Lattices with many congruences are planar

Gábor Czédli (University of Szeged)
http://www.math.u-szeged.hu/~czedli/

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September 4, 2018

For the present audience, no definition is necessary.

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Let n be a positive integer and let L be an n-element lattice. If L has many (that is, more than 2^{n-5}) congruences, then it is planar.

An easy result; 9 page long paper, 7 page long proof.

Remark (Sharpness of the Thm.)

For each integer $n \geq 8$, there exists a non-planar n-element lattice L with few (in fact, exactly 2^{n-5}) congruences. This L is not even dismantlable.

Coordinates:

- Submitted to AU.
- `http://arxiv.org/abs/1807.08384`
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Motivation (Numbers and *planar* lattices have already been connected in some ways; for example:)

We know from Czédli, Dékány, Gyenizse and Kulin's paper, AU **75** (2016) 33–50, that the number of n-element slim rectangular lattices (which are necessarily planar) is asymptotically

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Motivation (The five largest $|\mathrm{Con}(L)|$ provided $|L| = n$)

In the set $\{|\mathrm{Con}(L)| : L$ is an n-element lattice$\}$, for $n \geq 5$,

- the largest number is $16 \cdot 2^{n-5}$ congruences by Ralph Freese, 1997.
- the second largest number is $8 \cdot 2^{n-5}$ by Czédli, 2017.
- the third, fourth, and fifth largest numbers of are $5 \cdot 2^{n-5}$, $4 \cdot 2^{n-5}$, and $3.5 \cdot 2^{n-5}$, respectively, by Kulin and Mureșan, 2018 (long paper).

Moreover, these authors have **described** the lattices witnessing the numbers above.

Corollary (of Mureșan and Kulin's description)

If $n := |L|$ and $|\mathrm{Con}(L)| \geq 3.5 \cdot 2^{n-5}$, then L is planar.

A hopeless plan: 6th, 7th, 8th, 9th, 10th, ... ?

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Why? Because we can use Kelly and Rival: Planar lattices; Canad. J. Math. **27**, 636–665 (1975). Their main result is this:

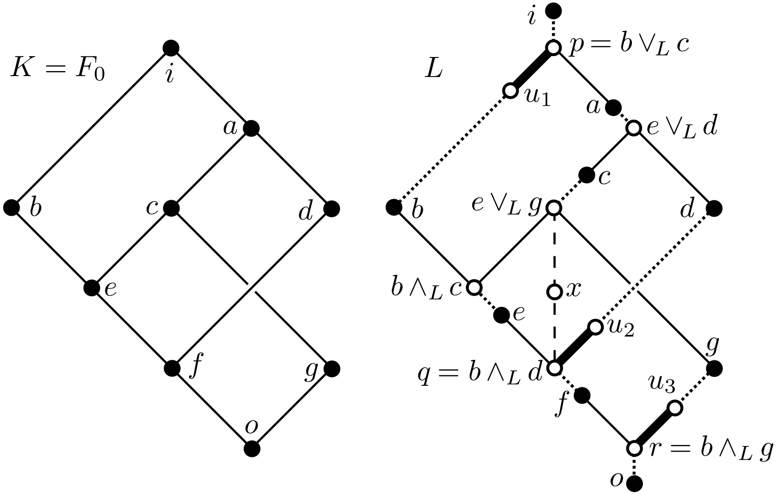
Theorem (The Kelly–Rival characterization of planar lattices)

A finite lattice L is planar if and only if none of the following lattices and their duals is a subposet of L.

These lattices will be displayed directly from their paper, that is, from another file! Observe: with two exceptions, each of them has at least 4 join-reducible elements or 4 meet-irreducible ones.

Answer: "if and only if none of the following lattices and their duals is a subposet of L"; subposets create much more problems than sublattices.
For example, look at $K := F_0$ from Kelly and Rival's list.

K as the "black-filled" subposet of L (an example)



(solid thin, solid thick, dotted) := ($<$, $\prec$, $\leq$). x may be missing.

$\mathrm{J}(L) := \{x \in L : x \text{ has exactly one lower cover}\}$.
$\mathrm{jR}(L) := \{x \in L : x \text{ has more than one lower cover}\}$.
Note that $0 \notin \mathrm{J}(L) \cup \mathrm{jR}(L)$.

Lemma (see Theorem 2.35 in Freese, Ježek and Nation's book)

For every finite lattice L, $|\mathrm{Con}(L)| \le 2^{|\mathrm{J}(L)|}$, and dually.

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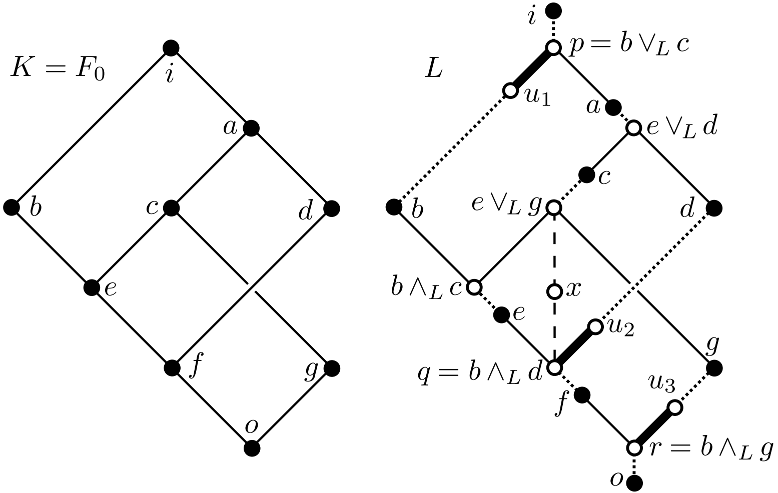

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Lemma

If K and L are finite lattices and K is a subposet of L, then $|\mathrm{jR}(K)| \le |\mathrm{jR}(L)|$.

Corollary

If K from Kelly and Rival's list is a subposet of L and $|\mathrm{jR}(K)| \ge 4$, or dually, then L has few congruences.

Proof.

Then $|\{0\} \cup \mathrm{jR}(L)| \ge 5$, whence $|\mathrm{J}(L)| \le |L| - 5 = n - 5$, and the previous lemma yields $|\mathrm{Con}(L)| \le 2^{|\mathrm{J}(L)|} = 2^{n-5}$, as required. $\square$

Separate (and much more involved) treatments are necessary for those two lattices of Kelly and Rival's list that have less than 4 join-reducible elements. Just for illustration (without details):

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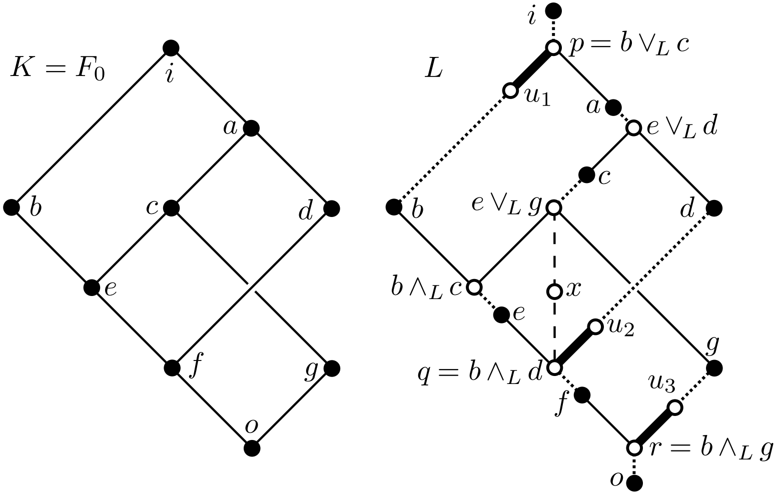

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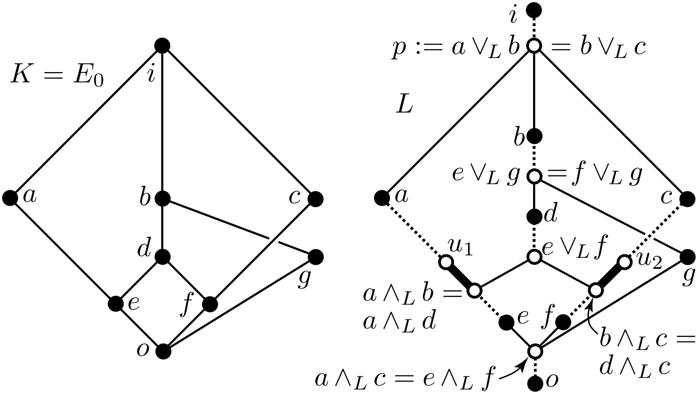

Finally, if L has many congruences, then no K from the list is a subposet, and L is planar by Kelly and Rival's theorem. Q.e.d.

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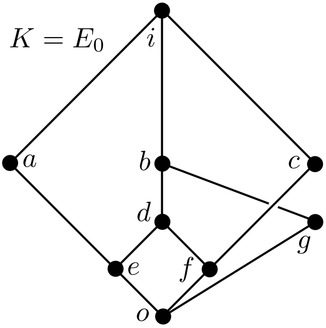

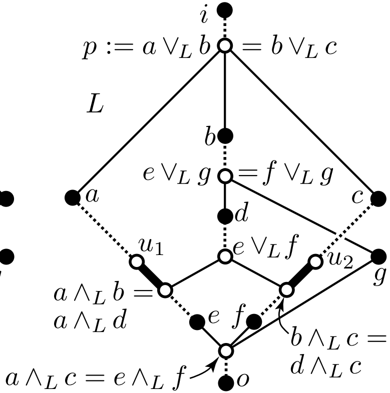

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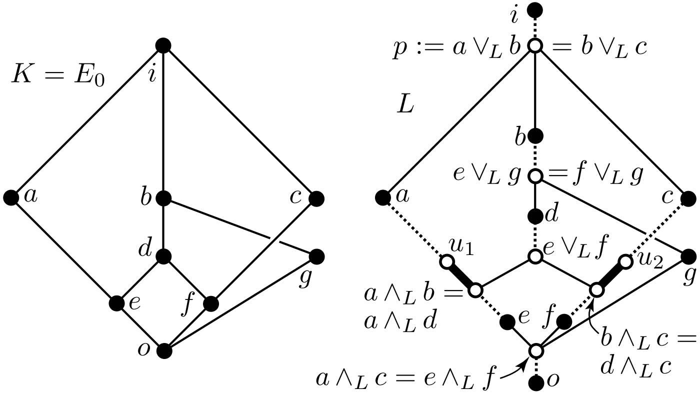

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