Left distributive biracks

Přemysl Jedlička[1], Agata Pilitowska[2]
Anna Zamojska-Dzienio[2]

[1] Faculty of Engineering, Czech University of Life Sciences
[2]Faculty of Mathematics and Information Science, Warsaw University of Technology [3]

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Biracks

$(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ is a birack if:

- $(X, \circ, \backslash_\circ)$ is a left quasigroup $[x \circ (x \backslash_\circ y) = y,\ x \backslash_\circ (x \circ y) = y]$
- $(X, \bullet, /_\bullet)$ is a right quasigroup $[(y /_\bullet x) \bullet x = y,\ (y \bullet x) /_\bullet x = y]$
- the following identities are satisfied:

$$\begin{aligned} x \circ (y \circ z) &= (x \circ y) \circ ((x \bullet y) \circ z) \\ (x \circ y) \bullet ((x \bullet y) \circ z) &= (x \bullet (y \circ z)) \circ (y \bullet z) \\ (x \bullet y) \bullet z &= (x \bullet (y \circ z)) \bullet (y \bullet z) \end{aligned}$$

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  $L_a \colon X \to X;\ x \mapsto a \circ x$ are invertible for any $a \in X$
- $(X, \bullet, /_\bullet)$ is a right quasigroup $[(y /_\bullet x) \bullet x = y,\ (y \bullet x) /_\bullet x = y]$
  $\boldsymbol{R}_a \colon X \to X;\ x \mapsto x \bullet a$ are invertible for any $a \in X$
- the following identities are satisfied:

$$x \circ (y \circ z) = (x \circ y) \circ ((x \bullet y) \circ z)$$
$$(x \circ y) \bullet ((x \bullet y) \circ z) = (x \bullet (y \circ z)) \circ (y \bullet z)$$
$$(x \bullet y) \bullet z = (x \bullet (y \circ z)) \bullet (y \bullet z)$$

A set-theoretical solutions of the Yang–Baxter equation

X - a set
$\sigma : X^2 \to X$ and $\tau : X^2 \to X$ - two mappings
$r\colon X^2 \to X^2;\ r(x,y) = (\sigma(x,y), \tau(x,y))$
(X, r) is a *set-theoretical solution of the Yang–Baxter equation* if the mapping r satisfies the *braid relation*:

$$(id \times r)(r \times id)(id \times r) = (r \times id)(id \times r)(r \times id).$$

$[id \times r, r \times id, id \times r\colon X^3 \to X^3]$

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Open problem

Find all (set-theoretical) solutions of the Yang-Baxter equation.

Biracks provide solutions to YBE

Theorem (Rump; Gateva-Ivanova; Dehornoy)

If $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ *is an birack then the mapping* $r\colon X^2 \to X^2$,
$r(x,y) := (x \circ y, x \bullet y)$ *satisfies the braid relation.*
(X, r) *is a set-theoretical solution of the Yang-Baxter equation.*

Involutive biracks

A birack $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ is called an *involutive* if it additionally satisfies for every $x, y \in X$:

$$(x \circ y) \circ (x \bullet y) = x,$$
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If a birack $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ is involutive then for every $x, y \in X$:

$$x \bullet y = L_{x \circ y}^{-1}(x) = (x \circ y)\backslash_\circ x$$

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Example (Lyubashenko)

X - a non-empty set

$f, g\colon X \to X$ - mappings

$$x \circ y = f(y) \quad \text{and} \quad x \bullet y = g(x)$$

$(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ is a birack if and only if f and g are bijections and $fg = gf$. It is involutive if and only if $g = f^{-1}$.

Right cyclic left quasigroups

A left quasigroup $(X, *, \backslash)$ is *right cyclic*, if it satisfies the *right cyclic* law:

$$(x * y) * (x * z) = (y * x) * (y * z).$$

Theorem (Rump)

Let $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ be an involutive birack.
Then the left quasigroup $(X, \backslash_\circ, \circ)$ is right cyclic.

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Example (Rump)

Let $(X, *, \backslash)$ be a finite right cyclic left quasigroup. Then defining

$$x \circ y = x * y,\ x \backslash_\circ y = x \backslash y,\ x \bullet y = (x * y) \backslash x,$$

and $x /_\bullet y = z$, where z is the unique one such that $z \backslash z = y * (x \backslash x)$, the algebra $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ is an involutive birack.

Left distributive biracks

A birack $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ is *left distributive*, if $(X, \circ, \backslash_\circ)$ is a rack

$$x \circ (y \circ z) = (x \circ y) \circ (x \circ z)$$

In a birack $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$, the left distributivity is equivalent to the condition:

$$L_x = L_{x \bullet y}$$

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Lemma

- *An involutive birack $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ is left distributive if and only if it is 2-reductive:*

$$(x \circ y) \circ z = y \circ z$$

- *An involutive left distributive birack $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ is medial:*

$$(x \circ y) \circ (z \circ t) = (x \circ z) \circ (y \circ t)$$

The Structure Theorem

Theorem

Each involutive left distributive birack $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ *is a disjoint union of abelian groups* A_j, $j \in I$, *with operations for* $x \in A_i$ *and* $y \in A_j$:

$$x \circ y = y + c_{i,j} \quad \text{and} \quad x \bullet y = x - c_{j,i},$$

where $A_j = \langle \{c_{i,j} \mid i \in I\} \rangle$, *for every* $j \in I$.
$(X, \circ, \backslash_\circ, \bullet, /_\bullet) = ((A_i)_{i \in I}; (c_{i,j})_{i,j \in I})$

Example

I - a set
$A_i = \langle g_i \rangle$, cyclic group for each $i \in I$
$c_{i,i} = g_i$, for each $i \in I$
$(X, \circ, \backslash_\circ, \bullet, /_\bullet) = ((A_i)_{i \in I}; (c_{i,j})_{i,j \in I})$ is an involutive left distributive birack.

Isomorphism Theorem

Theorem

Two involutive left distributive biracks $\mathcal{A} = ((A_i)_{i\in I};\ (c_{i,j})_{i,j\in I})$ and $\mathcal{B} = ((B_i)_{i\in I};\ (b_{i,j})_{i,j\in I})$, over the same index set I, are isomorphic if and only if there is a permutation $\pi \in S_n$ and group isomorphisms $\psi_i \colon A_i \to B_{\pi(i)}$ such that $\psi_j(c_{i,j}) = b_{\pi(i),\pi(j)}$.

Example

Up to isomorphism, there are exactly 5 involutive left distributive biracks of size 3:

- One group: $((\mathbb{Z}_3),(1))$.
- Two groups: $((\mathbb{Z}_2,\mathbb{Z}_1),\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right))$, $((\mathbb{Z}_2,\mathbb{Z}_1),\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right))$ and $((\mathbb{Z}_2,\mathbb{Z}_1),\left(\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right))$.
- Three groups: $((\mathbb{Z}_1,\mathbb{Z}_1,\mathbb{Z}_1),\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix})$.

2-permutational biracks

A birack $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ is *2-permutational* if for every $a, b, x, y \in X$,

$$(a \circ x) \circ y = (b \circ x) \circ y.$$

[A birack is 2-reductive if $(x \circ y) \circ z = y \circ z$]

Example

Each 2-reductive birack is 2-permutational.
Each 2-permutational, idempotent birack is 2-reductive.

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Theorem

Each involutive 2-permutational birack $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ defines, for $e \in X$, a left distributive one $(X, *, \backslash_*, \diamond, /_\diamond)$:

$$x * y := L_e^{-1} L_x(y) = L_e^{-1}(x \circ y) \quad \text{and} \quad x \backslash_* y := L_x^{-1} L_e(y) = L_x^{-1}(e \circ y).$$

$[x \diamond y = (x * y) \backslash_* x]$

2-permutational biracks

Theorem

Let $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ *be a 2-reductive involutive birack and* σ *be a bijection on the set* X *such that*

- $\forall x, y \in X \quad L_{\sigma(y)}\sigma L_x = L_{\sigma(x)}\sigma L_y.$

Then $(X, *, \backslash_*, \diamond, /_\diamond)$ *with operations:*

$$x * y := \sigma L_x(y), \quad \text{and} \quad x\backslash_* y := L_x^{-1}\sigma^{-1}(y).$$

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Theorem

Each involutive 2-permutational birack originates from a left distributive (2-reductive) involutive birack.

Enumeration

n	1	2	3	4	5	6	7	8
involutive biracks	1	2	5	23	88	595	3456	34528
2-permutational inv. biracks	1	2	5	19	70	359	2095	16332
2-reductive inv. biracks	1	2	5	17	65	323	1960	15421
2-permutational, not 2-reductive	0	0	0	2	5	36	135	911

Table: The number of involutive biracks of size n, up to isomorphism.

The retraction of a birack

$(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ - an involutive birack
The relation

$$a \sim b \quad \Leftrightarrow \quad L_a = L_b \quad \Leftrightarrow \quad \forall x \in X \quad a \circ x = b \circ x$$

is a congruence of $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$.

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The quotient birack of $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ is called the *retraction* of the birack:

$$\mathrm{Ret}^1(X) = (X/_\sim, \circ, \backslash_\circ, \bullet, /_\bullet)$$

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The quotient birack of $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ is called the *retraction* of the birack:

$$\mathrm{Ret}^1(X) = (X/_\sim, \circ, \backslash_\circ, \bullet, /_\bullet)$$

For any natural number $k > 1$

$$\mathrm{Ret}^k(X) := \mathrm{Ret}^1(\mathrm{Ret}^{k-1}(X))$$

Multipermutational biracks of level 2

Definition

A birack $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ is called a *multipermutational of level m* if m is the least nonnegative integer m such that

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- Each left distributive involutive birack $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ is a multipermutational birack of level less or equal 2.
- There are not left distributive multipermutational biracks of level 2.

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Proposition (Gateva-Ivanova)

Let $|X| \geq 2$. An involutive birack $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ is a multipermutational birack of level less or equal 2 if and only if $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ is 2-permutational.

Thank you for your attention.