TECHNISCHE UNIVERSITÄT WIEN

On a stronger reconstruction notion for clones

Mike Behrisch[×] Edith Vargas-García[†]

[×]Institute of Discrete Mathematics and Geometry, Algebra Group, TU Wien
[†]Departamento Académico de Matemáticas
ITAM, México, D.F.

6th September 2018 • Špindlerův Mlýn

Disclaimer

Warning 1

- Some model theory may be involved.
- I am not a model theorist.

Disclaimer

Warning 1

- Some model theory may be involved.
- I am not a model theorist.

Warning 2: even though this is a Summer School

- Some of my definitions may not be standard...
- Go and read a book on model theory for those...
- Don't use my queer perspective on some aspects of model theory as a reference.

What is reconstruction?

Model theoretic concept:

Given $\operatorname{Aut}\mathbb{A}$, what can be said about $\mathbb{A}$?

$\operatorname{Aut}\mathbb{A}$ can be considered as

- abstract group (alg.)
- topological group (alg. + top.) (Tichonov (=pointw. conv.) top.)
- permutation group ((alg. + top. +) act.)

$[(A, \operatorname{Aut}\mathbb{A})]_{\cong} \subseteq [\operatorname{Aut}\mathbb{A}]_{\cong,\mathrm{top}} \subseteq [\operatorname{Aut}\mathbb{A}]_{\cong}$

Reconstruction up to fo-interdefinability

$\mathbb{A} \in \mathcal{K}$ has reconstruction up to fo-interdefinability wrt $\mathcal{K}$

$$\iff [(A, \operatorname{Aut} \mathbb{A})]_{\cong} \text{ determines } [\mathbb{A}]_{\text{fo-interdef}} \text{ wrt. } \mathcal{K}.$$

(allow only structures from $\mathcal{K}$ on carriers of the same size)

Proposition (Cor. of Ryll-Nardzewski's Thm, 1959)

$\forall \mathbb{A}, \mathbb{B}$ $\aleph_0$-categorical:

$$(A, \operatorname{Aut} \mathbb{A}) \cong (B, \operatorname{Aut} \mathbb{B}) \iff \mathbb{A} \xleftrightarrow{\text{fo-interdefinable}} \mathbb{B}$$

In other words

$\aleph_0$-categorical str: reconstruction up to fo-interdefinability wrt the class $\mathcal{K}$ of $\aleph_0$-categorical structures

Reconstruction up to fo-biinterpretability

$\mathbb{A} \in \mathcal{K}$ has reconstruction up to fo-biinterpretability wrt $\mathcal{K}$

$\iff$ $[\mathrm{Aut}\,\mathbb{A}]_{\cong,\mathrm{top}}$ determines $[\mathbb{A}]_{\text{fo-biint}}$ wrt. $\mathcal{K}$.

(allow only structures from $\mathcal{K}$ on carriers of the same size)

Theorem (Coquand, 1980s)

$\forall \mathbb{A}, \mathbb{B}$ $\aleph_0$-categorical:

$$\mathrm{Aut}\,\mathbb{A} \cong_{\mathrm{top}} \mathrm{Aut}\,\mathbb{B} \iff \mathbb{A} \xleftrightarrow{\text{fo-biinterpretable}} \mathbb{B}$$

In other words

$\aleph_0$-categorical str: reconstruction up to fo-biinterpretability wrt the class $\mathcal{K}$ of $\aleph_0$-categorical structures

"Reconstruction"

$\mathbb{A} \in \mathcal{K}$ has reconstruction wrt $\mathcal{K}$

$\iff$ $[\operatorname{Aut}\mathbb{A}]_{\cong}$ determines $[\mathbb{A}]_{\text{fo-biint}}$ wrt. $\mathcal{K}$.

(allow only structures from $\mathcal{K}$ on carriers of the same size)

Proposition (folklore)

$\forall \mathbb{A}, \mathbb{B}$ $\aleph_0$-categorical: $\mathbb{A}$ has small index property (SIP)

$\implies$ $\operatorname{Aut}\mathbb{A} \cong \operatorname{Aut}\mathbb{B} \iff \mathbb{A} \xleftrightarrow{\text{fo-biinterpretable}} \mathbb{B}$

In other words

$\aleph_0$-categorical str with SIP: reconstruction wrt the class $\mathcal{K}$ of $\aleph_0$-categorical structures

SIP and automatic homeomorphicity

Closed $G \leq \operatorname{Sym} A$, $|A| = \aleph_0$

G has SIP $\iff$ $\forall U \leq G\colon |G/U| \leq \aleph_0 \implies U$ is open ($G_X \subseteq U$ for some finite $X \subseteq A$)

$\mathbb{A}$ has SIP $\iff$ $\operatorname{Aut}\mathbb{A}$ has SIP

Closed $G \leq \operatorname{Sym} A$ has autom. homeomorphicity/continuity† $\Leftrightarrow$

$\forall H \leq \operatorname{Sym} B$ closed, $|B| = |A|$:‡ $\forall \xi\colon G \cong H\colon$ ξ homeo/cont

Proposition

For $G \leq \operatorname{Sym} A$ closed, $|A| = \aleph_0$:

G SIP $\iff$ G has autom. cont.† $\overset{\text{Lascar}}{\iff}$ G has autom. homeo.

† should be called "weak automatic continuity" since "automatic continuity" requires this even for any homomorphism $\xi\colon G \to H$, not just $G \cong H$

‡ some people restrict this only to $|A| = \aleph_0$

"Reconstruction revisited"

$\mathbb{A} \in \mathcal{K}$ has reconstruction wrt $\mathcal{K}$

$\iff$ $[\operatorname{Aut} \mathbb{A}]_{\cong}$ determines $[\mathbb{A}]_{\text{fo-biint}}$ wrt. $\mathcal{K}$.

(allow only structures from $\mathcal{K}$ on carriers of the same size)

Proposition

$\forall \mathbb{A}, \mathbb{B}$ $\aleph_0$-categorical: Aut $\mathbb{A}$ has SIP $\implies$
$\operatorname{Aut} \mathbb{A} \cong \operatorname{Aut} \mathbb{B} \iff \mathbb{A} \xleftrightarrow{\text{fo-biinterprtbl}} \mathbb{B}$

In other words

$\aleph_0$-categorical str with automatic homeo: reconstruction wrt the class $\mathcal{K}$ of $\aleph_0$-categorical structures

“Reconstruction revisited”

$\mathbb{A} \in \mathcal{K}$ has reconstruction wrt $\mathcal{K}$

$\iff$ $[\operatorname{Aut}\mathbb{A}]_{\cong}$ determines $[\mathbb{A}]_{\text{fo-biint}}$ wrt. $\mathcal{K}$.

(allow only structures from $\mathcal{K}$ on carriers of the same size)

Proposition (replace SIP

$\forall \mathbb{A}, \mathbb{B}$ $\aleph_0$-categorical: Aut $\mathbb{A}$ has autom. homeo. $\implies$
$\operatorname{Aut}\mathbb{A} \cong \operatorname{Aut}\mathbb{B} \iff \mathbb{A} \xleftrightarrow{\text{fo-biinterprtbl}} \mathbb{B}$

In other words

$\aleph_0$-categorical str with automatic homeo: reconstruction wrt the class $\mathcal{K}$ of $\aleph_0$-categorical structures

“Reconstruction revisited”

$\mathbb{A} \in \mathcal{K}$ has reconstruction wrt $\mathcal{K}$

$\iff$ $[\operatorname{Aut}\mathbb{A}]_{\cong}$ determines $[\mathbb{A}]_{\text{fo-biint}}$ wrt. $\mathcal{K}$.

(allow only structures from $\mathcal{K}$ on carriers of the same size)

Proposition (replace SIP + Coquand's Thm)

$\forall \mathbb{A}, \mathbb{B}$ $\aleph_0$-categorical: Aut $\mathbb{A}$ has autom. homeo. $\implies$
$\operatorname{Aut}\mathbb{A} \cong \operatorname{Aut}\mathbb{B} \iff \mathbb{A} \xleftrightarrow{\text{fo-biinterprtbl}} \mathbb{B} \iff \operatorname{Aut}\mathbb{A} \cong_{\text{top}} \operatorname{Aut}\mathbb{B}$

In other words

$\aleph_0$-categorical str with automatic homeo: reconstruction wrt the class $\mathcal{K}$ of $\aleph_0$-categorical structures

"Reconstruction revisited"

$\mathbb{A} \in \mathcal{K}$ has reconstruction wrt $\mathcal{K}$

$\iff$ $[\operatorname{Aut}\mathbb{A}]_{\cong}$ determines $[\mathbb{A}]_{\text{fo-biint}}$ wrt. $\mathcal{K}$.

(allow only structures from $\mathcal{K}$ on carriers of the same size)

Proposition (replace SIP + Coquand's Thm)

$\forall \mathbb{A}, \mathbb{B}$ $\aleph_0$-categorical: Aut $\mathbb{A}$ has autom. homeo. $\implies$

$\operatorname{Aut}\mathbb{A} \cong \operatorname{Aut}\mathbb{B} \iff \operatorname{Aut}\mathbb{A} \cong_{\text{top}} \operatorname{Aut}\mathbb{B} \iff \mathbb{A} \xleftrightarrow{\text{fo-biinterprtbl}} \mathbb{B}$

In other words

$\aleph_0$-categorical str with automatic homeo: reconstruction wrt the class $\mathcal{K}$ of $\aleph_0$-categorical structures

The advantage of removing SIP

- SIP needs index $\implies$ only makes sense for groups
- automatic homeomorphicity: available for (endomorphism) monoids, (polymorphism) clones
- from structures $\rightsquigarrow$ closed groups, monoids, clones, i.e., from model theory $\rightsquigarrow$ algebra

Instead of

$\forall \mathbb{A}, \mathbb{B}$ $\aleph_0$-categorical: $\quad$ $\operatorname{Aut}\mathbb{A}$ has autom. homeo. $\implies$

$$\operatorname{Aut}\mathbb{A} \cong \operatorname{Aut}\mathbb{B} \iff \operatorname{Aut}\mathbb{A} \cong_{\text{top}} \operatorname{Aut}\mathbb{B} \iff \mathbb{A} \xleftrightarrow{\text{fo-biinterprtbl}} \mathbb{B}$$

Proposition (keeping structures out of the game)

$G = \operatorname{Aut}\mathbb{A}$ has autom. homeo. $\implies$ $[G]_{\cong} = [G]_{\cong_{\text{top}}}$ wrt. the class of automorphism groups of $\aleph_0$-categorical str

The advantage of removing SIP

- SIP needs index $\implies$ only makes sense for groups
- automatic homeomorphicity: available for (endomorphism) monoids, (polymorphism) clones
- from structures $\rightsquigarrow$ closed groups, monoids, clones, i.e., from model theory $\rightsquigarrow$ algebra

Instead of

$\forall \mathbb{A}, \mathbb{B}$ $\aleph_0$-categorical: Aut $\mathbb{A}$ has autom. homeo. $\implies$

$\operatorname{Aut}\mathbb{A} \cong \operatorname{Aut}\mathbb{B} \iff \operatorname{Aut}\mathbb{A} \cong_{\text{top}} \operatorname{Aut}\mathbb{B} \iff \mathbb{A} \xleftrightarrow{\text{fo-biinterprtbl}} \mathbb{B}$

Proposition (trivial, hard part in the assumption)

$G = \operatorname{Aut}\mathbb{A}$ has autom. homeo. $\implies [G]_{\cong} = [G]_{\cong_{\text{top}}}$

~~wrt. the class of automorphism groups of $\aleph_0$-categorical str~~

The advantage of removing SIP

- SIP needs index $\implies$ only makes sense for groups
- automatic homeomorphicity: available for (endomorphism) monoids, (polymorphism) clones
- from structures $\rightsquigarrow$ closed groups, monoids, clones, i.e., from model theory $\rightsquigarrow$ algebra

Instead of

$\forall \mathbb{A}, \mathbb{B}$ $\aleph_0$-categorical: Aut $\mathbb{A}$ has autom. homeo. $\implies$

$\operatorname{Aut}\mathbb{A} \cong \operatorname{Aut}\mathbb{B} \iff \operatorname{Aut}\mathbb{A} \cong_{\text{top}} \operatorname{Aut}\mathbb{B} \iff \mathbb{A} \xleftrightarrow{\text{fo-biinterprtbl}} \mathbb{B}$

Proposition (still trivial, but well. . .)

$G = \operatorname{Aut}\mathbb{A}$ has autom. homeo. wrt $\mathcal{K} \implies [G]_{\cong} = [G]_{\cong_{\text{top}}}$ wrt $\mathcal{K}$

$\mathcal{K}$. . . . . .the class of automorphism groups of $\aleph_0$-categorical str

Towards something stronger

In analogy to automatic homeomorphicity:

Closed $G \leq \operatorname{Sym} A$ has automatic action compatibility $\iff$

$\forall H \leq \operatorname{Sym} B$ closed, $|B| = |A|$: $\forall \xi\colon G \cong H$:

ξ induced by perm. grp. isom.

i.e.: $\exists \theta\colon A \to B$ bij. $\forall g \in G\colon \xi(g) = \theta \circ g \circ \theta^{-1}$

Proposition (trivial)

$G = \operatorname{Aut} \mathbb{A}$ has autom. action comp. wrt $\mathcal{K}$

$\implies [G]_{\cong} = [(A, G)]_{\cong}$ wrt $\mathcal{K}$

Towards something stronger

In analogy to automatic homeomorphicity:

Closed $G \leq \operatorname{Sym} A$ has automatic action compatibility $\iff$

$\forall H \leq \operatorname{Sym} B$ closed, $|B| = |A|$: $\forall \xi\colon G \cong H$:

ξ induced by perm. grp. isom.

i.e.: $\exists \theta\colon A \to B$ bij. $\forall g \in G\colon \xi(g) = \theta \circ g \circ \theta^{-1}$

Proposition (back to structures by Ryll-Nardzewski)

$G = \operatorname{Aut} \mathbb{A}$ has autom. action comp. wrt $\mathcal{K}$

$\implies [G]_{\cong} = [(A, G)]_{\cong}$ wrt $\mathcal{K}$

$\implies [G]_{\cong}$ determines $[\mathbb{A}]_{\text{fo-interdef}}$ wrt $\mathcal{K}$

for $\mathcal{K}$ the class of (automorphism groups of) $\aleph_0$-categorical str

Are there any examples with this strong property?

Theorem (Rubin, 1994)

$G = \operatorname{Aut}\mathbb{A} \in \mathcal{K}$ for $\mathbb{A}$ with weak $\forall\exists$-interpretations

$\implies$ G has automatic action compatibility wrt $\mathcal{K}$

for $\mathcal{K}$ . . . . the class of automorphism groups of $\aleph_0$-categorical str without algebraicity

Theorem (Paolini & Shelah, 2017)

$G = \operatorname{Aut}\mathbb{A} \in \mathcal{K}$

$\implies$ G has automatic action compatibility wrt $\mathcal{K}$

for $\mathcal{K}$ . . . . the class of automorphism groups of $\aleph_0$-categorical str without algebraicity and strong SIP

Our contribution

Automatic action compatibility for monoids / clones

- Given closed clone $F \leq \mathcal{O}_A$,
- $M = F^{(1)}$ closed monoid,
- $G \subseteq M$ group of invertible monoid elements,
- How to extend automatic action compatibility:

$$G \rightsquigarrow M \rightsquigarrow F?$$

Groups to monoids

Lifting lemma

Suppose

- $M \leq A^A$, $M' \leq B^B$ closed transformation monoids
- $G \subseteq M$, $G' \subseteq M'$ groups of invertibles
- G dense: $\quad M \subseteq \overline{G}$
- $\varphi\colon M \to M'$ monoid isomorphism
- $\varphi\restriction_G^{G'}\colon G \to G'$ is induced by bijection $\theta\colon A \to B$
- $\forall \psi\colon M \cong \overline{G'}\colon \psi\restriction_G = \varphi\restriction_G \implies \psi = \varphi$

Then φ is induced by θ: $\quad \forall f \in M\colon \varphi(f) = \theta \circ f \circ \theta^{-1}$.

Groups to monoids

Lifting lemma

Suppose

- $M \leq A^A$, $M' \leq B^B$ closed transformation monoids
- $G \subseteq M$, $G' \subseteq M'$ groups of invertibles
- G dense: $M \subseteq \overline{G}$
- $\varphi\colon M \to M'$ monoid isomorphism
- $\varphi\restriction_G^{G'}\colon G \to G'$ is induced by bijection $\theta\colon A \to B$
- $\forall \psi\colon M \to \overline{G'}\colon \psi\restriction_G = \varphi\restriction_G \implies \psi = \varphi$

Then φ is induced by θ: $\forall f \in M\colon \varphi(f) = \theta \circ f \circ \theta^{-1}$.

Corollary for $G \subseteq M \subseteq \overline{G} \subseteq A^A$ as above

If $\forall G' \subseteq M' \leq B^B, |B| = |A| \, \forall \varphi\colon M \cong M' \, \forall \psi\colon M \cong \overline{G'}$:

$\psi\restriction_G = \varphi\restriction_G \implies \psi = \varphi$, then

G has autom action comp. $\implies$ M has autom action comp.

Groups to monoids

Lifting lemma

Suppose

- $M \leq A^A$, $M' \leq B^B$ closed transformation monoids
- $G \subseteq M$, $G' \subseteq M'$ groups of invertibles
- G dense: $M \subseteq \overline{G}$
- $\varphi\colon M \to M'$ monoid isomorphism
- $\varphi\restriction_G^{G'}\colon G \to G'$ is induced by bijection $\theta\colon A \to B$
- $\forall \psi\colon M \to \overline{G'}\colon \psi\restriction_G = \varphi\restriction_G \implies \psi = \varphi$

Then φ is induced by θ: $\forall f \in M\colon \varphi(f) = \theta \circ f \circ \theta^{-1}$.

Corollary for $G \subseteq M \subseteq \overline{G} \subseteq A^A$ as above

If M has automatic homeomorphicity, then
G has autom action comp. $\implies$ M has autom action comp.

Groups to monoids

Lifting lemma

Suppose

- $M \leq A^A$, $M' \leq B^B$ closed transformation monoids
- $G \subseteq M$, $G' \subseteq M'$ groups of invertibles
- G dense: $\quad M \subseteq \overline{G}$
- $\varphi\colon M \to M'$ monoid isomorphism
- $\varphi\restriction_G^{G'}\colon G \to G'$ is induced by bijection $\theta\colon A \to B$
- $\forall \psi\colon M \to \overline{G'}\colon \psi\restriction_G = \varphi\restriction_G \implies \psi = \varphi$

Then φ is induced by θ: $\quad \forall f \in M\colon \varphi(f) = \theta \circ f \circ \theta^{-1}$.

Corollary for $G \subseteq M \subseteq \overline{G} \subseteq A^A$ as above

If $\forall \varphi\colon M \to M$ inj: $\varphi\restriction_G = \mathrm{id}_M\restriction_G \implies \varphi = \mathrm{id}_M$, then
G has autom action comp. $\implies$ M has autom action comp.

Monoids to clones

Theorem

Suppose for **any** sets A, B

- $F \leq \mathcal{O}_A$, $F' \leq \mathcal{O}_B$ clones (closedness not needed)
- $\xi\colon F \twoheadrightarrow F'$ surjective clone homomorphism
- $F^{(1)}$ weakly directed
  $(\forall a_1, a_2 \in A\, \exists a_0 \in A\, \exists f_1, f_2 \in F^{(1)}\colon f_1(a_0) = a_1 \wedge f_2(a_0) = a_2)$
- $\xi\restriction_{F^{(1)}}^{F'^{(1)}}\colon F^{(1)} \twoheadrightarrow F'^{(1)}$ induced by bijection $\theta\colon A \to B$

Then ξ is induced by θ: ($\implies \xi$ unif. homeo)
$\forall f \in F\colon \qquad \xi(f) = \theta \circ f \circ (\theta^{-1} \times \cdots \times \theta^{-1})$.

Corollary for closed $F \leq \mathcal{O}_A$ with $F^{(1)}$ weakly directed

$F^{(1)}$ has autom action comp. $\implies$ F has autom action comp.

Making things concrete

Corollary

- $F \leq \mathcal{O}_A$ closed clone, $M := F^{(1)}$ weakly directed
- $G \subseteq M \subseteq \overline{G}$ dense group of invertibles
- $\mathbb{A}$ $\aleph_0$-categorical, no algebraicity, weak $\forall\exists$-interpretations
- $\operatorname{Aut} \mathbb{A} = G$
- $\forall \varphi\colon M \to M$ inj: $\varphi{\restriction_G} = \mathrm{id}_M{\restriction_G} \implies \varphi = \mathrm{id}_M$, or M has automatic homeomorphicity (wrt `technical`)

$\implies$ F has automatic action compatibility wrt $\mathcal{C}$,
$\implies$ M has automatic action compatibility wrt $\mathcal{K}$,
$\mathcal{C}$ . . . . all polymorphism clones of $\aleph_0$-categorical str w/o algebraicity
$\mathcal{K}$. all endomorphism monoids of $\aleph_0$-categorical str w/o algebraicity

Making things very concrete

Let $\mathbb{A}$ be one of

- $(\mathbb{Q}, <)$
- the random (Rado) graph
- the random directed graph
- the countable universal homogeneous tournament
- the countable universal k-uniform hypergraph for $k \geq 2$
- the countable universal homogeneous $\mathbb{K}_n$-free graph, for $n \geq 3$
- any (countable universal homogeneous) Henson digraph

$\Longrightarrow$ $M = \operatorname{Emb} \mathbb{A}$ has automatic action compatibility wrt endomorphism monoids of $\aleph_0$-categ. str. without algebraicity

$\Longrightarrow$ every $F = \overline{F} \leq \mathcal{O}_A$ with $F^{(1)} = \operatorname{Emb} \mathbb{A}$, e.g. $F = \operatorname{Pol} \mathbb{A}^{\complement}$, has automatic action compatibility wrt polymorphism clones of $\aleph_0$-categ. str. without algebraicity

$\Longrightarrow$ $\operatorname{Pol}(\mathbb{Q}, <)$ has automatic action compatibility

Open problems

Problem 1

Which of $(\mathbb{Q}, \mathrm{betw})$, $(\mathbb{Q}, \mathrm{circ})$ and $(\mathbb{Q}, \mathrm{sep})$ have a weak $\forall\exists$-interpretation?

Problem 2

For which $\mathbb{A}$ among

- random strict poset
- countable universal homogeneous bipartite graph
- countable dense local order $\mathbb{S}_2$
- countable myopic local order $\mathbb{S}_3$

does $\operatorname{Emb}\mathbb{A}$ have automatic homeomorphicity / satisfy the condition the only injective monoid endo fixing $\operatorname{Aut}\mathbb{A}$ pointwise is $\mathrm{id}_{\operatorname{Emb}\mathbb{A}}$?

The end

Th∀nk you for your ∀tt∃ntion.