Graph of walks in universal algebra

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General question

Under what circumstances a graph compatible with an algebra has to contain a loop?

- Graph = set of ordered pairs (directed edges)
- Compatible an algebra $\boldsymbol{A}$ = subuniverse of $\boldsymbol{A}^2$
- Loop = Edge on one node, i.e. pair (x,x)

- Historically motivated by Constraint Satisfaction Problem

Approaches

1) Compare loop conditions in varieties
2) Use idempotency

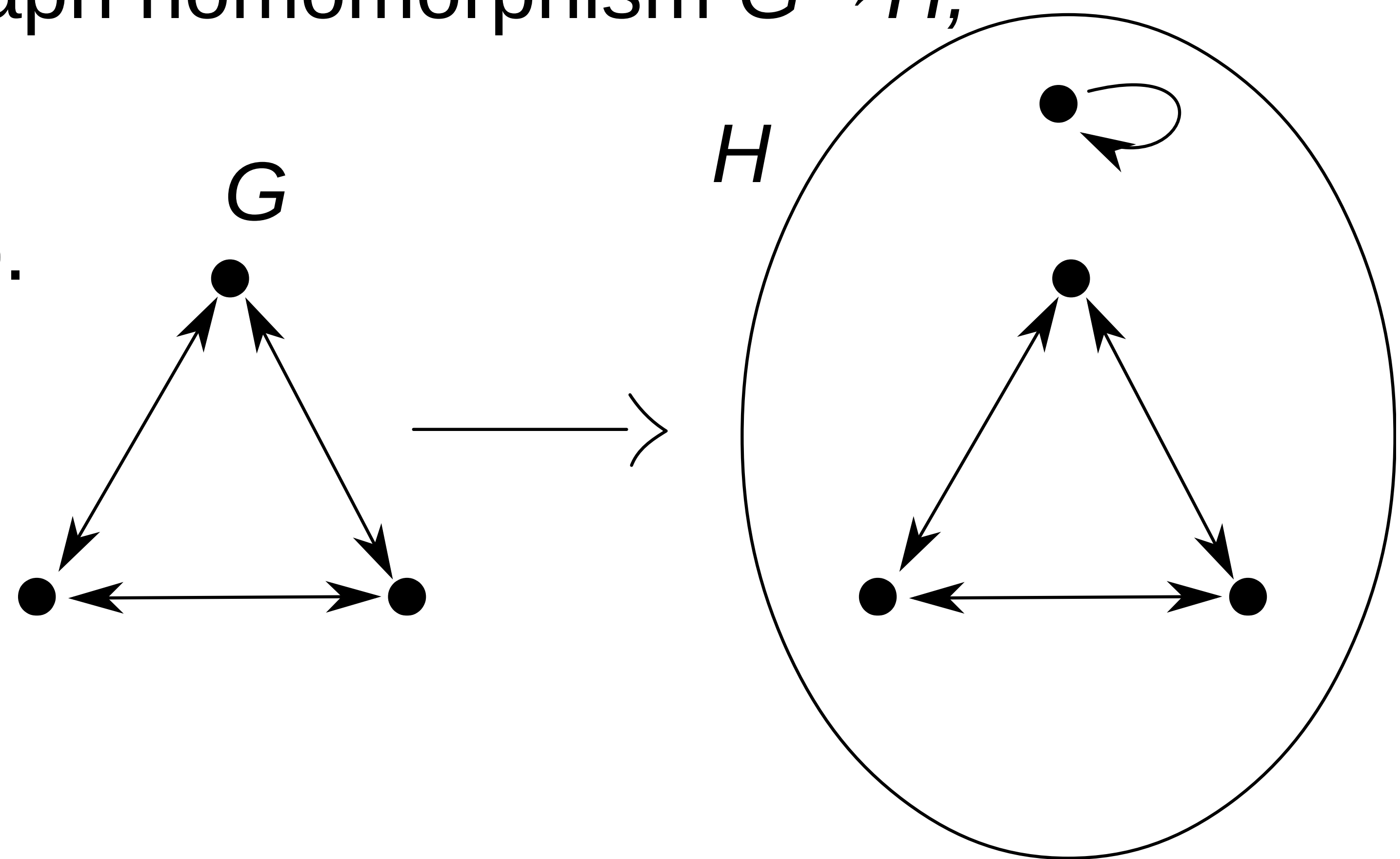
The G loop condition for a variety V

If

- H is a graph compatible with an algebra in V,
- there is a graph homomorphism $G \to H$,

then

- H has a loop.



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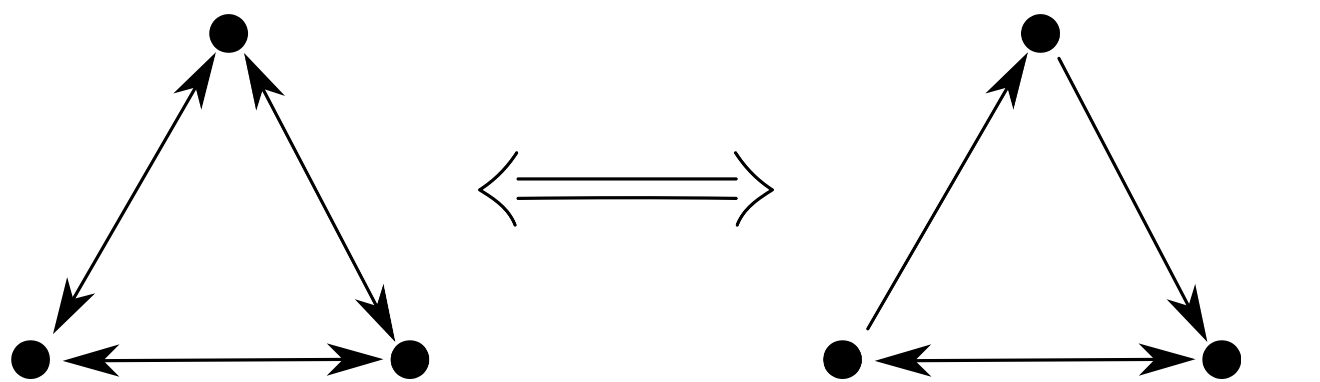

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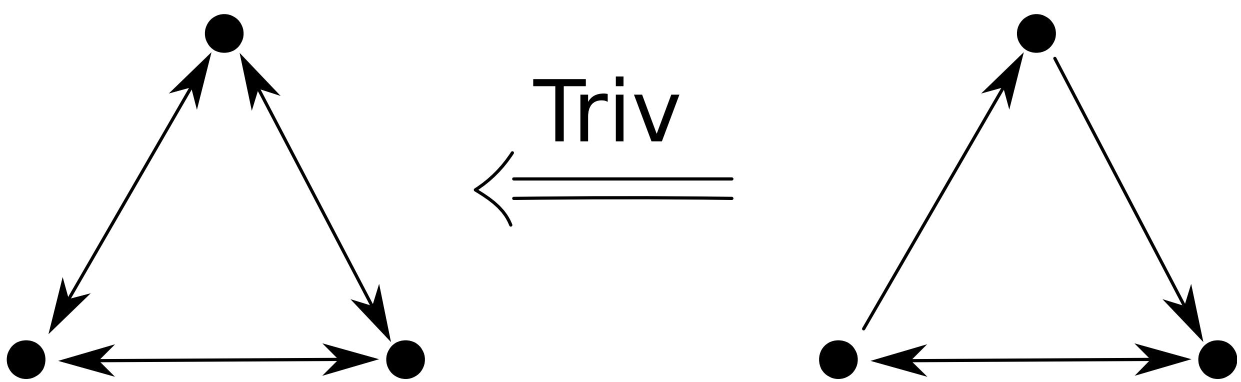
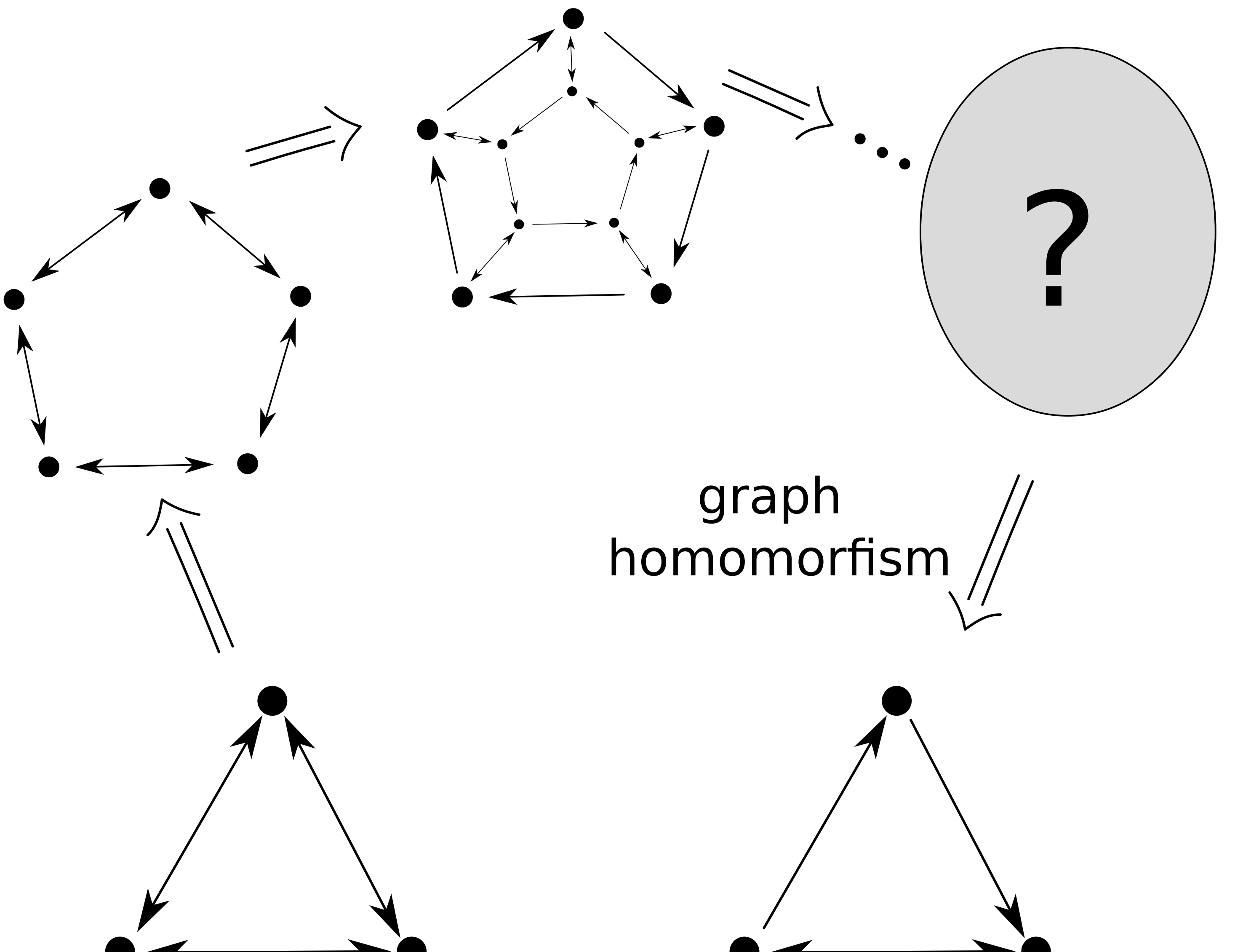


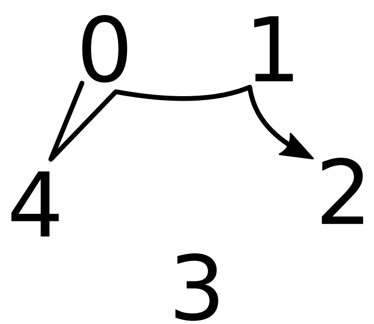
Hard implication idea



The graph of cycle walks

Parameters: cycle size (odd), walk length

Nodes = walks



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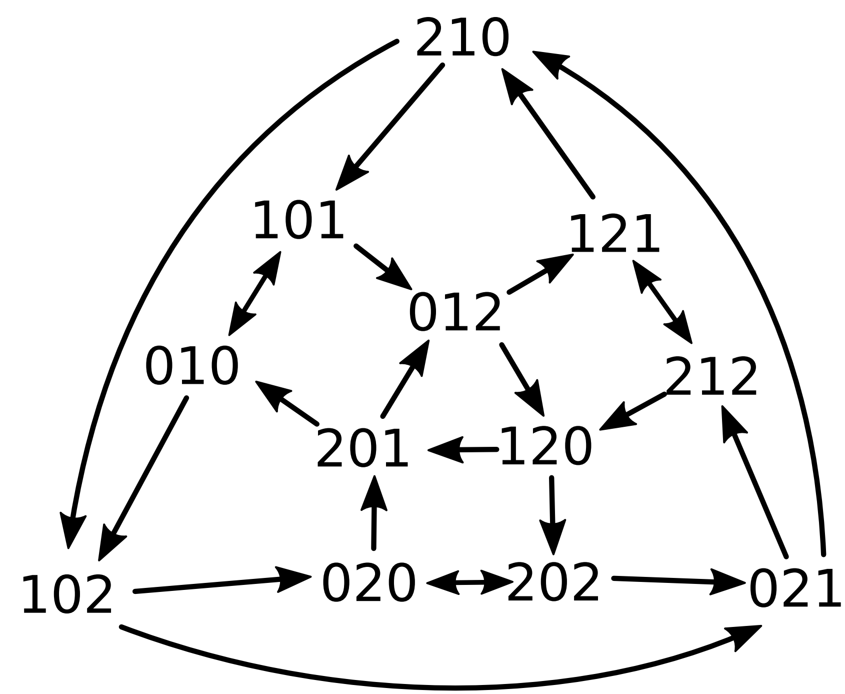
Example (3,3)

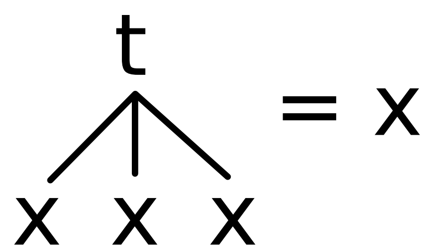


Edges

$$(x_1\ x_2\ x_3\ldots\ x_n)$$
$$\downarrow$$
$$(x_2\ x_3\ x_4\ldots\ x_{n+1})$$

Idempotency

- $t(x,x,x) = x$

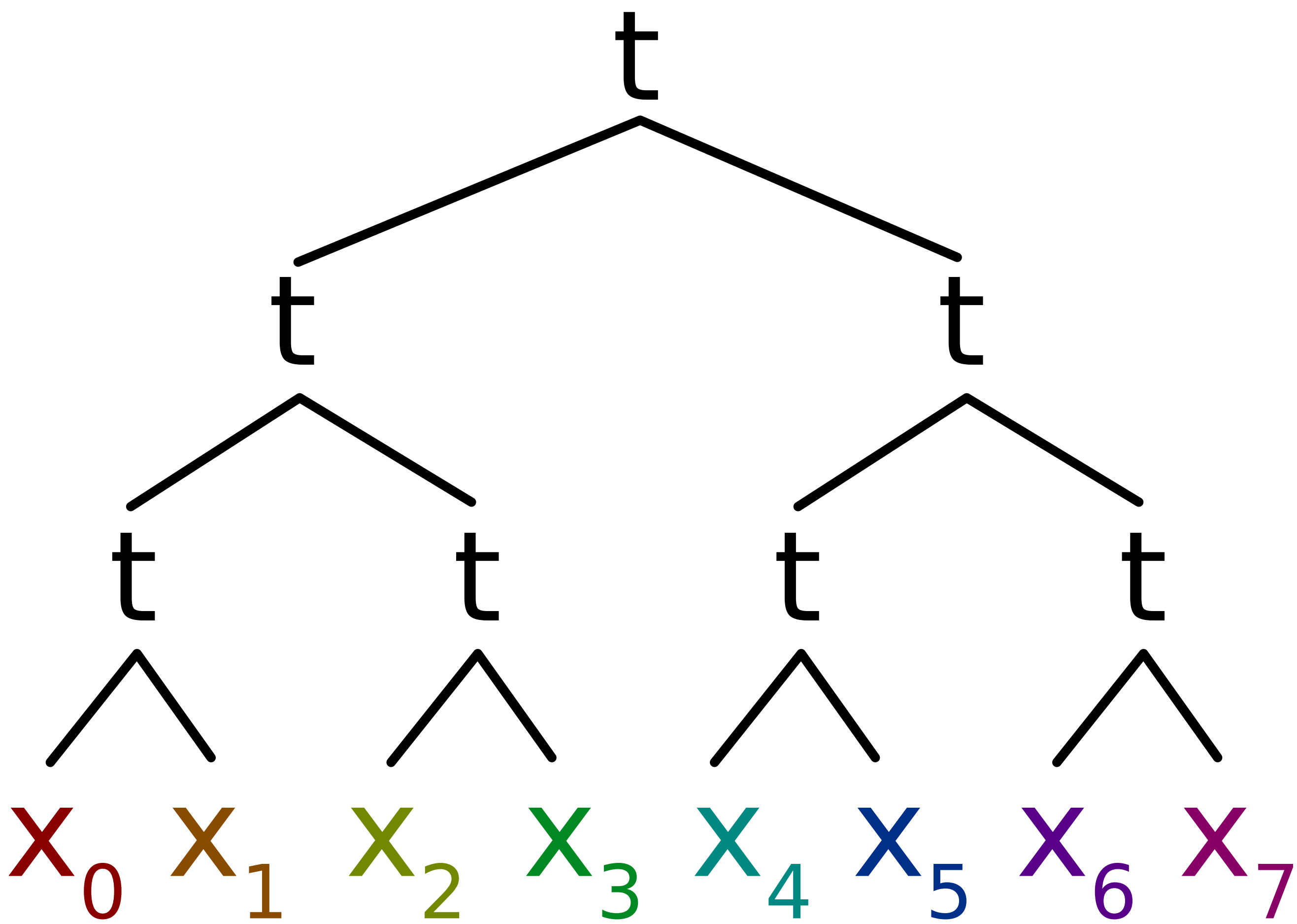


Consider a graph

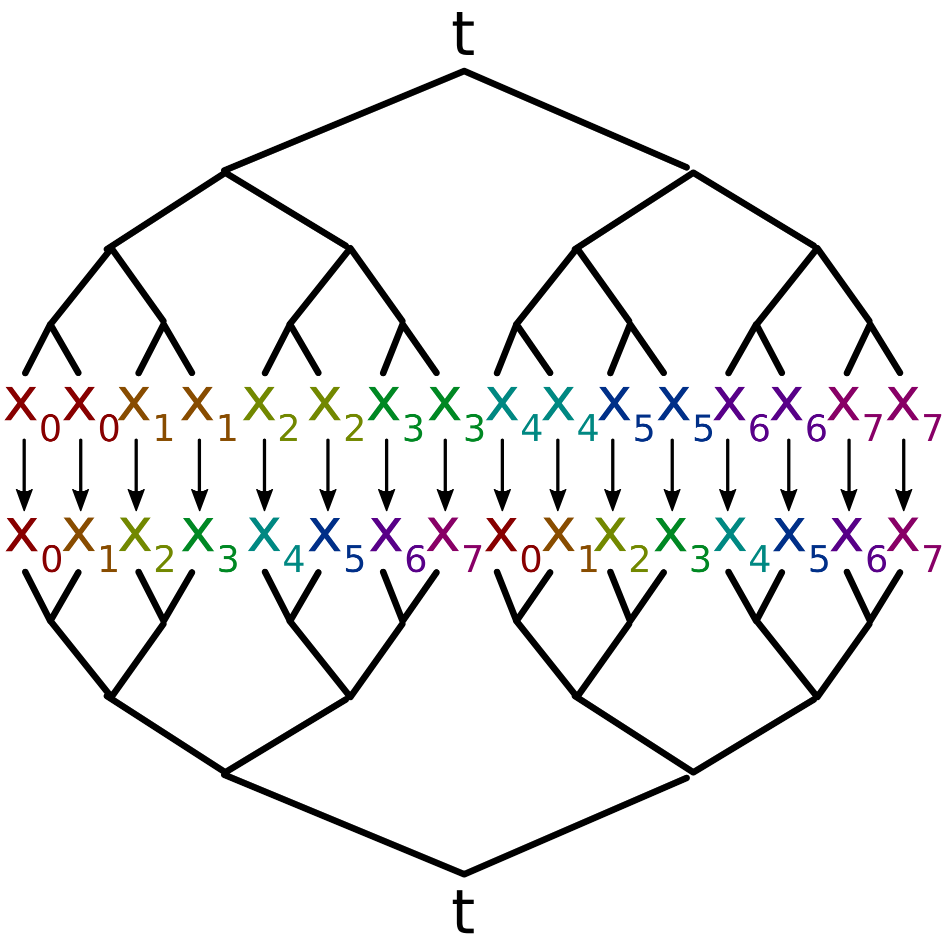
- compatible with an idempotent operation,
- + certain weak technical assumption,

Then, the graph contains a loop.

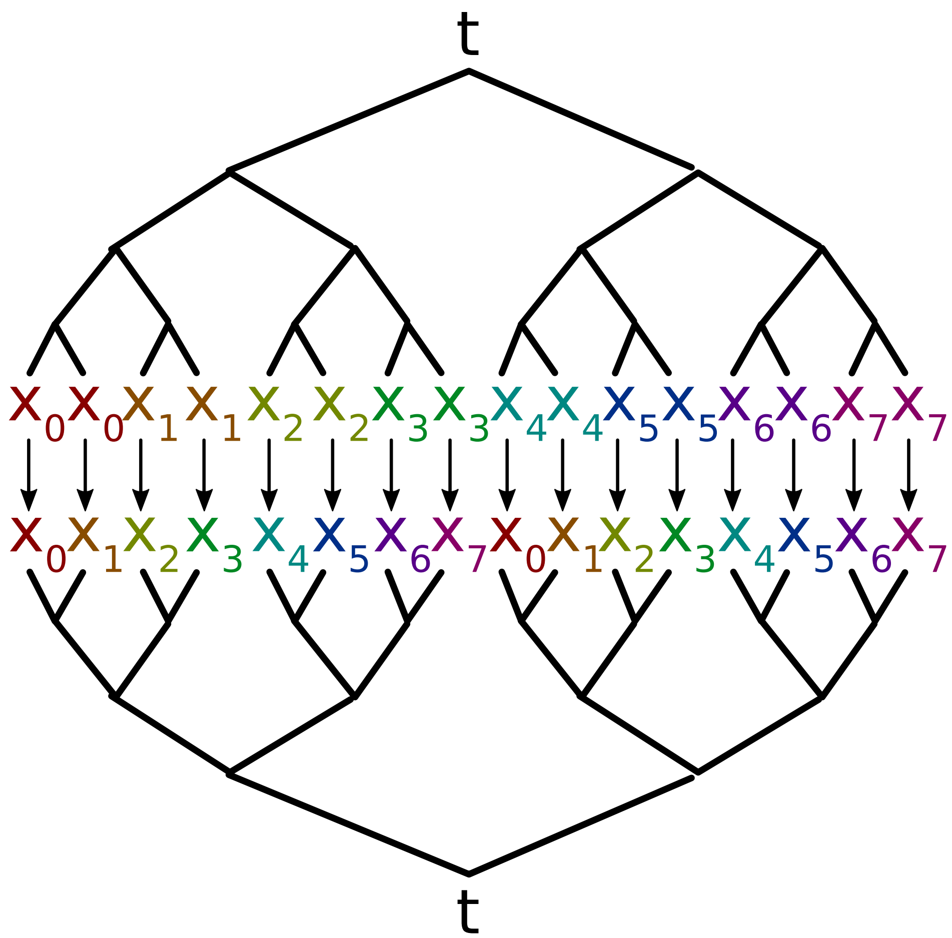
The loop

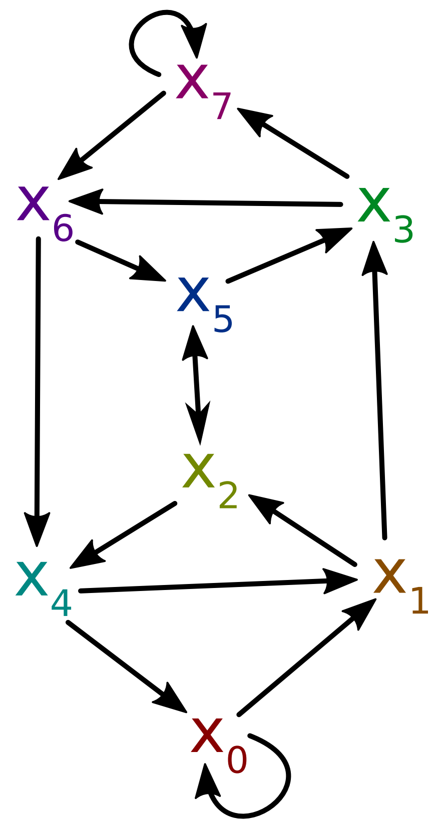


The loop

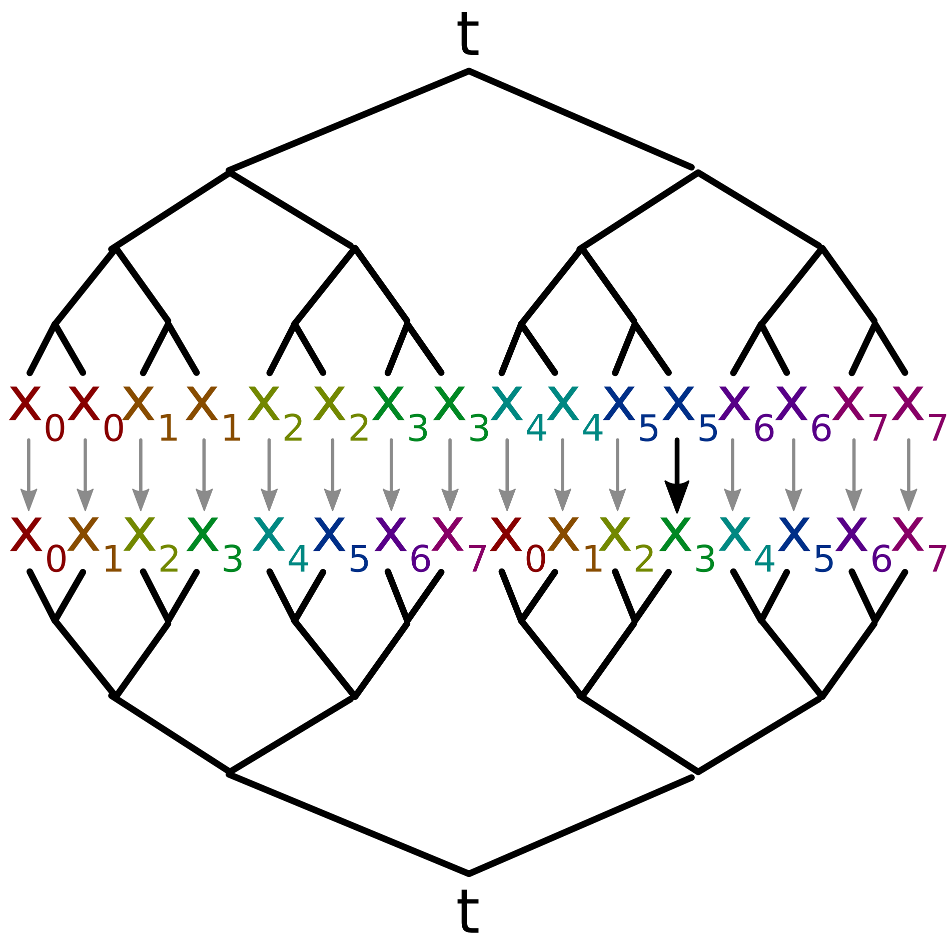

t
x0 x0 x1 x1 x2 x2 x3 x3 x4 x4 x5 x5 x6 x6 x7 x7
x0 x1 x2 x3 x4 x5 x6 x7 x0 x1 x2 x3 x4 x5 x6 x7
t

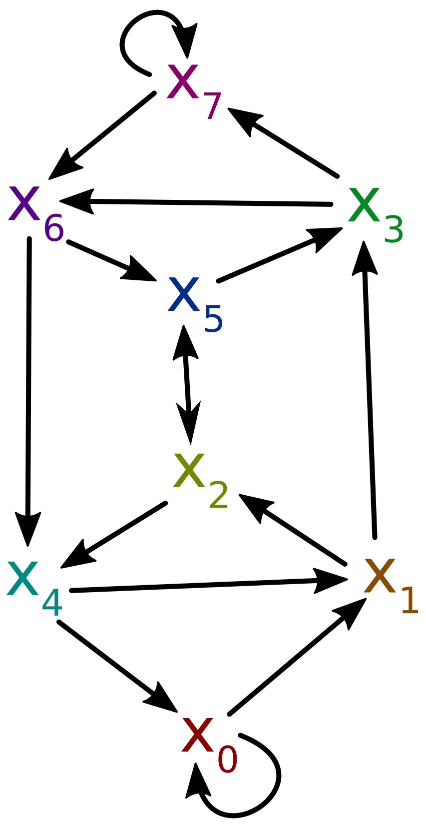
The loop


t
$x_0\ x_0\ x_1\ x_1\ x_2\ x_2\ x_3\ x_3\ x_4\ x_4\ x_5\ x_5\ x_6\ x_6\ x_7\ x_7$
$x_0\ x_1\ x_2\ x_3\ x_4\ x_5\ x_6\ x_7\ x_0\ x_1\ x_2\ x_3\ x_4\ x_5\ x_6\ x_7$
t

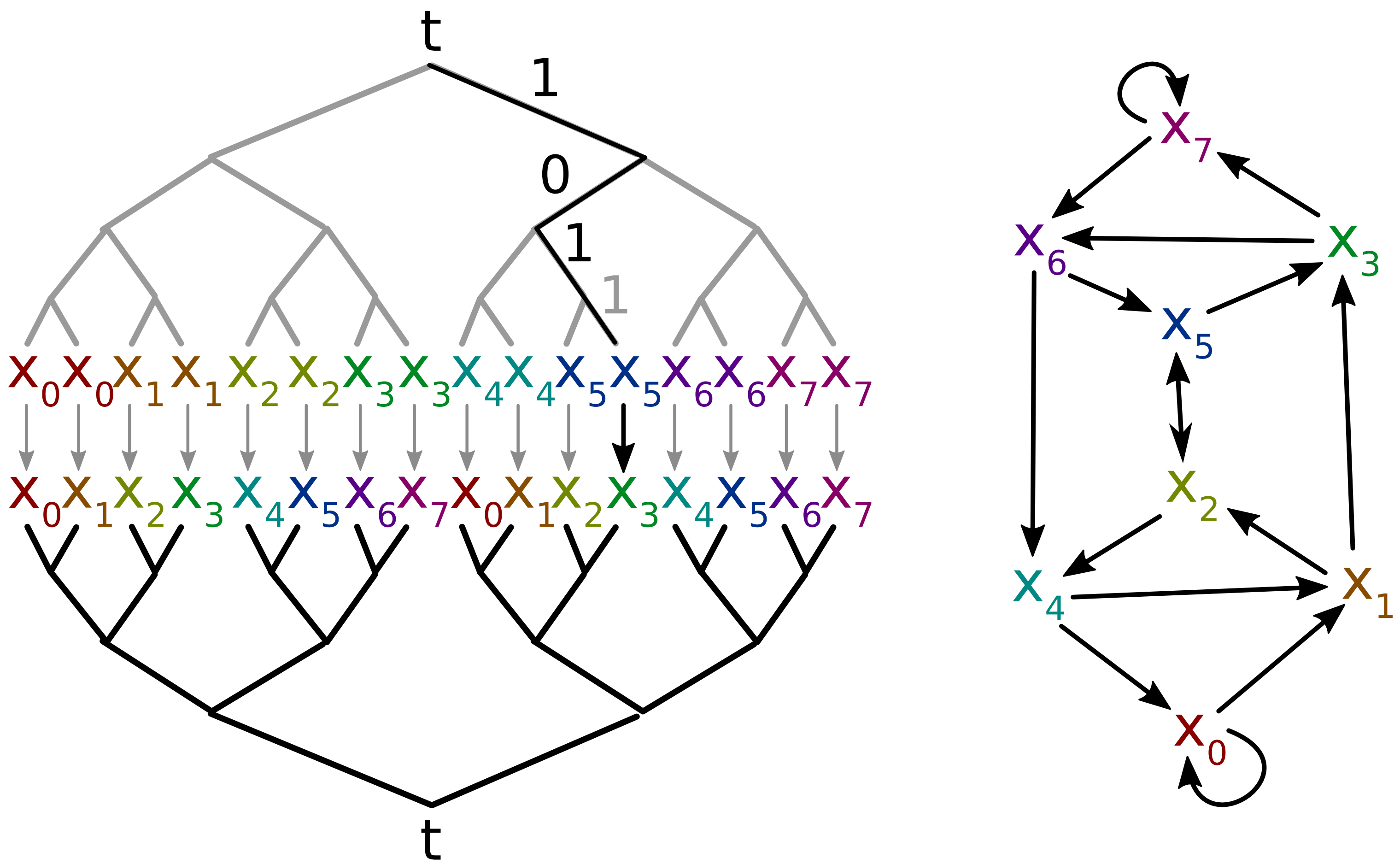

x_7
x_6
x_3
x_5
x_2
x_4
x_1
x_0

The loop

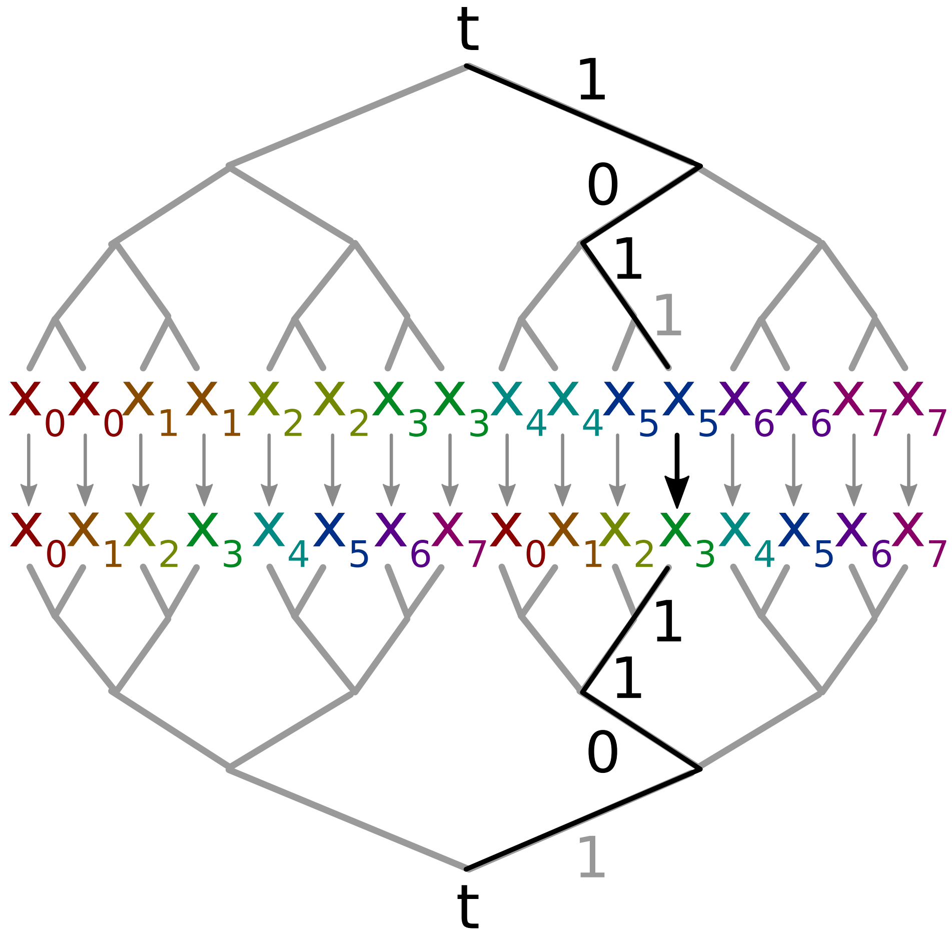

t
x_0 x_0 x_1 x_1 x_2 x_2 x_3 x_3 x_4 x_4 x_5 x_5 x_6 x_6 x_7 x_7
x_0 x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_0 x_1 x_2 x_3 x_4 x_5 x_6 x_7
t


x_7
x_6
x_3
x_5
x_2
x_4
x_1
x_0

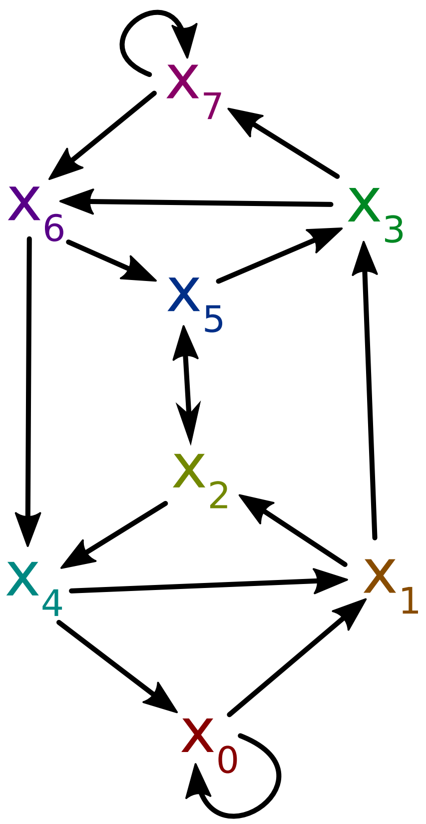
The loop


t
1
0
1
1
x_0 x_0 x_1 x_1 x_2 x_2 x_3 x_3 x_4 x_4 x_5 x_5 x_6 x_6 x_7 x_7
x_0 x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_0 x_1 x_2 x_3 x_4 x_5 x_6 x_7
t
x_7
x_6
x_3
x_5
x_2
x_4
x_1
x_0

The loop



t
1
0
1
1
x_0 x_0 x_1 x_1 x_2 x_2 x_3 x_3 x_4 x_4 x_5 x_5 x_6 x_6 x_7 x_7
x_0 x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_0 x_1 x_2 x_3 x_4 x_5 x_6 x_7
1
1
0
1
t



x_7
x_6
x_3
x_5
x_2
x_4
x_1
x_0

Conclusion

- Graphs of walks / sequences are useful for their “freeness” in a certain class of graphs.
- Perhaps something known, but not for me…
- Generalizations are open:
  - Multiple graphs
  - Hypergraphs