Commutativity in the lattice of topologizing filters of a commutative semiartinain ring

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Outline

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Throughout this talk:

- R denotes an associative ring with identity,
- Mod-R the category of unital right R-modules.
  - If X, Y are nonempty subsets of M, we define $Y^{-1}X \overset{\text{def}}{=} \{r \in R : Yr \subseteq X\}$.

    If I and J are ideals of R, then
  - $IJ^{-1} = \{r \in R \mid rJ \subseteq I\}$ and
  - $J^{-1}I = \{r \in R \mid Jr \subseteq I\}$
    are left (resp. right) residual of I by J.

Topologizing filters on a ring

Definition

A nonempty family $\mathfrak{F}$ of right ideals of a ring R is called a right topologizing filter on R if:

F1. $I \in \mathfrak{F}$ and $I \subseteq J \leq R_R$ implies $J \in \mathfrak{F}$;

F2. $I, J \in \mathfrak{F}$ implies $I \cap J \in \mathfrak{F}$;

F3. $I \in \mathfrak{F}$ and $r \in R$ implies $r^{-1}I \stackrel{\text{def}}{=} \{x \in R \,:\, rx \in I\} \in \mathfrak{F}$.

- $\operatorname{Id} R \stackrel{\text{def}}{=} \{$(two-sided) ideals of $R\}$.
  $\operatorname{Fil} R_R \stackrel{\text{def}}{=}$ set of all right topologizing filters on R.

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$\mathrm{Fil}\, R_R$ admits a binary operation $:$ defined by

- $\mathfrak{F} : \mathfrak{G} \stackrel{\text{def}}{=} \{K \leq R_R \ : \ \exists H \in \mathfrak{F}$ s.t. $K \subseteq H$ & $h^{-1}K \in \mathfrak{G}\, \forall h \in H\}$.
  - We say $[\mathrm{Fil}\, R_R]^{du}$ is left (resp. right) residuated if given $\mathfrak{F}, \mathfrak{G} \in \mathrm{Fil}\, R_R$ there exists a smallest filter $\mathfrak{H}$ in $\mathrm{Fil}\, R_R$ satisfying $\mathfrak{H} : \mathfrak{G} \supseteq \mathfrak{F}$ (resp. $\mathfrak{G} : \mathfrak{H} \supseteq \mathfrak{F}$).
  - We denote the left (resp. right) residual of $\mathfrak{F}$ by $\mathfrak{G}$ $\mathfrak{F}\mathfrak{G}^{-1}$ (resp. $\mathfrak{G}^{-1}\mathfrak{F}$).

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Theorem (J. Golan, 1987)

If R is any ring then $\langle \mathrm{Fil}\, R_R; \subseteq^{du}; :\rangle$ is an integral, left residuated, complete lattice ordered monoid. ∎

Question. What does the structure $\mathrm{Fil}\, R_R$ tell us about the ring R and its category of modules?

$\mathrm{Fil}\, R_R$ is larger than $\mathrm{Id}\, R$ and there is an embedding of lattice ordered monoid $\mathrm{Id}\, R$ into $\mathrm{Fil}\, R_R$.

$$\begin{array}{ccc} \mathrm{Id}\, R & \underset{\text{order reversing}}{\overset{\text{one-to-one}}{\hookrightarrow}} & [\mathrm{Fil}\, R_R]^{du} \\ I & \longmapsto & \eta(I) \overset{\text{def}}{=} \{K \leq R_R \;:\; K \supseteq I\}. \end{array}$$

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Question. When is $\langle \mathrm{Fil}\, R_R, \subseteq^{du}, : \rangle$ both right and left residuated?

Theorem (Beachy and Blair, 1978)

The following statements are equivalent for a ring R:

(a) R is right artinian (satisfies DCC on right ideals);

(b) $\mathrm{Fil}\, R_R = \mathrm{Id}R$, i.e., every $\mathscr{F} \in \mathrm{Fil}\, R_R$ has the form $\mathscr{F} = \eta(I)$, for some ideal $I \in \mathrm{Id}R$.

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Corollary

If R is right artinian, then $\operatorname{Fil} R_R$ is right and left residuated.

Theorem (J. van den Berg, 1991)

If R is a commutative noetherian ring, then $\operatorname{Fil} R_R$ is commutative, to mean the monoid operation ':' on $\operatorname{Fil} R_R$ is commutative.

Theorem (J. van den Berg, 1999)

For any ring R, if : is commutative then $\operatorname{Fil} R_R$ is two-sided residuated.

Two-sided residuation in a commutative ring

Let P be a poset. $X \subseteq P$ is said to be downward (upward) directed if, given any pair x_1, x_2 in X, $\exists y \in X$ s.t $x_1 \geq y$ and $x_2 \geq y$ ($x_1 \leq y$ and $x_2 \leq y$).

Theorem (N.Arega and J. vanden Berg, 2017)

Let R be a commutative ring for which $[\mathrm{Fil}\, R_R]^{du}$ is two-sided residuated. Then $\mathrm{Fil}\, R_R$ is commutative, i.e., $\mathfrak{F} : \mathfrak{G} = \mathfrak{G} : \mathfrak{F}$ $\forall \mathfrak{F}, \mathfrak{G} \in \mathrm{Fil}\, R_R$.

Continue...

Then

$$
\begin{aligned}
\mathfrak{F} : \mathfrak{G} &= \mathfrak{F} : \left[\bigcup_{I \in \mathfrak{G}} \eta(I) \right] \\
&= \bigcup_{I \in \mathfrak{G}} [\mathfrak{F} : \eta(I)] \\
&= \bigcup_{I \in \mathfrak{G}} [\eta(I) : \mathfrak{F}] \\
&= \left[\bigcup_{I \in \mathfrak{G}} \eta(I) \right] : \mathfrak{F} \\
&= \mathfrak{G} : \mathfrak{F}, \text{ as required.}
\end{aligned}
$$

Definition

An ideal I of an arbitrary ring R is said to be a right annihilator ideal [resp. left annihilator ideal] if, for some ideal K of R, $I = K^{-1}0 \overset{\text{def}}{=} \{r \in R : Kr = 0\}$ [resp. $I = 0K^{-1} \overset{\text{def}}{=} \{r \in R : rK = 0\}$]

Definition

A submodule U of a right R-module M is called an annihilator submodule if $U = \{x \in M : xI = 0\}$ for some $I \in \mathrm{Id}\, R$.

For a commutative ring R, we denote:

- AIs $\overset{\text{def}}{=}$ annihilator ideal(s)
- ASs $\overset{\text{def}}{=}$ annihilator submodule(s).
- hps $\overset{\text{def}}{=}$ hereditary pretorsion class.

Theorem (N. Arega and J.van den Berg, 2017)

For a commutative ring R, TFAE:

(a) $\mathrm{Fil}\, R_R$ *is commutative;*

(b) $[\ Fil\ R_R]^{du}$ *is two-sided residuated;*

(c) The ring R/I *satisfies the ACC on AIs for all* $I \lhd R$*;*

(d) The ring R/I *satisfies the DCC on AIs for all* $I \lhd R$*;*

(e) $(R/I)_R$ *satisfies the ACC on ASs for all* $I \lhd R$*;*

(f) $(R/I)_R$ *satisfies the ACC on hps for all* $I \lhd R$*;*

(g) $(R/I)_R$ *satisfies the DCC on ASs for all* $I \lhd R$*;*

(h) $(R/I)_R$ *satisfies the DCC on hps for all* $I \lhd R$*.*

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Proof. Sketch: (a)⇔(b)
(c)⇔(d) For any commutative ring R, the map $K \mapsto K^{-1}0$ constitutes a Galois connection on the set of annihilator ideals of R.
(c)⇔(e) and (d)⇔(g).

(f)⇒(e) and (h)⇒(g) follows from the fact that every annihilator submodule is a hereditary pretorsion submodule.

(e)⇒(f) taking $M = (R/I)_R$, and noting that hereditary pretorsion and annihilator submodules coincide if (e) holds.

(g)⇒(h) since the chain of implications, (g) ⇒(d)⇒(c)⇒ e. As in the proof of (e)⇒(f), the equivalence of (g) and (h) follows. Finally, we prove (b)⇔(h) and (h)⇒(b).

Theorem (N. Arega and J.van den Berg, preprint 2018)

The conditions FilR is commutative and R is noetherian, coincide for large classes of rings R such as Prüfer domains and commutative von Neumann regular rings.

Question: Are there non-noetherian commutative rings R for which FilR is commutative that is strictly more general than the noetherian case?

The Answer is YES.

- The class of commutative semiartinian rings are such examples.

Theorem (N.Arega and J.van den Berg, 2017)

Recall that for a commutative ring R TFAE:

(a) FilR is commutative;

(b) the ring R/I satisfies the ACC on annihilator ideals for all proper ideals of I of R, i.e., for each ideal I of R, the family $\{K^{-1}I : K \in Id\ R\}$ *satifies the ACC;*

(c) the ring R/I satisfies the DCC on annihilator ideals for all proper ideals I of R, i.e., for each ideal I of R, the family $\{K^{-1}I : K \in Id\ R\}$ *satisfies the DCC.*

Commutative Semiartinian rings

Let R be an arbitrary ring (not Necessarily commutative) and M a right R-module.

- The (ascending) Loewy series of M is the ordinal-indexed family $\{soc^{\alpha}(M)\}_{\alpha}$ of submodules of M defined recursively as follows:

$$soc^{0}(M) \stackrel{\text{def}}{=} 0 \text{ and } soc^{1}(M) \stackrel{\text{def}}{=} soc(M),$$

$$soc^{\alpha+1}(M)/soc^{\alpha}(M) \stackrel{\text{def}}{=} soc(M/soc^{\alpha}(M)), \text{ for ordinals } \alpha \geq 0,$$

$$soc^{\beta}(M) \stackrel{\text{def}}{=} \bigcup_{\alpha<\beta} soc^{\alpha}(M), \text{ for a limit ordinals.}$$

Continue...

- We say that M is semiartinain, or a Loewy module if $soc^{\alpha}(M) = M$ for some ordinal α.
- The smallest such α is called the Loewy length of M.
- For each M $\in$ Mod-R and ordinal α, the module soc$^{\alpha+1}$(M)/soc$^{\alpha}$(M) is called the α^{th} Loewy factor of M.
- If R is commutative, then the number of summands in a direct sum decomposition of soc $^{\alpha+1}$(M)/soc $^{\alpha}$(M) into simples is an invariant of M called the α^{th} Loewy invariant of M, and denoted d$^{\alpha}$(M).

Continue...

- If S is any nonempty class of simple right R-modules and M $\in$ Mod-R, we define soc$_S$(M) to be the sum of all simple submodules of M that are isomorphic to some member of S.
- The (ascending) S-Loewy series $\{soc_S^\alpha(M)\}_\alpha$ of M is defined in a manner entirely analogous to $\{soc^\alpha(M)\}_\alpha$, as are the notions S-Loewy module, S-Loewy length, α^{th} S-Loewy factor and α^{th} S-Loewy invariant, d_S^α(M).

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Theorem (B. Stenstr$\ddot{o}m$, 1975)

M is semiartinian if and only if every nonzero factor module of M has a nonzero socle.

- We call a ring R right semiartinian if the module R_R is semiartinian.
- It is known that if K is a right ideal of arbitrary ring R, then R will be right artinian if K_R and $(R/K)_R$ are both artinian

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Theorem (T.Shores, 1974)

TFAE for a commutative semiartinian ring R:

(a) R is artinian;

(b) R has an ideal K such that $soc^2(K_R)$ *and* $(R/K)_R$ *are artinian;*

(c) The Loewy invariants $d^0(R_R)$ *and* $d^1(R_R)$ *are finite, i.e.,* $soc\,(R_R)$ *and* $soc^{\,2}(R_R)/soc\,(R_R) = soc(R_R/soc(R_R))$ *are both finitely generated right R-modules.*

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Theorem (N.Arega and J.van den Berg, preprint 2018)

Let R be a commutative ring for which FilR is commutative and let S be a simple right R-module. Then TFAE:

(a) $soc_S(R_R)$ *is f.g;*

(b) $soc_S^n(R_R)/soc_S^{n-1}(R_R)$ *is f.g for all* $n \in \mathbb{N}$*;*

(c) $soc_S^n(R_R)$ *f.g for all* $n \in \mathbb{N}$*.*

Continue...

Theorem (T.shore, 1974)

Let R be a commutative ring and M a semiartinian right R-module that contains only finitely many nonisomorphic simple submodules, that is to say, soc (M) = soc_S(M) for some finite nonempty family $\mathbb{S}$ of nonisomorphic simple right R-modules. Then for each ordinal α:

(a)$soc^{\alpha}(M)=\bigoplus_{S\in\mathbb{S}} soc_S^{\alpha}(M)$.

(b) $d^{\alpha} = \sum_{S\in\mathbb{S}} d_{\mathbb{S}}^{\alpha}$.

Continue...

Theorem (N.Arega and J.van den Berg, preprint 2018)

TFAE for a commutative semiartinian ring R:

(a) R is artinian;

(b) $\operatorname{soc}(R_R)$ *and* $\operatorname{soc}^2(R_R)$ *are both f.g;*

(c) $\operatorname{Fil} R$ *is commutative and* $\operatorname{soc}(R_R)$ *is f.g.*

Sketch of proof: $(b \Rightarrow a)$; $(a \Rightarrow c)$ and $(c \Rightarrow b)$.

A class of examples of commutative semiartinian rings

- Let $R = \langle F, V, U, \mu \rangle$ where F is a field, V and U F-spaces, and $\mu : V \times V \to U$ a symmetric F-bilinear map with $\mu(v, v') = v \cdot v'$.

- We equip the set $F \times V \times U$ with an F-algebra structure by taking addition to be natural and defining multiplication by:

$$(a_1, v_1, u_1) \cdot (a_2, v_2, u_2) \stackrel{\text{def}}{=} (a_1 a_2, a_1 v_2 + a_2 v_1, a_1 u_2 + v_1 \cdot v_2 + a_2 u_1).$$

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Alternatively:

$$\begin{pmatrix} a_1 & v_1 & u_1 \\ 0 & a_1 & v_1 \\ 0 & 0 & a_1 \end{pmatrix} \begin{pmatrix} a_2 & v_2 & u_2 \\ 0 & a_2 & v_2 \\ 0 & 0 & a_2 \end{pmatrix} =$$

$$\begin{pmatrix} a_1 a_2 & a_1 v_2 + a_2 v_1 & a_1 u_2 + v_1 . v_2 + a_2 u_1 \\ 0 & a_1 a_2 & a_1 v_2 + a_2 v_1 \\ 0 & 0 & a_1 a_2 \end{pmatrix}.$$

Continue...

- It is easily checked that R is an F-algebra.
- The symmetry of μ guarantees that R is commutative.

Theorem (N.Arega and J.van den Berg, preprint 2018)

Let $R = \langle F, V, U, \mu \rangle$ with V and U nonzero F-spaces. Then

(a) R is a commutative local F-algebra with unique maximal proper ideal J(R) = $\langle 0, V, U \rangle = \{(0, v, u) : v \in V, u \in U\}$.

(b) $\operatorname{soc}(R_R) = \langle 0, 0, U \rangle = \{(0, 0, u) : u \in U\}$ and

(c) $\operatorname{soc}^2(R_R) = \langle 0, V, U \rangle = J(R)$ and $\operatorname{soc}^3(R_R) = R_R$.

- Hence R_R is semiartinian with Loewy length 3.

- If X and Y are F-subspaces of V and U, respectively, we define F-subspace $X^{-1}Y$ of V as follows:

$$X^{-1}Y \stackrel{\text{def}}{=} \{v \in V : X \cdot v \subseteq Y\} \\ = \{v \in V : x \cdot v \in Y \ \forall x \in X\}$$

Lemma

The following assertions are equivalent for a proper nonzero ideal A of $R = \langle F, V, U, \mu \rangle$:

(a) $A = K^{-1}I = \{r \in R : Kr \subseteq I\}$ for some $I, K \in \mathrm{Id}\, R$;

(b) $A = \langle 0, X^{-1}Y, U \rangle$ for some F-subspaces $X$and $Y$of $V$and U respectively.

Theorem*. Let $R = \langle F, V, U, \mu \rangle$. TFAE:

(a) $\mathrm{Fil}\, R$ is commutative;

(b) For each ideal I of R, the family $\{K^{-1}I : K \in \mathrm{Id}\, R\}$ satisfies the ACC;

(c) For each F-subspace $Y$of U, the family $\{X^{-1}Y : X$ is an F-subspace of $V\}$ of F-subspaces of V, satisfies the ACC.

Corollary

Let $R = \langle F, V, U, \mu \rangle$. If $\dim_F V$ is finite, then $\mathrm{Fil}\, R$ is commutative.

There is no finiteness requirement on $\mathrm{soc}\,(R_R) = \langle 0, 0, U \rangle$ in order that $\mathrm{Fil}\, R$ be commutative.

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Remark: The consequence is a plentiful supply of commutative non-artinian semiartinian rings R for which $\mathrm{Fil}\, R_R$ is commutative.

The sufficient condition of the previous corollary, is not a necessary condition for $\mathrm{Fil}\, R_R$ to be commutative.

Example: We show that choices for V, U and μ can be made such that $dim_F V$ and $dim_F U$ are both infinite, but the ring $R = \langle F, V, U, \mu \rangle$ is such that $\mathrm{Fil}\, R_R$ is commutative.

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Let T be the commutative F-algebra defined by

$$T \stackrel{\text{def}}{=} \left\{ \begin{pmatrix} a & w \\ 0 & a \end{pmatrix} : a \in F, w \in W \right\}$$

where W is any infinite dimensional F-space.

Take $V = U = T$ with symmetric F-bilinear map $\mu : T \times T \to T$ the usual multiplication map on T.

Observe that $dim_F T$ is infinite because $dim_F W$ is infinite.

Let $R = \langle F, V, U, \mu \rangle = \langle F, T, T, \mu \rangle$. We use Theorem* ((c)$\Rightarrow$ (a)), to show that $\mathrm{Fil}\, R_R$ is commutative.

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Let Y be an F-subspace of T. It is easily seen that Y may be written as

$$Y = \left\{ \begin{pmatrix} ax & sx \\ 0 & ax \end{pmatrix} : x \in F \right\} + \begin{pmatrix} 0 & Z \\ 0 & 0 \end{pmatrix}$$

for some fixed $\begin{pmatrix} a & s \\ 0 & a \end{pmatrix} \in T$ and F-subspace Z of W.

Continue....

A routine calculation shows that if X is any F-subspace of T, then $X^{-1}Y$ has one of the following four forms:

$$X^{-1}Y = \begin{cases} 0; \\ T; \\ \begin{pmatrix} 0 & W \\ 0 & 0 \end{pmatrix} = J(T); \\ \left\{ \begin{pmatrix} x & tx \\ 0 & x \end{pmatrix} : x \in F \right\} + \quad \begin{pmatrix} 0 & Z \\ 0 & 0 \end{pmatrix} \\ \textit{for some } t \in W. \end{cases}$$

Continue....

It is clear from the above that the family $\{X^{-1}Y : X$ is an F-subspace of $T\}$ admits no strictly ascending chain of F-subspace, so by Theorem*((c)$\Leftrightarrow$(a)), $\mathrm{Fil}\, R_R$ is commutative.

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