Expansions of $(\mathbb{Z}_p \times \mathbb{Z}_q, +)$

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Supported by the Austrian Science Fund FWF (P29931).

September 3, 2018

Clones

Definition

A *clone (closed set of operations)* on a set A is a subset of $\bigcup_{n\in\mathbb{N}} A^{A^n}$ which contains all the projections and is closed under composition.

Definition

Let $\mathbf{A}$ be an algebra with a basic binary operation of sum $+_{\mathbf{A}}$. A $+$-*clone* of $\mathbf{A}$ is a clone which contains $+_{\mathbf{A}}$.

(p, q)-linear closed clonoids

Definition

Let p and q be prime numbers. A (p, q)-**linear closed clonoid** is a non-empty subset C of $\bigcup_{n\in\mathbb{N}} \mathbb{Z}_p^{\mathbb{Z}_q^n}$ such that:

(1) if $f, g \in C^{[n]}$ then:

$$f +_p g \in C^{[n]};$$

(2) if $f \in C^{[m]}$ and $A \in \mathbb{Z}_q^{m\times n}$ then:

$$g : (x_1, \ldots, x_n) \mapsto f(A \cdot_q (x_1, \ldots, x_n)^t) \text{ is in } C^{[n]}.$$

The generators for every (p, q)-linear closed clonoid

Definition

Let f be an n-ary function from $\mathbb{Z}_q$ to $\mathbb{Z}_p$. A **restriction of f to a line passing through the origin**, or **one dimensional restriction**, is a function obtained with the substitution:

$$g(x) = f(a_1 x, ..., a_n x),$$

where $(a_1, ..., a_n) \in \mathbb{Z}_q^n - \{(0, .., 0)\}$.

A characterization

Theorem

Let p and q be two distinct prime numbers. Then every (p, q)-linear closed clonoid is generated by its one dimensional restriction.

Corollary

Let p and q be two distinct prime numbers. Then every (p, q)-linear closed clonoid has a set of finitely many unary functions as generators. Hence there are only finitely many distinct (p, q)-linear closed clonoids.

Notations

We investigate clones containing $\mathrm{Clo}(\mathbb{Z}_p \times \mathbb{Z}_q, +)$. Hence $+$-clones.

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We will always consider the functions of a clone of an expansions of $\mathbb{Z}_p \times \mathbb{Z}_q$ like functions from $\mathbb{Z}_p^n \times \mathbb{Z}_q^n$ split in the two components.

$$f(x_1, \dots, x_n, y_1, \dots, y_n) = (f_1(x_1, \dots, x_n, y_1, \dots, y_n),\ f_2(x_1, \dots, x_n, y_1, \dots, y_n)).$$

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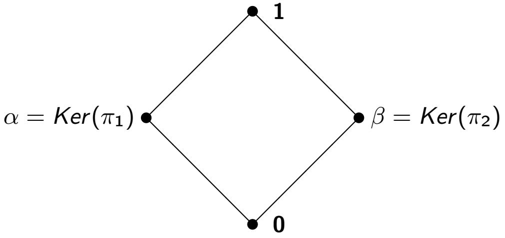
We will always consider congruence lattices that are sublattices of:

$\mathrm{Con}((\mathbb{Z}_p \times \mathbb{Z}_q, +))$



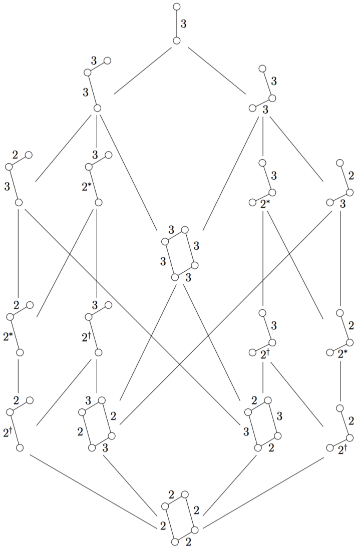


Figure: (P. Mayr's picture) Simple factors are labelled 2 if they are abelian and 3 otherwise. $2^\dagger$ means the factor is central; 2*means it is not central.

Known results

E. Aichinger, P. Mayr, *Polynomial clones on groups of order pq*, in: Acta Mathematica Hungarica, Volume 114, Number 3, Page(s) 267-285, 2007.
(All 17 clones containing $(\mathbb{Z}_p \times \mathbb{Z}_q, +, (1,1))$);

A. A. Bulatov, *Polynomial clones containing the Mal'cev operation of the groups $\mathbb{Z}_{p^2}$ and $\mathbb{Z}_p \times \mathbb{Z}_p$*, in: Mult.-Valued Log. 8(2) (2002) 193-221 (Multiple-valued logic in Eastern Europe).
(All infinitely many clones containing $(\mathbb{Z}_p \times \mathbb{Z}_p, +, (1,0), (0,1))$;

S. Kreinecker, *Closed function sets on groups of prime order*, Manuscript, 2018.
(All finitely many clones containing $(\mathbb{Z}_p, +)$).

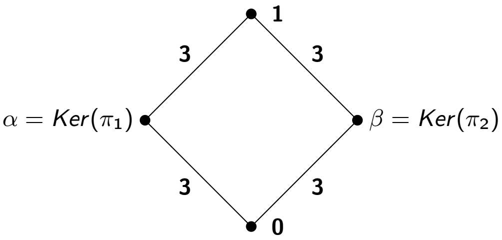
General expression of functions of $\mathbb{Z}_p \times \mathbb{Z}_q$

Lemma

Let p and q be distinct prime numbers. Then for every function f from $\mathbb{Z}_p^n \times \mathbb{Z}_q^n$ to $\mathbb{Z}_p \times \mathbb{Z}_q$ there exist two sequences of functions $\{f_{\mathbf{m}}\}_{\mathbf{m}\in\mathbb{Z}_p^n}$ from $\mathbb{Z}_q^n$ to $\mathbb{Z}_p$ and $\{s_{\mathbf{h}}\}_{\mathbf{h}\in\mathbb{Z}_q^n}$ from $\mathbb{Z}_p^n$ to $\mathbb{Z}_q$ such that for all $\mathbf{x} \in \mathbb{Z}_p^n$, $\mathbf{y} \in \mathbb{Z}_p^n$:

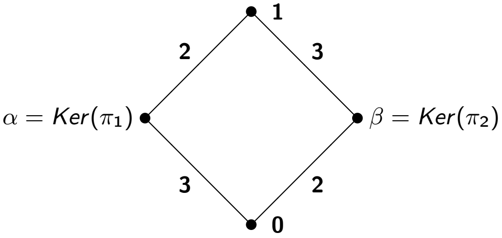
$$f(\mathbf{x}, \mathbf{y}) = \Big(\sum_{\mathbf{m}\in\mathbb{Z}_p^n} f_{\mathbf{m}}(\mathbf{y})\mathbf{x}^{\mathbf{m}} \, , \, \sum_{\mathbf{h}\in\mathbb{Z}_q^n} s_{\mathbf{h}}(\mathbf{x})\mathbf{y}^{\mathbf{h}} \Big).$$

Con(**V**)



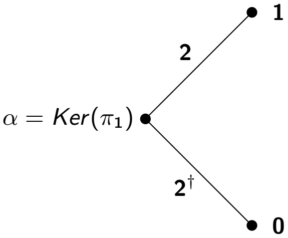
$$f(\mathbf{x},\mathbf{y}) = \Big(\sum_{\mathbf{m}\in\mathbb{Z}_p^n} a_{\mathbf{m}}\mathbf{x}^{\mathbf{m}} \, , \, \sum_{\mathbf{h}\in\mathbb{Z}_q^n} b_{\mathbf{h}}\mathbf{y}^{\mathbf{h}} \Big).$$

$\mathrm{Con}(\mathbf{V})$



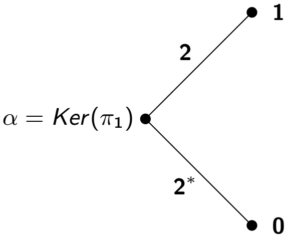
$$f(\mathbf{x}, \mathbf{y}) = \Big(\sum_{\mathbf{m} \in \mathbb{Z}_p^n} a_{\mathbf{m}} \mathbf{x}^{\mathbf{m}} \ , \ \mathbf{b}\mathbf{y} + d \Big).$$

Con(**V**)



$$f(\mathsf{x},\mathsf{y}) = \Big(\mathbf{a}\mathsf{x} + c\ ,\ \mathbf{b}\mathsf{y} + f_0(\mathsf{x})\Big).$$

Con($\mathbf{V}$)



$$f(\mathsf{x}, \mathsf{y}) = \Big(\mathbf{a}\mathsf{x} + c \;,\; \mathbf{f}_1(\mathsf{x})\mathsf{y} + f_0(\mathsf{x})\Big).$$

Case of independent algebras

Definition

Two algebras **A** and **B** of the same variety **V** are *independent* if there exists a binary term in $\mathrm{Clo}(\mathbf{V})$ such that $\mathbf{A} \models t(x, y) \approx x$ and $\mathbf{B} \models t(x, y) \approx y$.

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Theorem (E. Aichinger, P. Mayr 2015; SF)

Let p and q be distinct prime numbers. Then for every group expansion **V** of $\mathbb{Z}_p \times \mathbb{Z}_q$ which has congruence lattice form by the congruences $\{0, \alpha, \beta, 1\}$ it follows that:

(1) $\mathrm{Clo}(\mathbf{V}) = \mathrm{Clo}(\mathbf{V}_1) \times \mathrm{Clo}(\mathbf{V}_2)$ for some $\mathbf{V}_1$ and $\mathbf{V}_2$ group expansion of $\mathbb{Z}_p$, and $\mathbb{Z}_q$;

(2) $\mathrm{Clo}(\mathbf{V}_2)$ is composed by affine functions $\Leftrightarrow [\alpha, \alpha] = 0$;

(3) $\mathrm{Clo}(\mathbf{V}_1)$ is composed by affine functions $\Leftrightarrow [\beta, \beta] = 0$.

Number of expansions in the case: $Con(\mathbb{Z}_p \times \mathbb{Z}_q, +, \mathcal{C}) = \{0, \alpha, \beta, 1\}$

$t(x, y) = mx + ny$ where

$$\begin{aligned} m &\equiv_p 1 & \qquad n &\equiv_p 0 \\ m &\equiv_q 0 & \qquad n &\equiv_q 1 \end{aligned}$$

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Corollary

Let p and q be distinct prime numbers. Let $n(x) := \sharp$ of divisors of $x - 1$. Then there are $(n(q) + 3) * (n(p) + 3)$ many group expansions of $\mathbb{Z}_p \times \mathbb{Z}_q$ with $\{0, \alpha, \beta, 1\}$ as congruence lattice. Moreover $(n(q) + 3) * 2$ of these expansions respect $[\alpha, \alpha] = 0$, $(n(p) + 3) * 2$ of these expansions respect $[\beta, \beta] = 0$, and 4 respect both.

Case: $Con(\mathbb{Z}_p \times \mathbb{Z}_q, +, \mathcal{C}) \supseteq \{0, \alpha, 1\}$ and $[\alpha, \alpha] = 0$

Lemma

Let p and q distinct prime numbers. Then $\mathcal{C}$ is a clone of $\mathbb{Z}_p \times \mathbb{Z}_q$ which respects $\{\alpha, 1, 0\}$ and $[\alpha, \alpha] = 0$ if and only if for every n-ary function $f \in \mathcal{C}$ there exist: a sequence $\{a_{\mathbf{m}}\}_{\mathbf{m}}$ from $\mathbb{Z}_p$, $\mathbf{f}_1 : \mathbb{Z}_p^n \mapsto \mathbb{Z}_q^n$, and $\mathbf{f}_0 : \mathbb{Z}_p^n \mapsto \mathbb{Z}_q$ such that:

$$\forall \mathbf{x} \in \mathbb{Z}_p^n, \mathbf{y} \in \mathbb{Z}_q^n \quad f(\mathbf{x}, \mathbf{y}) = \Big(\sum_{\mathbf{m} \in \mathbb{Z}_p^n} a_{\mathbf{m}} \mathbf{x}^{\mathbf{m}} \ , \ \mathbf{f}_1(\mathbf{x})\mathbf{y} + f_0(\mathbf{x}) \Big).$$

Case: $Con(\mathbb{Z}_p \times \mathbb{Z}_q, +, \mathcal{C}) \supseteq \{0, \alpha, 1\}$ and $[\alpha, \alpha] = 0$

Definition

Let $\mathcal{C}_{pq}$ be a clone of $\mathbb{Z}_p \times \mathbb{Z}_q$. We define $\mathcal{A}, \mathcal{B}, \mathcal{C} \subseteq \mathcal{C}_{pq}$ as:

$\mathcal{A} := \{f | f \in \mathcal{C}_{pq}, \exists p : \mathbb{Z}_p^n \mapsto \mathbb{Z}_p, \forall \mathbf{x} \in \mathbb{Z}_p^n, \mathbf{y} \in \mathbb{Z}_q^n \quad f(\mathbf{x}, \mathbf{y}) = (p(\mathbf{x}), 0)\}$

$\mathcal{B} := \{f | f \in \mathcal{C}_{pq}, \exists \mathbf{f}_1 : \mathbb{Z}_p^n \mapsto \mathbb{Z}_q^n, \forall \mathbf{x} \in \mathbb{Z}_p^n, \mathbf{y} \in \mathbb{Z}_q^n \quad f(\mathbf{x}, \mathbf{y}) = (0, \mathbf{f}_1(\mathbf{x})\mathbf{y})\}$

$\mathcal{C} := \{f | f \in \mathcal{C}_{pq}, \exists f_c : \mathbb{Z}_p^n \mapsto \mathbb{Z}_q, \forall \mathbf{x} \in \mathbb{Z}_p^n, \mathbf{y} \in \mathbb{Z}_q^n \quad f(\mathbf{x}, \mathbf{y}) = (0, f_0(\mathbf{x}))\}.$

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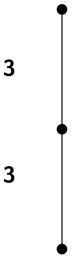
Lemma

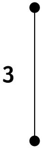
Let p and q be distinct prime numbers. Then every clone $\mathcal{C}_{pq}$ of $\mathbb{Z}_p \times \mathbb{Z}_q$, which respects $\{\alpha, 1, 0\}$ and $[\alpha, \alpha] = 0$, is generated by its three subsets $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$. Hence $\mathcal{C}_{pq}$ is completely characterized by $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$

Other cases

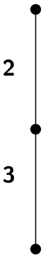
Case 1



Case 2



Case 3



Theorem (K. Kearnes, A. Szendrei)

If **A** is a finite algebra with a k-parallelogram term $(k > 1)$ such that **A** generates a residually small variety, then the relational clone of compatible relations of **A** is generated by relations of arity $\leqslant c$, where

$$c = max(k, c_0) \text{ and } c_0 = |A|^{|A|+1}(B(|A|+1)-1).$$

where $B(n)$ denotes the nth Bell's number.

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Let p and q be different prime numbers. Then there exist only finitely many term expansions of $(\mathbb{Z}_p \times \mathbb{Z}_q, +)$.

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THANK YOU!!!!