Congruence FD-maximal algebras

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Problem. For a given class $\mathcal{K}$ of algebras describe Con $\mathcal{K}$ =all lattices isomorphic to Con A for some $A \in \mathcal{K}$.

In this lecture we concentrate on

Problem. Let $\mathcal{K}$ be a finitely generated CD variety. Describe finite members of Con $\mathcal{K}$.

In this case, the membership problem for Con $\mathcal{K}$ is decidable.

But...it is not a description.

Membership testing

In the sequel: $\mathcal{V} \ldots$ a finitely generated CD variety;
SI$(\mathcal{V}) \ldots$ the family of subdirectly irreducible members;
$\mathrm{M}(L) \ldots$ completely $\wedge$-irreducible elements of a lattice L.

Let $L = \operatorname{Con} A$ with A finite. For every $\alpha \in \mathrm{M}(L)$ we have $A/\alpha \in \mathrm{SI}(\mathcal{V})$. For $\alpha \leq \beta$ we have a natural projection homomorphism $f_{\alpha,\beta} : A/\alpha \to A/\beta$. Let B be the limit of the commutative diagram

$$(A/\alpha, f_{\alpha,\beta} \mid \alpha \leq \beta \text{ in } \mathrm{M}(L)).$$

Then $\operatorname{Con} B \cong L$.

Membership testing

To check if a finite distributive lattice L belongs to $\operatorname{Con}\mathcal{V}$ we have to
1. find a valuation $v: \mathrm{M}(L) \to \mathrm{SI}(\mathcal{V})$ such that $\operatorname{Con} v(t)$ is isomorphic to $\uparrow_L t$ (for every $t \in \mathrm{M}(L)$) and the isomorphism

$$\varphi: \operatorname{Con} v(t) \to \uparrow_L t$$

satisfies

$$v(t)/\alpha \cong v(\varphi(\alpha))$$

for every $\alpha \in \mathrm{M}(\operatorname{Con} v(t))$;
2. choose surjective homomorphisms $f_{s,t}: v(s) \to v(t)$ and check if the limit algebra has the reqired congruence lattice.

Congruence FD-maximality

Lemma

Let $L \in \mathrm{Con}\,\mathcal{V}$. Then for every $x \in \mathrm{M}(L)$, the lattice $\uparrow_L x$ is isomorphic to $\mathrm{Con}\,T$ for some $T \in \mathrm{SI}(\mathcal{V})$.

We say that $\mathcal{V}$ is congruence FD-maximal, if for every *finite distributive* lattice L the following two conditions are equivalent:

(i) $L \in \mathrm{Con}\,\mathcal{V}$;

(ii) for every $x \in \mathrm{M}(L)$, the lattice $\uparrow_L x$ is isomorphic to $\mathrm{Con}\,T$ for some $T \in \mathrm{SI}(\mathcal{V})$.

In other words, $\mathcal{V}$ is congruence FD-maximal iff the class of all finite members of $\mathrm{Con}\,\mathcal{V}$ is as large as possible by the necessary condition. For these varieties we have a nice description of finite members of $\mathrm{Con}\,\mathcal{V}$.

Congruence FD-maximal algebras

Let A be a finite algebra generating a CD variety. We say that A is *congruence FD-maximal*, if for every finite distributive lattice L the following two conditions are equivalent:

(i) $L \in \operatorname{Con} \mathrm{P}_s \mathrm{H}(A)$;

(ii) for every $x \in \mathrm{M}(L)$, the lattice $\uparrow_L x$ is isomorphic to $\operatorname{Con} T$ for some $T \in \mathrm{H}(A)$.

In other words, A is congruence FD-maximal iff the class of all finite members of $\operatorname{Con} \mathrm{P}_s \mathrm{H}(A)$ is as large as possible by the necessary condition.

The understanding of congruence FD-maximal algebras is essential for understanding congruence FD-maximal varieties.

Strategy

The ordered sets P satisfying

(ii) for every $x \in P$, the lattice $\uparrow x$ is isomorphic to $\uparrow_{\mathrm{M}(\mathrm{Con}\, A)} \alpha$ for some $\alpha \in \mathrm{M}(\mathrm{Con}\, A)$

will be called *eligible*.

Now, A is congruence FD-maximal iff for every eligible P

1. there exists a suitable valuation $v : \; P \to \mathrm{SI}(\mathrm{H}(A))$;
2. it is possible to choose suitable homomorphisms $f_{s,t} : \; v(s) \to v(t)$.

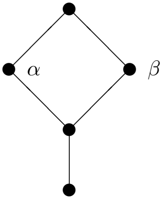
Easy case

Theorem

Every finite algebra generating a CD variety, whose congruence lattice is a chain, is congruence FD-maximal.

The simplest of the difficult cases

Let A be a finite algebra generating CD variety such that $\operatorname{Con} A$ is as follows:



The simplest of the difficult cases

Theorem

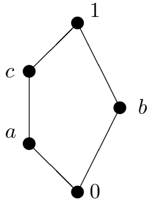
The following are equivalent:

(i) *every eligible ordered set has a suitable valuation;*

(ii) $A/\alpha \cong A/\beta$.

Negative example 1

The lattice N_5 with the distinguished element (nullary operation) b is not congruence FD-maximal, because the quotients N_5/α and N_5/β are not isomorphic.



N_5

Compatible families

Let E be a subset of B^2, let X be a set and let $\mathcal{F} = \{f_1, \dots, f_k\}$ be a set of functions $X \to B$. We say that $\mathcal{F}$ is E-*compatible* if

$$\{(f_{i_1}(x), f_{i_2}(x)) \mid x \in X\} = E$$

whenever $i_1 < i_2$.

Characterization theorem

Let A be as above.

Theorem

A is congruence FD-maximal iff

(i) *all quotients $A/\alpha \cong A/\beta \cong B$;*

(ii) *there are homomorphisms $h_1, h_2 : \ A \to B$ with*

$$\{\mathrm{Ker}(h_1), \mathrm{Ker}(h_2)\} = \{\alpha, \beta\}$$

such that the relation $E = \{(h_1(x), h_2(x)) \mid x \in A\} \subseteq B^2$ admits arbitrarily large E-compatible sets of functions.

Large compatible families

Lemma

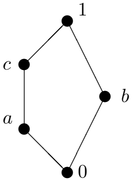
Suppose that $E \subseteq B^2$ contains a non-diagonal pair. Then the following condition are equivalent.

(i) *There exist arbitrarily large finite E-compatible sets of functions.*

(ii) *For every $(a_2, a_4) \in E$ there are $a_1, a_3, a_5 \in B$ such that $(a_{i_1}, a_{i_2}) \in E$ whenever $i_1 \leq i_2$ and every every even k appears at most once among i_1, i_2.*

Positive example

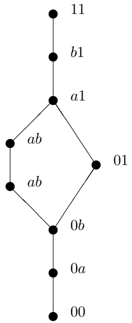
For $A = N_5$ we have $B = \{0, 1\}$, $E = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$ so almost every family of functions is E-compatible and A is congruence FD-maximal.



N_5

Negative example2

Consider the following lattice A with two additional unary operations.



$f(00) = 00, f(0a) = 0b, f(0b) = f(01) = 01$

$f(ab) = f(a1) = b1, f(b1) = f(11) = 11$

$g(11) = 11, g(b1) = a1, g(a1) = g(01) = 01$

$g(ab) = g(0b) = 0a, g(0a) = g(00) = 00$

Negative example 2

This algebra satisfies the condition of having a suitable valuation for every eligible set P, as A/α and A/β are both isomorphic to the 4-element chain $B = \{0 < a < b < 1\}$.
However, the algebra fails the second test. There is only one choice of homomorphisms $A \to B$ and the corresponding relation

$$E = \{(0,0), (0,a), (0,b), (a,b), (0,1), (a,1), (b,1), (1,1)\}$$

(the labels on the elements of A), and the pair (a, b) violates the condition. Thus, A is not congruence FD-maximal.

General case - stage 1

The previous examples suggest the conjecture that the following conditions for an algebra A are equivalent:

(i) Every eligible ordered set has a suitable valuation;

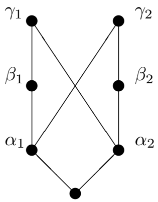
(ii) For every $\alpha, \beta \in \mathrm{M}(\mathrm{Con}\, A)$,

$$\uparrow\alpha \cong \uparrow\beta \iff A/\alpha \cong A/\beta.$$

The implication (ii)$\Longrightarrow$ (ii) is easy, but the converse fails. (See the example on the next slide.)

General case - stage 1

Let A be an algebra with $\mathrm{M}(\mathrm{Con}\, A)$ depicted below. The condition that every eligible ordered set has a suitable valuation implies that $A/\gamma_1 \cong A/\gamma_2$, $A/\beta_1 \cong A/\beta_2$, but not necessarily $A/\alpha_1 \cong A/\alpha_2$.



Problem. Characterize finite ordered sets Q (with a least element) with a valuation $v:\ Q \to \mathbb{N}$ (satisfying $v(x) = v(y) \implies {\uparrow}x \cong {\uparrow}y$) having the following property:

(*) Every eligible ordered set P has a valuation $\mu:\ P \to \mathbb{N}$ such that for every $p \in P$ there exists $q \in Q$ and a valuation-preserving isomorphism ${\uparrow}p \to {\uparrow}q$.

(Recall that P is eligible if for every $p \in P$ there exists $q \in Q$ and an isomorphism ${\uparrow}p \to {\uparrow}q$.)

General case - stage 2

This stage depends on the result of stage 1 and, in general, requires large compatible families of functions on heterogeneous relational systems.