k$ or $S = D_{k+1}$. This means that $D_{k+1} \prec D_k$ holds in $\mathcal{D}(P)$, too.
Thus we conclude by induction that the chain $D_1 \succ \ldots \succ D_k \succ \ldots$ can be continued as long as the condition $|B \setminus \{m_1, \ldots, m_{k-1}\}| \geq 2$ is still valid. Since P is finite, the process stops after finite - let say $n-1$ steps, when $|B \setminus \{m_1, \ldots, m_{n-1}\}| = 1$, and the last set is $D_n = B \setminus \{m_1, \ldots, m_{n-1}\}$. As $0 \in B$, and since $0 \notin \max(X)$ whenever $|X| \geq 2$, we get $\{0\} = B \setminus \{m_1, \ldots, m_{n-1}\} = D_n$. As D_1 is a maximal element and $D_n = \{0\}$ is the least element in $\mathcal{D}(P)$, $D_1 \succ \ldots \succ D_n$ is a maximal chain in $\mathcal{D}(P)$. Since $B = \{m_1, \ldots, m_{n-1}, 0\}$, we obtain $|B| = n$.

Bizonyítás

To prove the second part of Theorem 1.5, assume that the disjoint sets $D_1, \dots, D_m$ form a maximal chain $\mathcal{C}$:

$$D_1 \prec \dots \prec D_m$$

in $\mathcal{D}(P)$. Then $D_1 = \{0\}$. Let $D = \bigcup_{i=1}^{m} D_i$. First, we prove that the set D is CD-independent. Indeed, take any $x, y \in D$, i.e. $x \in D_i$ and $y \in D_j$ for some $1 \leq i \leq j \leq m$. Then $x \leq z$ for some $z \in D_j$ by (A). Assume that x and y are not comparable. Then $z \neq y$, and $z \perp y$ implies $x \perp y$ by (1). This means that D is CD-independent.
Now, assume that D is not a CD-base. Then there is an $x \in P \setminus D$ such that $D \cup \{x\}$ is CD-independent. Next, consider the set

$$\mathcal{E} = \{D_i \in \mathcal{C} \mid x \nleq d \text{ for all } d \in D_i\}.$$

Clearly, $D_1 = \{0\} \in \mathcal{E}$ since $x \nleq 0$. Let $D_i \in \mathcal{E}$. Then $d \perp x$ or $d < x$ holds for each $d \in D_i$ because $D \cup \{x\}$ is CD-independent. Thus $T_i := \{x\} \cup \{d \in D_i \mid d \nless x\}$ is a disjoint set, and $d < x$ or $d \in T_i$ holds for all $d \in D_i$. Hence

$$D_i < T_i, \qquad (6)$$

in view of (A) and $x \notin D_i$. Observe that $D_m \notin \mathcal{E}$ since $D_m < T_m$ is not possible because $\mathcal{C}$ is a maximal chain. Thus, there exists a $k \leq m-1$ such that $D_k \in \mathcal{E}$ but $D_{k+1} \notin \mathcal{E}$. This means that $x \nleq d$ for all $d \in D_k$, and $x \leq z$ holds for some $z \in D_{k+1}$. Then $T_k = \{x\} \cup \{d \in D_k \mid d \nless x\} \in \mathcal{D}(P)$ satisfies $D_k < T_k$ in virtue of (6). Since $T_k \setminus \{x\} \subseteq D_k < D_{k+1}$ and $x \leq z$, for each $t \in T_k$ there is a $v \in D_{k+1}$ with $t \leq v$. In view of (A) we get $D_k < T_k < D_{k+1}$ because $x \notin D_{k+1} \subseteq D$. Since this fact contradicts $D_k \prec D_{k+1}$, we conclude that D is a CD-base.
Further, in view of (4), it follows that any set $D_i \setminus D_{i-1}$, $2 \leq i \leq m$ contains exactly one element, let say, a_i. Observe also that

$$D = \bigcup_{i=1}^{m} D_i = D_1 \cup \left(\bigcup_{i=2}^{m} (D_i \setminus D_{i-1}) \right).$$

Since $D_1 = \{0\}$ and $D_i \setminus D_{i-1} = \{a_i\}$, we get $D = \{0, a_2, \dots, a_m\}$. We prove that all the elements $0, a_2, \dots, a_m$ are different: Clearly, $0 \notin \{a_2, \dots, a_m\}$. Take any $i, j \in \{2, \dots, m\}$, $i < j$. Then $D_i \leqslant D_{j-1} \prec D_j$. As $a_i \in D_i$, there is a $b \in D_{j-1}$ with $0 < a_i \leq b$ by (A). As $a_j \in D_j \setminus D_{j-1}$, $b < a_j$ or $b \perp a_j$ holds by (2). Since both facts imply $a_i \neq a_j$, we conclude that D contains m different elements. $\square$

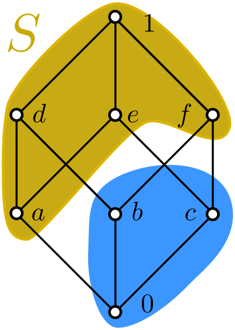
Lemma 1

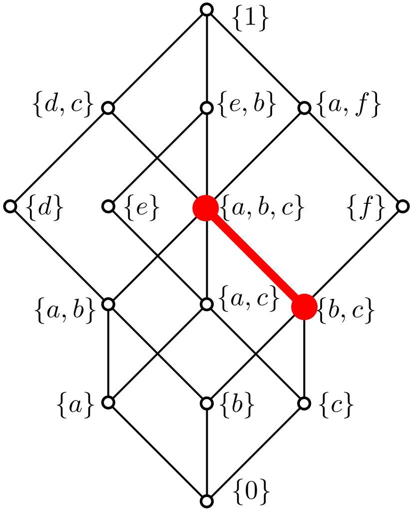
If $D_1 \prec D_2$ *in* $\mathcal{D}(P)$, *then* $D_2 = \{a\} \cup \left\{y \in D_1 \setminus \{0\} \mid y \perp a\right\}$ *for some minimal element* a *of the set*

$$S = \left\{s \in P \setminus (D_1 \cup \{0\}) \mid y \perp s \text{ or } y < s \text{ for all } y \in D_1\right\}.$$

Moreover, $D_1 \prec \{a\} \cup \left\{y \in D_1 \setminus \{0\} \mid y \perp a\right\}$ *holds for any minimal element* a *of the set* S.

Illustration for Lemma 1





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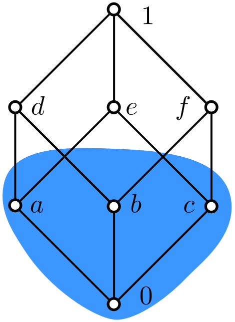

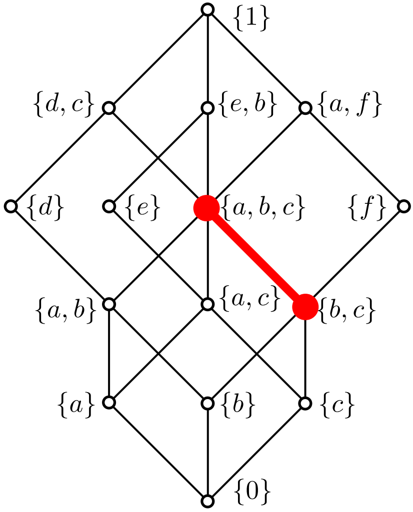

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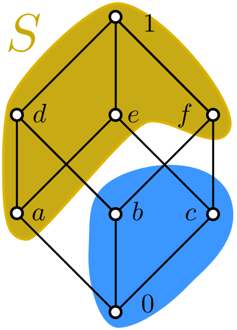

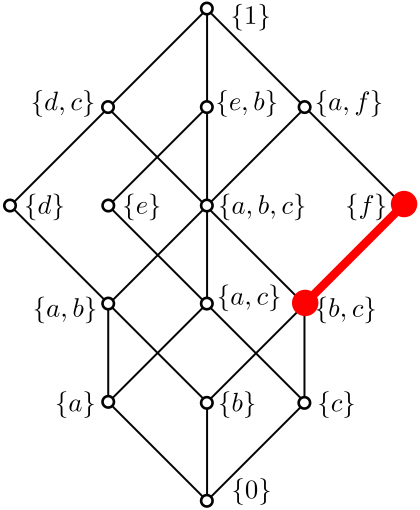

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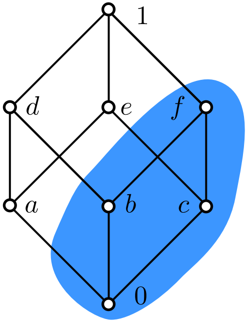
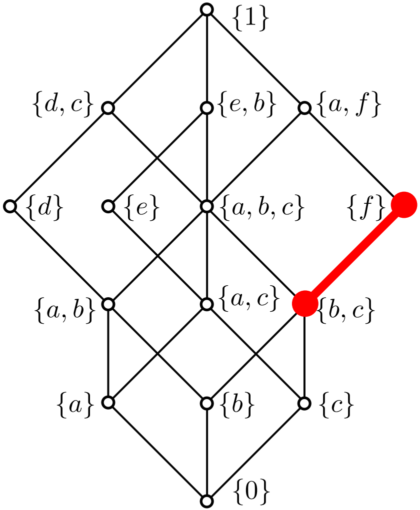


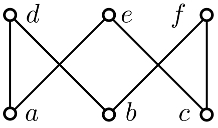


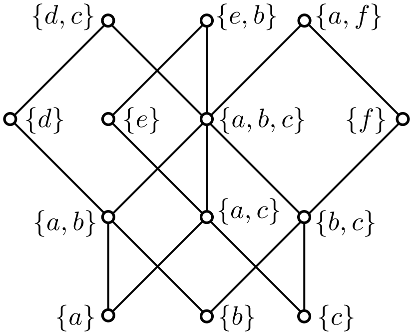
Lemma 2

Assume that B is a CD-base with at least two elements in a finite poset $\mathbb{P} = (P, \leq)$, $M = \max(B)$, and $m \in M$. Then M and $N := \max(B \setminus \{m\})$ are disjoint sets.

Moreover M is a maximal element in $\mathcal{D}(P)$, and $N \prec M$ holds in $\mathcal{D}(P)$.

Illustration for Lemma 2





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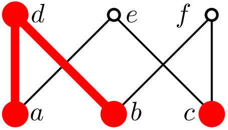

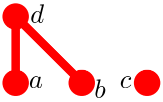

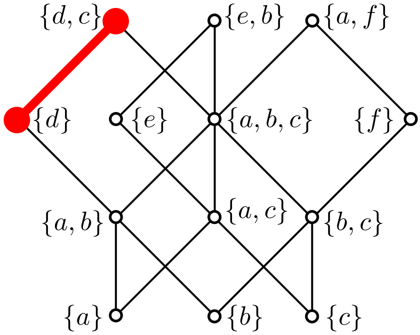

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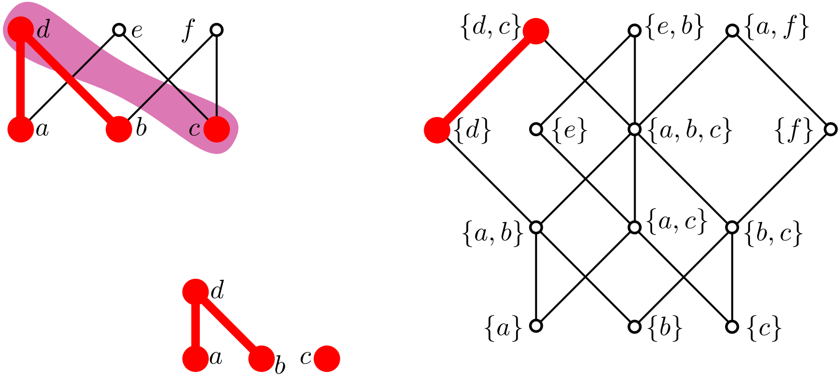

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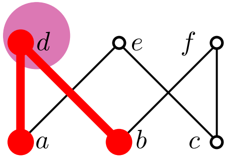

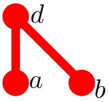

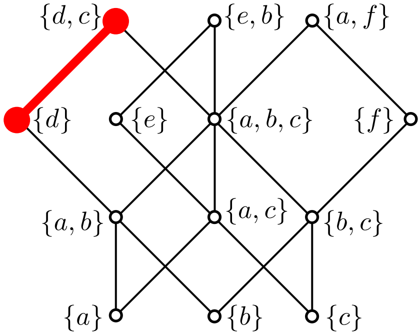

Corollary

Let $\mathbb{P} = (P, \leq)$ be a finite poset.

The poset $\mathbb{P}$ is called *graded*, if all its maximal chains have the same cardinality.

The CD-bases of $\mathbb{P}$ have the same number of elements if and only if the poset $\mathcal{D}(P)$ is graded.

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Corollary

Let $B \subseteq P$ be a CD-base of $\mathbb{P}$, and $(B, \leq)$ the poset under the restricted ordering.

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Illustration: P and $\mathcal{D}(P)$

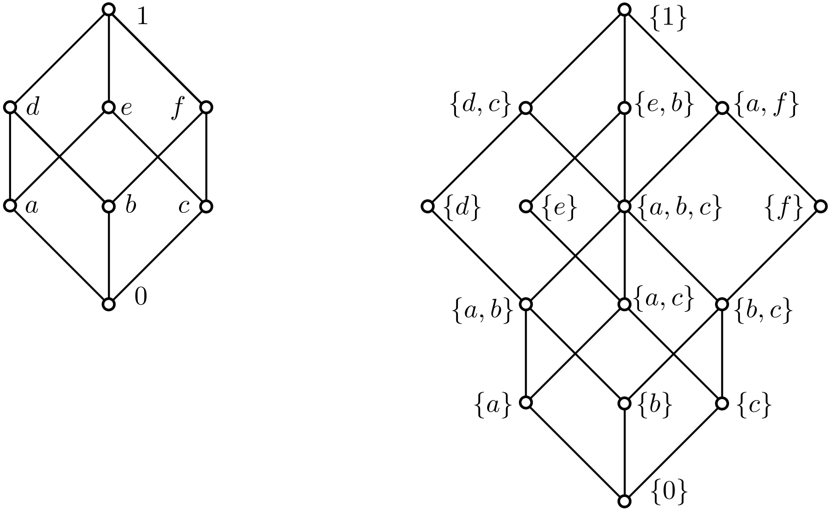
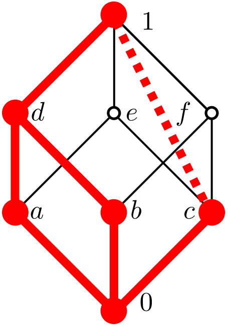
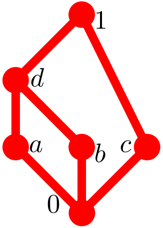


Illustration: P and $\mathcal{D}(P)$, B and $\mathcal{D}(B)$





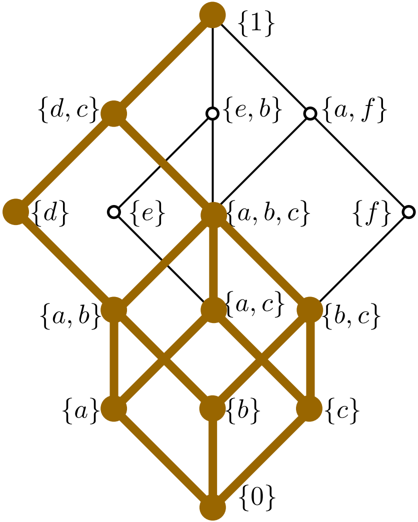


Illustration: P and $\mathcal{D}(P)$, B and $\mathcal{D}(B)$; a maximal chain

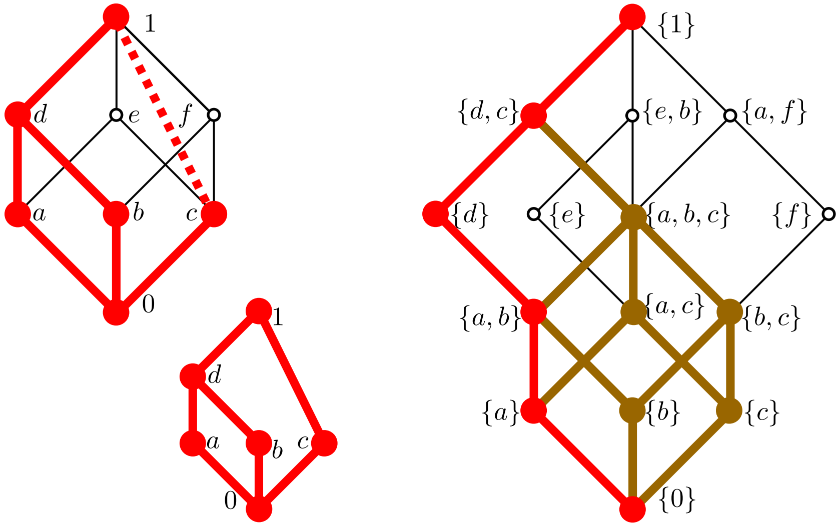
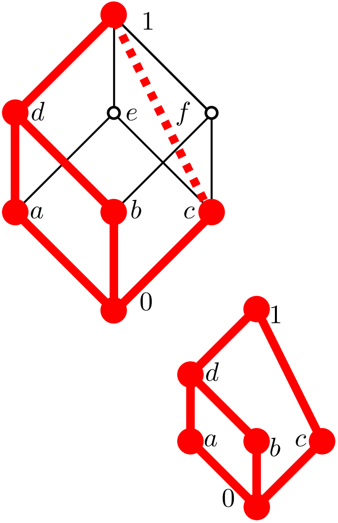
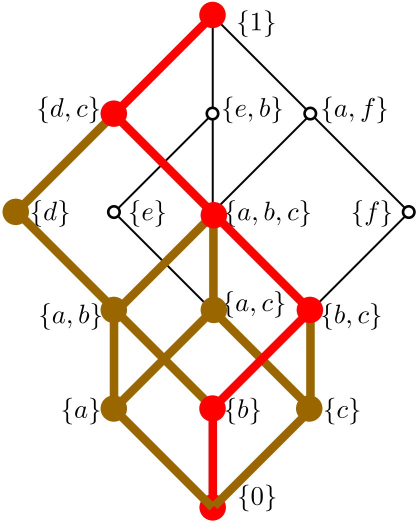


Illustration: P and $\mathcal{D}(P)$, B and $\mathcal{D}(B)$; other

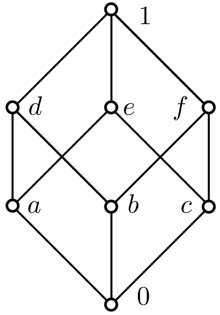


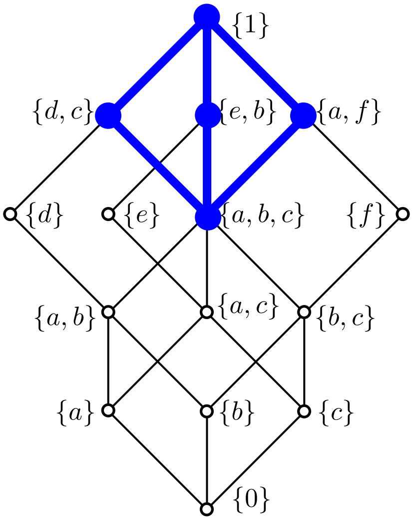


$\mathcal{DC}(P)$

A set of pairwise disjoint elements D of a poset $(P, \leq)$ is called *complete,* if there is no $p \in P \setminus D$ such that $D \cup \{p\}$ is also a set of pairwise disjoint elements.

P, $\mathcal{D}(P)$ and $\mathcal{DC}(P)$





Equivalent conditions

Let $\mathbb{P} = (P, \leq)$ *be a finite poset with* 0. *Then the following conditions are equivalent:*

(i) *The CD-bases of* $\mathbb{P}$ *have the same number of elements,*

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Let $(P, \leq)$ be a poset and $A \subseteq P$. $(A, \leq)$ is called a *sublattice* of $(P, \leq)$, if $(A, \leq)$ is a lattice such that for any $a, b \in A$ the infimum and the supremum of $\{a, b\}$ is the same in the subposet $(A, \leq)$ and in $(P, \leq)$.

Theorem (E. K. H., S. Radeleczki)

Let $\mathbb{P} = (P, \leq)$ *be a poset with* 0 *and* B *a CD-base of it. Then* $(\mathcal{D}(B), \leqslant)$ *is a distributive cover-preserving sublattice of the poset* $(\mathcal{D}(P), \leqslant)$.

If $\mathbb{P}$ *is a* $\wedge$*-semilattice, then for any* $D \in \mathcal{D}(P)$ *and* $D_1, D_2 \in \mathcal{D}(B)$ *we have*

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CD-bases in semilattices and lattices

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CD-bases in particular lattice classes

An *interval system* $(V, \mathcal{I})$ is an algebraic closure system satisfying the axioms:

(I_0) $\{x\} \in \mathcal{I}$ for all $x \in V$, and $\emptyset \in \mathcal{I}$;

(I_1) $A, B \in \mathcal{I}$ and $A \cap B \neq \emptyset$ imply $A \cup B \in \mathcal{I}$;

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Corollary

If $(V, \mathcal{I})$ is a finite interval system, then the CD-bases of the lattice $(\mathcal{I}, \subseteq)$ contain the same number of elements.

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Island domain

$U \in \mathcal{C} \subseteq \mathcal{K} \subseteq \mathcal{P}(U)$

Let $h \colon U \to \mathbb{R}$ be a height function and let $S \in \mathcal{C}$ be a nonempty set.

We say that S is an *pre-island* with respect to the triple $(\mathcal{C}, \mathcal{K}, h)$, if every $K \in \mathcal{K}$ with $S \prec K$ satisfies

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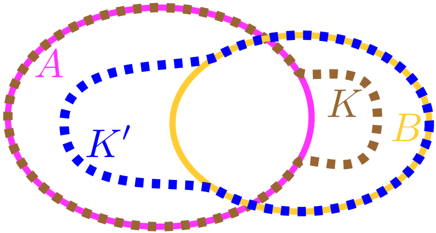
Connective island domains

Definition

A pair $(\mathcal{C}, \mathcal{K})$ is an *connective island domain* if

$$\forall A, B \in \mathcal{C} : \ (A \cap B \neq \emptyset \text{ and } B \not\subseteq A) \implies \exists K \in \mathcal{K} : A \subset K \subseteq A \cup B.$$



Connective island domains

Theorem 5. (S. Foldes, E. K. H., S. Radeleczki, T. Waldhauser)

The following twoo conditions are equivalent for any pair $(\mathcal{C}, \mathcal{K})$:

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Thank you for your attention!