Cancellable varieties of semigroups

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Special elements

Definition

An element $x \in L$ is called

distributive if $(\forall y, z \in L)\quad x \vee (y \wedge z) = (x \wedge y) \vee (x \wedge z)$,

standard if $(\forall y, z \in L)\quad (x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$,

modular if $(\forall y, z \in L)\quad y \leq z \longrightarrow (x \vee y) \wedge z = (x \wedge z) \vee y$

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\end{array}$$

An element is standard if and only if it is distributive and modular.

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Each cancellable element is modular.

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A variety **V** is standard [distributive, modular, cancellable] if and only if the variety $\mathbf{V} \vee \mathbf{SL}$ has the same property.

Theorem (J. Ježek, R. N. McKenzie, 1993)

If a semigroup variety $\mathbf{V}$ *is modular then either* $\mathbf{V} = \mathbf{SEM}$ *or* $\mathbf{V}$ *has the form* $\mathbf{M} \vee \mathbf{N}$ *where* $\mathbf{M} \in \{\mathbf{T}, \mathbf{SL}\}$ *while* $\mathbf{N}$ *is a nilvariety.*

Modular varieties: sufficiency

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Theorem (B. M. Vernikov, M. V. Volkov, 1988)

Each 0-reduced variety is modular.

Distributive and standard varieties

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Theorem (V. Yu. Shaprynskii, B. M. Vernikov, 2010)

For a semigroup variety $\mathbf{V}$*, the following are equivalent:*

1) $\mathbf{V}$ *is standard;*
2) $\mathbf{V}$ *is distributive;*
3) *either* $\mathbf{V} = \mathbf{SEM}$ *or* $\mathbf{V} = \mathbf{M} \vee \mathbf{N}$ *where* $\mathbf{M} \in \{\mathbf{T}, \mathbf{SL}\}$ *and* $\mathbf{N}$ *is one of the varieties* $\mathbf{P}$, $\mathbf{P}_n$, $\mathbf{Q}$, $\mathbf{Q}_n$.

Cancellable varieties

A *permutational identity of the langth n* is an identity of the form

$$x_1 x_2 \dots x_n \approx x_{\sigma(1)} x_{\sigma(2)} \dots x_{\sigma(n)}$$

where $\sigma \in S_n$. We denote it by $p_n[\sigma]$.

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A semigroup variety $\mathbf{V}$ is cancellable if and only if either $\mathbf{V} = \mathbf{SEM}$ or $\mathbf{V} = \mathbf{M} \vee \mathbf{N}$ where $\mathbf{M} \in \{\mathbf{T}, \mathbf{SL}\}$ and $\mathbf{N}$ is one of the varieties $\mathbf{X}_{m,n}$ and $\mathbf{Y}_{m,n}$.

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Corollary

There exist:

- *cancellable non-standard varieties;*
- 0*-reduced non-cancellable varieties;*
- *modular non-cancellable varieties.*