On the lattice of complete join-endomorphisms of a complete lattice

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1. Preliminaries

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Let L be a complete lattice. A map $f : L \to L$ is called a complete join-endomorphism, if it preserves arbitrary joins, i.e.

$$f\left(\bigvee_{i\in I} x_i\right) = \bigvee_{i\in I} f(x_i), \text{ for all } x_i \in L \text{ and any index set } I.$$

In particular, $f(0) = 0$. It was proved by G. Grätzer and E.T. Schmidt [GS] that the complete join-endomorphisms of L also form a complete lattice $\mathrm{End}_{\vee}(L)$. Here $f \vee g$ for $f, g \in \mathrm{End}_{\vee}(L)$ is defined as a pointwise join.

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- (T1) $\mathrm{End}_\vee(L)$ is distributive if and only if L is distributive.
- (T2) If L is not distributive, then $\mathrm{End}_\vee(L)$ is not even semimodular.

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- (T2) If L is not distributive, then $\mathrm{End}_\vee(L)$ is not even semimodular.

S. Valentini(1994) have shown that

- (Thm3) The complete join-endomorphisms of L form a quantale with respect to the composition and the pointwise join.

Our aim here is to determine the completely join and meet-irreducible elements in the lattice $\mathrm{End}_\vee(L)$, because these can be viewed as some "building stones" of a complete lattice.

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A **quantale** is an algebraic structure $\mathbb{Q} = (L, \vee, \odot)$, such that $(L, \leq)$ is a complete lattice (induced by $\vee$) and $(L, \odot)$ is a semigroup satisfying

$$a \odot \left(\bigvee_{i \in I} b_i \right) = \bigvee_{i \in I} (a \odot b_i) \text{ and } \left(\bigvee_{i \in I} b_i \right) \odot a = \bigvee_{i \in I} (b_i \odot a).$$

for all $a \in L$ and $b_i \in L$, $i \in I$.

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for all $a \in L$ and $b_i \in L$, $i \in I$. $\mathbb{Q}$ is called commutative, if $\odot$ is commutative, and $\mathbb{Q}$ is unital, whenever $(L, \odot)$ is a monoid. A unital quantale in which the neutral element of $\odot$ coincides to the greatest element 1 of the lattice L is called integral.

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(b) *The two-sided ideals* of a ring $(R, +, \cdot)$ with unit form an integral quantale $(\mathcal{I}(R), \vee, \bullet)$, where $(\mathcal{I}(R), \vee, \cap)$ is the complete lattice of the ideals of R and $\bullet$ is their usual multiplication.

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for all $a \in L$ and $b_i \in L$, $i \in I$. $\mathbb{Q}$ is called commutative, if $\odot$ is commutative, and $\mathbb{Q}$ is unital, whenever $(L, \odot)$ is a monoid. A unital quantale in which the neutral element of $\odot$ coincides to the greatest element 1 of the lattice L is called integral.

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(b) *The two-sided ideals* of a ring $(R, +, \cdot)$ with unit form an integral quantale $(\mathcal{I}(R), \vee, \bullet)$, where $(\mathcal{I}(R), \vee, \cap)$ is the complete lattice of the ideals of R and $\bullet$ is their usual multiplication.
(c) *For any complete lattice L,* $(\mathrm{End}_{\vee}(L), \vee, \cdot)$ is a unital quantale (its unit is the identity mapping on L.)

The complete join-endomorphisms of lattice L are strongly related with the so-called ordered relations of L.

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Definition 2.

A **$\leq$-ordered relation** is a binary relation $R \subseteq L \times L$ such that
(1) For any $u, x, y, z \in L$, if $u \leq x, (x, y) \in R$ and $y \leq z$, then $(u, z) \in R$;
(2) For any $t \in L$, $A \subseteq L$, if $(a, t) \in R$ for all $a \in A$, then $(\bigvee A, t) \in R$;
(3) For any $t \in L$, $A \subseteq L$, if $(t, a) \in R$ for all $a \in A$, then $(t, \bigwedge A) \in R$.

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Let us denote by $OR_{\leq}(L)$ the ordered relations on L. Notice, that the smallest ordered relation with respect to $\subseteq$ is the "*corner*" relation

$$\Gamma(L) = \{(x, y) \in L^2 \mid x = 0 \text{ or } y = 1\}.$$

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The following characterization of ordered relations was given by K. Kaarli and V. Kuchmei:

Proposition 1.

Let R be a binary relation on a complete lattice L. Then the following are equivalent:

(1) *R is a $\leq$-ordered relation;*

(2) *R is the universe of a complete subdirect square of L containing the "corner" element $(0, 1)$;*

(3) *R is a complete compatible binary relation on L containing $\Gamma(L)$.*

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Proposition 2.

For any complete lattice L, $(OR_{\leq}(L), \cap, \circ)$ is a unital quantale, with the multiplicative identity $\leq$.
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Proposition 3.

For any complete lattice L, the quantales $(OR_{\leq}(L), \cap, \circ)$ and $(\mathrm{End}_{\vee}(L), \vee, \cdot)$ are dually isomorphic.

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The mapping ψ: $\mathrm{OR}_{\leq}(L) \to \mathrm{End}_{\vee}(L)$, $\psi(R) = f^R$, where for each $R \in \mathrm{OR}_{\leq}(L)$ and $x \in L$,

$$f^R(x) := \bigwedge\{z \in L \mid (x, z) \in R\},$$

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An element $p \in L\backslash\{0\}$ of a complete lattice L is called completely join-irreducible if for any system $X \subseteq L$ the equality $p = \bigvee X$ implies $p \in X$. The set of completely join-irreducible elements of L is denoted by $J(L)$. The completely meet-irreducible elements of L are defined dually, consisting a set $M(L)$.

In what follows, for any elements a, b of the lattice L we will use the notations $\downarrow a = \{x \in L \mid x \leq a\}$ and $\uparrow b = \{x \in L \mid x \geq b\}$.

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(i) *The completely join-irreducible elements of* $\mathrm{OR}_{\leq}(L)$ *are exactly the relations*

$$\Gamma(L) \cup (\downarrow j \times \uparrow m) \subseteq L^2$$

where $j \in J(L)$ *and* $m \in M(L)$.

(ii) *The completely meet-irreducible elements of* $\mathrm{End}_{\vee}(L)$ *are exactly the maps of the form (with* $j \in J(L)$, $m \in M(L)$*):*

$$f(x) = \begin{cases} 0, \text{ if } x = 0 \\ m, \text{ if } x \leq j,\ x \neq 0 \\ 1, \text{ otherwise} \end{cases}$$

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Concerning the join-irreducible elements of $\mathrm{End}_{\vee}(L)$ we have obtained the following results:

Theorem 2.

(i) *Let* $p \in J(L)$ *and* $m \in M(L)$. *Then any mapping* $f : L \to L$ *of the form*

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A **complete tolerance** of a complete lattice L is a reflexive, symmetric relation T on L compatible with arbitrary suprema and infima, i.e. for any system of pairs $(a_i, b_i) \in T$, $i \in I$ we have

$$\left(\bigwedge_{i \in I} a_i, \bigwedge_{i \in I} b_i \right) \in T \text{ and } \left(\bigvee_{i \in I} a_i, \bigvee_{i \in I} b_i \right) \in T.$$

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If in addition the above T is transitive, then it is called a *complete congruence* of L.

If $T \subseteq L^2$ is a complete tolerance of L, then for any $x \in L$ we define:
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It is known that for any complete tolerance T of L the map $\lambda_T : x \mapsto x_T$, $x \in L$ is a decreasing complete $\vee$-endomorphism, and $\mu_T : x \mapsto x^T$, $x \in L$ is an increasing complete $\wedge$-endomorphism of the lattice L.

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(a) A lattice L with 0 is called pseudocomplemented if for each $x \in L$ there exists an $x^* \in L$ such that for any $y \in L$, $y \wedge x = 0 \Leftrightarrow y \leq x^*$.
(b) A bounded lattice L is called double-pseudocomplemented if both L, both its dual L^d are pseudocomplemented.

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Proposition 4.

Let L be a finite lattice and $\theta \subseteq L^2$ a congruence of it which is maximal with respect to the property $\theta^* \neq \triangle$. Then the mapping $\lambda_\theta \colon L \to L$, $\lambda_\theta(x) = x_\theta$, is a decreasing join-irreducible element of $\mathrm{End}_\vee(L)$.

As an immediate consequence of the first theorem (Thm1.) we obtain

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Corollary 1.

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Corollary 2.

The dual atoms of $\mathrm{End}_{\vee}(L)$ *are exactly the maps of the form:*

$$f(x) = \begin{cases} 0, \ \textit{if } x = 0, \\ d, \ \textit{if } x = a, \\ 1, \ \textit{otherwise} \end{cases}$$

where a is an atom, and d is a dual atom of L.

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