Special Elements in Pseudocomplemented Lattice Effect Algebras

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Lattice effect algebras - special class of basic algebras

Definition (Chajda, Halaš, and Kühr AU 2009)

A basic algebra is an algebra $\mathcal{A} = (A, \oplus, \neg, 0, 1)$ of type $(2, 1, 0, 0)$ that satisfies the equations

$$x \oplus 0 = x,$$
$$\neg\neg x = x,$$
$$\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x,$$
$$\neg\big(\neg(\neg(x \oplus y) \oplus y) \oplus z\big) \oplus (x \oplus z) = 1.$$

BA = bounded lattices with antitone involutions

- The relation $\leq = \{(x,y) \in A^2 \mid \neg x \oplus y = 1\}$ is a partial order on A such that 0 and 1 are the least and the greatest element of A.
- The poset $(A, \leq)$ is a bounded lattice $(A, \vee, \wedge, 0, 1)$ where

$$x \vee y = \neg(\neg x \oplus y) \oplus y \text{ and } x \wedge y = \neg(\neg x \vee \neg y)$$

- For each $a \in A$, the map $\gamma_a \colon x \mapsto \neg x \oplus a$ is an antitone involution on $[a, 1]$.
- For each $a \in A$, the map $\delta_a \colon x \mapsto \neg(x \oplus \neg a)$ is an antitone involution on $[0, a]$.
- $(A, \oplus, \neg, 0, 1)$ is determined by $(A, \vee, \wedge, 0, 1, (\gamma_a)_{a \in A})$ as follows:

$$\neg x = \gamma_0(x) \text{ and } x \oplus y = \gamma_y(\neg x \vee y).$$

- $(A, \oplus, \neg, 0, 1)$ is determined by $(A, \vee, \wedge, 0, 1, (\delta_a)_{a \in A})$ as follows:

$$\neg x = \delta_1(x) \text{ and } x \oplus y = \neg\delta_{\neg y}(x \wedge \neg y).$$

What is behind the axioms?

- Every basic algebra satisfies the following conditions:

  1) $0 \oplus x = x$,
  2) $\neg x \oplus x = 1$,
  3) $x \oplus 1 = 1 \oplus x = 1$,
  4) $x \leq y \;\Rightarrow\; \neg y \leq \neg x$,
  5) $x \leq y \;\Rightarrow\; x \oplus z \leq y \oplus z$,
  6) $\neg x \leq y \oplus z$ iff $\neg y \leq x \oplus z$,
  7) $(x \wedge y) \oplus z = (x \oplus z) \wedge (y \oplus z)$,
  8) $y \leq x \oplus y$,
  9) $\neg(\neg(x \oplus y) \oplus y) \oplus y = x \oplus y$,

- The variety of basic algebras is arithmetical and congruence regular.

What is behind the axioms?

In addition to the negation $\neg$ and the addition $\oplus$, it is useful to define multiplication $\odot$ and two substractions ($\ominus$, $\oslash$) by

$$x \odot y = \neg(\neg x \oplus \neg y),\ x \ominus y = \neg(y \oplus \neg x),\ x \oslash y = \neg(\neg x \oplus y).$$

- Every basic algebra satisfies the following conditions:

1) $0 \odot x = 0 = x \odot 0$,
2) $\neg x \odot x = 0$,
3) $x \odot 1 = x = 1 \odot x$,
4) $x \odot y \leq y$,
5) $x \leq y \;\Rightarrow\; x \odot z \leq y \odot z, \quad x \oslash z \leq y \oslash z, \quad z \ominus x \leq z \ominus y$,
6) $x \leq y$ iff $x \odot \neg y = 0$ iff $x \ominus y = 0$ iff $x \oslash y = 0$,
7) $(x \vee y) \odot z = (x \odot z) \vee (y \odot z)$,
$(x \vee y) \oslash z = (x \oslash z) \vee (y \oslash z)$,
8) $x \ominus (y \wedge z) = (x \ominus y) \vee (x \ominus z)$,
9) $\neg x \odot y \leq z$ iff $\neg z \odot y \leq x$.

What is behind the axioms?

The following (dual) identities 1) - 4) are equivalent to one another and they are equivalent to lattice distributivity:

1) $(x \vee y) \oplus z = (x \oplus z) \vee (y \oplus z)$,
2) $(x \wedge y) \odot z = (x \odot z) \wedge (y \odot z)$,
3) $(x \wedge y) \oslash z = (x \oslash z) \wedge (y \oslash z)$,
4) $x \ominus (y \vee z) = (x \ominus y) \wedge (x \ominus z)$.

The identities 5) - 8) are equivalent to one another and they are stronger than lattice distributivity:

5) $x \oplus (y \wedge z) = (x \oplus y) \wedge (x \oplus z)$, (M)
6) $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$,
7) $x \oslash (y \wedge z) = (x \oslash y) \vee (x \oslash z)$,
8) $(x \vee y) \ominus z = (x \ominus z) \vee (y \ominus z)$.

Interval algebras

For each $a \in A$, the map $\delta_a \colon x \mapsto \neg(x \oplus \neg a) = a \ominus x$ is an antitone involution on $[0, a]$.
Corresponding **interval basic algebra** ... $([0, e], \oplus^e, \neg^e, 0, e)$, where

$$x \oplus^e y = e \ominus ((e \ominus y) \ominus x) \text{ and } \neg^e x = e \ominus x \text{ for } x, y \in [0, e].$$

For each $a \in A$, the map $\gamma_a \colon x \mapsto \neg x \oplus a$ is an antitone involution on $[a, 1]$.
Corresponding **interval basic algebra** ... $([e, 1], \oplus_e, \neg_e, e, 1)$, where

$$x \oplus_e y = \neg(\neg x \oplus e) \oplus y \text{ and } \neg_e x = \neg x \oplus e \text{ for } x, y \in [e, 1].$$

Lemma

For every $e \in A$, the interval algebras $[0, e]$ and $[\neg e, 1]$ are isomorphic.

MV-algebras as special class of basic algebras

MV-algebras were introduced by C.C.Chang (in 1950's) as an algebraic counterpart of Lukasiewicz multiple valued logic (in 1920's) .
By an **MV-algebra** is meant an algebra $\mathcal{A} = (A, \oplus, \neg, 0)$ of type $(2, 1, 0)$ satisfying the axioms:
(MV1) $a \oplus (b \oplus c) = (a \oplus b) \oplus c$
(MV2) $a \oplus b = b \oplus a$
(MV3) $a \oplus 0 = a$
(MV4) $\neg\neg a = a$
(MV5) $a \oplus \neg 0 = \neg 0$
(MV6) $\neg(\neg a \oplus b) \oplus b = \neg(\neg b \oplus a) \oplus a.$

Usually we denote $\neg 0$ by 1 and we read (MV5) as $a \oplus 1 = 1$.

- Antitone involution on $[a, 1]$: $\gamma_a : x \mapsto \neg x \oplus a$
  MV-algebras = associative basic algebras

The logic of quantum mechanic is axiomatized by means of orthomodular lattices (G. Birkhoff, J. von Neumann in 1940's)

Orthomodular lattice is a bounded complemented lattice $\mathcal{A} = (A, \vee, \wedge, ', 0, 1)$ satisfying the orthomodular law:

$$x \leq y \;\Rightarrow\; x \vee (x' \wedge y) = y$$

- Antitone involution on $[a, 1]$: $\gamma_a \colon x \mapsto x' \vee a$
  OML's: $x \oplus y = (x \wedge y') \vee y$

Orthomodular lattices = basic algebras satisfying

$$x \leq y \;\Rightarrow\; y \oplus x = y$$

Effect basic algebras as special class of basic algebras

An **effect algebra (Foulis, Bennett, 1994)** is a partial structure $(A; +, 0, 1)$ satisfying:

- $x + y = y + x$ if one side is defined;
- $(x + y) + z = x + (y + z)$ if one side is defined;
- for every x there is a unique x' such that $x' + x = 1$;
- if $x + 1$ is defined, then $x = 0$.
  The underlying order: $x \leq y$ iff $y = x + z$ for some z.
  Lattice-ordered effect algebras **(lattice effect algebras)** are equivalent to **effect basic algebras**, i.e., basic algebras satisfying

$$x \oplus y \leq \neg z \quad \Rightarrow \quad (x \oplus y) \oplus z = x \oplus (z \oplus y).$$

Antitone involution on $[a, 1]$: $\gamma_a \colon x \mapsto x' + a$

Effect basic algebras as special class of basic algebras

The class of lattice effect algebras includes both MV-algebras and orthomodular lattices:

Relative to the variety of effect basic algebras:
MV- algebras ... $x \oplus y = y \oplus x$
orthomodular lattices ... $x \oplus x = x$

The smallest variety containing both the variety of MV-algebras and the variety of orthomodular lattices was recently axiomatized by Kühr et. al. (2015).

Distributive, standard and neutral elements in lattice

Definition

Let $(L, \vee, \wedge)$ be a lattice and let a be an element of L.

- The element a is called distributive if $a \vee (x \wedge y) = (a \vee x) \wedge (a \vee y)$, for all $x, y \in L$.
- The element a is called standard if $x \wedge (a \vee y) = (x \wedge a) \vee (x \wedge y)$, for all $x, y \in L$.
- The element a is called neutral if $(a \wedge x) \vee (x \wedge y) \vee (y \wedge a) = (a \vee x) \wedge (x \vee y) \wedge (y \vee a)$, for all $x, y \in L$.

Dually distributive and dually standard elements are defined dually.

$$Neutr(L) \subseteqq Stand(L) \subseteqq Distr(L)$$
$$Neutr(L) \subseteqq Stand^{\delta}(L) \subseteqq Distr^{\delta}(L)$$

$Neutr(L)$ forms a distributive sublattice in L

Definition

Let $(A, \oplus, \neg, 0, 1)$ be a basic algebra and let a be an element of A.

- The element $a \in A$ is said to be sharp if $a \wedge \neg a = 0$; equivalently, $a \vee \neg a = 1$, (equivalently, $a \oplus a = a$). The sharp elements form neither a subalgebra of $(A, \oplus, \neg, 0, 1)$ nor sublattice of $(A, \vee, \wedge)$ in general.
- The element $a \in A$ is said to be central if the mapping $x \mapsto (x \wedge a, x \wedge \neg a)$ is an isomorphism of A onto $[0, a] \times [0, \neg a]$, or equivalently, $(x, y) \mapsto x \vee y$ is an isomorphism of $[0, a] \times [0, \neg a]$ onto A. The central elements form a subalgebra of A (Boolean algebra).
- An element $a \in A$ is said to be boolean if $a \oplus x = a \vee x$ for all $x \in A$. The boolean elements form a subalgebra of A (Boolean algebra).

$\mathcal{C}(A) \subseteq \mathcal{B}(A) \subseteq \mathcal{S}(A)$ $\qquad$ $\mathcal{C}(A) = \mathcal{S}(A)$ for any MV-algebra A

Characterization of distributive elements

Lemma

Let $(A, \oplus, \neg, 0, 1)$ *be a basic algebra. For any* $a \in A$*, the following are equivalent:*
(i) $a \in Distr(A)$ *in the lattice* $(A, \vee, \wedge)$*;*
(ii) $\neg a \in Distr^{\delta}(A)$ *in the lattice* $(A, \vee, \wedge)$*;*
(iii) $(x \vee y) \oplus a = (x \oplus a) \vee (y \oplus a)$ *for all* $x, y \in A$*;*
(iv) $f_a : x \mapsto x \vee a$ *is a lattice homomorphism from* A *onto* $[a, 1]$*;*
(v) $\alpha_a = \{(x, y) \in A^2 \mid a \vee x = a \vee y\}$ *(the kernel of* f_a*) is a lattice congruence of* A*.*
(vi) $f_{\neg a} : x \mapsto x \wedge a$ *is a lattice homomorphism from* A *onto* $[0, \neg a]$*;*
(vii) $\beta_{\neg a} = \{(x, y) \in A^2 \mid \neg a \wedge x = \neg a \wedge y\}$ *(the kernel of* $f_{\neg a}$*) is a lattice congruence of* A*.*

Lemma

Let $(A, \oplus, \neg, 0, 1)$ be a basic algebra. For any $a \in A$, the following are equivalent:
(i) $a \in Stand(A)$ in the lattice $(A, \vee, \wedge)$;
(ii) $\neg a \in Stand^{\delta}(A)$ in the lattice $(A, \vee, \wedge)$;
(iii) $(a \vee x) \oplus y = (x \oplus y) \vee (a \oplus y)$ for all $x, y \in A$.
(iv) $\widetilde{\boldsymbol{\alpha}_a} = \{(x, y) \in A^2 \colon x \vee y = (x \wedge y) \vee a_1$ for some $a_1 \leq a\}$ is a congruence of $(A, \vee, \wedge)$, in which case $a \in Distr(A)$ and $\widetilde{\boldsymbol{\alpha}_a} = \boldsymbol{\alpha}_a$;

Lemma

Let $(A, \oplus, \neg, 0, 1)$ *be a basic algebra. For any* $a \in A$*, the following are equivalent:*

(i) $a \in Neutr(A)$ *in the lattice* $(A, \vee, \wedge)$*;*

(ii) $a \in Distr(A) \cap Distr^{\delta}(A)$*, and for all* $x, y \in A$*, whenever* $x \vee a = y \vee a$ *and* $x \wedge a = y \wedge a$*, then* $x = y$*.*

(iii) $\alpha_a = \{(x, y) \in A^2 \mid a \vee x = a \vee y\}$ *and* $\beta_a = \{(x, y) \in A^2 \mid a \wedge x = a \wedge y\}$ *are lattice congruences such that* $\boldsymbol{\alpha}_a \cap \boldsymbol{\beta}_a = \Delta_A$*.*

$Neutr(A) \cap \mathcal{S}(A)$ is a Boolean algebra

Lemma

Let $(A, \oplus, \neg, 0, 1)$ be a basic algebra. Then $Neutr(A) \cap \mathcal{S}(A)$ is a subalgebra of $(A, \vee, \wedge, \neg, 0, 1)$, and $Neutr(A) \cap \mathcal{S}(A)$ is a Boolean algebra in its own right.

Remark. In general, $Neutr(A) \cap \mathcal{S}(A)$ is not a subalgebra of the basic algebra $(A, \oplus, \neg, 0, 1)$.

$\mathcal{B}(A) \subsetneqq Neutr(A) \cap \mathcal{S}(A)$

Theorem

Let $(A, \oplus, \neg, 0, 1)$ be a basic algebra. For any $a \in A$, the following are equivalent:
(i) $a \in \mathcal{B}(A)$;
(ii) $a \vee (x \oplus y) = (a \vee x) \oplus y$ *for all* $x, y \in A$;
(ii) $a \in \mathcal{S}(A)$ *and* $a \oplus (x \oplus y) = (a \oplus x) \oplus y$ *for all* $x, y \in A$;
(iii) the equivalence $\alpha_a = \{(x, y) \in A^2 \mid a \vee x = a \vee y\}$ *is a weak congruence of the algebra* $(A, \oplus, \neg, 0, 1)$.

Weak congruence of a basic algebra is an equivalence relation $\boldsymbol{\theta}$ with the property that

$$(x, y) \in \boldsymbol{\theta} \text{ implies } (\neg x, \neg y) \in \boldsymbol{\theta} \text{ and } (x \oplus z, y \oplus z) \in \boldsymbol{\theta}.$$

Any weak congruence is a lattice congruence (compatible with $\neg$), but the converse fails to be true in general. Thus weak congruences of $(A, \oplus, \neg, 0, 1)$ are a special case of congruences of $(A, \vee, \wedge, \neg, 0, 1)$.

When $\boldsymbol{\alpha}_a$ is a congruence of $(A, \oplus, \neg, 0, 1)$?

Lemma

Let $(A, \oplus, \neg, 0, 1)$ be a basic algebra. The following are equivalent for any $a \in A$:

(i) $x \oplus (y \vee a) = (x \oplus y) \vee a$ *for all* $x, y \in A$;

(ii) $a \in \mathcal{S}(A)$ *and* $x \oplus (y \oplus a) = (x \oplus y) \oplus a$ *for all* $x, y \in A$;

(iii) $f_a : x \mapsto x \vee a$ *is a homomorphism of* $(A, \oplus, \neg, 0, 1)$ *onto the interval basic algebra* $[a, 1]$;

(iv) $\boldsymbol{\alpha}_a$ *is a congruence of* $(A, \oplus, \neg, 0, 1)$.

If $a \in A$ satisfies these conditions, then $a \in \mathcal{B}(A)$.

Theorem

Let A be a basic algebra. For any $a \in A$, the following are equivalent:

(i) $a \in \mathcal{C}(A)$;

(ii) $x \oplus (y \vee z) = (x \oplus y) \vee z$ *for all* $x, y \in A$ *and* $z \in \{a, \neg a\}$;

(iii) $a \in \mathcal{S}(A)$ *and* $x \oplus (y \oplus z) = (x \oplus y) \oplus z$ *for all* $x, y \in A$ *and* $z \in \{a, \neg a\}$;

(iv) $\boldsymbol{\alpha}_a$ *and* $\boldsymbol{\alpha}_{\neg a}$ *are congruences of* $(A, \oplus, \neg, 0, 1)$.

Definition

Two elements x, y in a lattice effect algebra are said to be compatible (in symbols $x \leftrightarrow y$) if there exist $x_1, y_1, z \in A$ such that $x = x_1 + z$, $y = y_1 + z$ and $x_1 + y_1 + z$ is defined. In the language of basic algebras we have

$$x \leftrightarrow y \quad \text{iff} \quad x \oplus y = y \oplus x \quad \text{iff} \quad x \leq x \oplus y;$$

.

$K(A) = \{a \in A : a \leftrightarrow x \text{ for all } x \in A\}$ ... compatibility center (subalgebra of A, MV-algebra)

Riečanová, Z. (1997) ... $\mathcal{C}(A) = \mathcal{S}(A) \cap K(A)$

$a \in \mathcal{C}(A)$ iff $(x \wedge a) \vee (x \wedge \neg a) = x$ for all $x \in A$

Lemma

Let A be a lattice effect algebra. If $a \in \mathcal{B}(A)$, then $(x \wedge a) \vee (x \wedge \neg a) = x$ for all $x \in A$.

Corollary

For any lattice effect algebra A, $\mathcal{C}(A) = \mathcal{B}(A)$.

Theorem

If $(A, \oplus, \neg, 0, 1)$ is an effect basic algebra, then

$$\begin{aligned}\mathcal{C}(A) = \mathcal{B}(A) = \mathcal{S}(A) \cap \mathrm{Distr}(A) &= \mathcal{S}(A) \cap \mathrm{Distr}^{\partial}(A) \\ = \mathcal{S}(A) \cap \mathrm{Stand}(A) &= \mathcal{S}(A) \cap \mathrm{Stand}^{\partial}(A) \\ = \mathcal{S}(A) \cap \mathrm{Neutr}(A). &\end{aligned}$$

Definition

By a basic algebra with pseudocomplementation or a pseudocomplemented basic algebra we mean an algebra $\mathcal{A} = (A, \oplus, \neg, ^*, 0, 1)$ of type $(2, 1, 1, 0, 0)$ where $(A, \oplus, \neg, 0, 1)$ is a basic algebra and * is pseudocomplementation on its underlying lattice, i.e., for every $x \in A$, x^* is the pseudocomplement of x (i.e. $y \leq x^*$ iff $x \wedge y = 0$) .

$(A, \vee, \wedge, ^*, 0, 1)$ ... p-**algebra**

In fact, for every $x \in A$,

$$x^+ = \neg(\neg x)^*$$

is the **dual pseudocomplement** of x because $y \geq \neg(\neg x)^*$ iff $\neg y \leq (\neg x)^*$ iff $\neg x \wedge \neg y = 0$ iff $x \vee y = 1$. Thus $(A, \vee, \wedge, ^*, ^+, 0, 1)$ is a **double** p-**algebra**. Obviously,

$$x^* = \neg(\neg x)^+.$$

Basic algebras with pseudocomplementation

$A^* = \{x^* : x \in A\}$ ... boolean algebra, $x \sqcup y = (x \vee y)^{**}$, the meet in A^* agrees with $x \wedge y$ in A

$A^+ = \{x^+ : x \in A\}$ ... boolean algebra, join in A^+ agrees with $x \vee y$ in A and the meet in A^+ is given by $x \sqcap y = (x \wedge y)^{++}$

Theorem

The class of pseudocomplemented basic algebras is a variety which can be axiomatized by the axioms of basic algebras together with the identities

$$0^* = 1, \quad 1^* = 0 \quad \textit{and} \quad x \wedge (x \wedge y)^* = x \wedge y^*.$$

Basic algebras with pseudocomplementation

Lemma

Let $(A, \oplus, \neg, {}^*, 0, 1)$ *be a basic algebra with pseudocomplementation.*

(i) *If* $a \in \mathcal{S}(A)$, *then* $a^+ \leq \neg a \leq a^*$.

(ii) *If* $a \in \mathcal{S}(A) \cap Distr(A)$, *then* $a = (\neg a)^*$. *Dually, if* $a \in \mathcal{S}(A) \cap Distr^\delta(A)$, *then* $a = (\neg a)^+$. *Hence* $\mathcal{S}(A) \cap Distr(A) \subseteq A^*$ *and* $\mathcal{S}(A) \cap Distr^\delta(A) \subseteq A^+$.

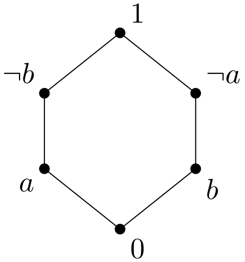
(iii) *If* $a \in \mathcal{S}(A) \cap Distr(A) \cap Distr^\delta(A)$, *then* $\neg a = a^* = a^+$.

Corollary

Let $\mathcal{A}$ *be a pseudocomplemented basic algebra. Then* $\mathcal{C}(A) \subseteq \mathcal{B}(A) \subseteq A^* \cap A^+$.

Basic algebras with pseudocomplementation

Example.
Let $(A, \oplus, \neg, 0, 1)$ be the basic algebra with the following underlying lattice (the so-called "benzene"- is neither distributive nor lattice effect algebra):



The linearly ordered intervals bear unique antitone involutions, thus $\oplus$ is determined by the lattice and $\neg$. The basic algebra is obviously pseudocomplemented and $\mathcal{S}(A) = A$, but $A^* = \{0, \neg a, \neg b, 1\}$ and $A^+ = \{0, a, b, 1\}$. So it can happen that $\mathcal{S}(A) \not\subseteq A^*$ and $\mathcal{S}(A) \not\subseteq A^+$. Moreover, $(\neg b)^* = \neg a \neq \neg\neg b = b$ and $a^+ = b \neq \neg a$.

Basic algebras with pseudocomplementation

Lemma

Let $(A, \oplus, \neg, {}^*, 0, 1)$ *be a basic algebra with pseudocomplementation. Then the underlying lattice is modular if and only if it is distributive, in which case* $\mathcal{S}(A) \subseteq A^* \cap A^+$ *and* $\neg a = a^* = a^+$ *for every* $a \in \mathcal{S}(A)$*.*

Riečanová, Z. (2009) ... For any lattice effect algebra A, $\mathcal{S}(A) \subseteq A^* \cap A^+$ and $\neg a = a^* = a^+$ for every $a \in \mathcal{S}(A)$.

Lemma

Let $\mathcal{A}$ be a pseudocomplemented basic algebra such that $A^* \subseteq \mathcal{S}(A)$ or, equivalently, $A^+ \subseteq \mathcal{S}(A)$. Then $\mathcal{A}$ satisfies the Stone identity

$$x^{**} \vee x^* = 1,$$

as well as the dual Stone identity

$$x^{++} \wedge x^+ = 0.$$

Pseudocomplemented basic algebras satisfying (M)

Lemma

Let $(A, \oplus, \neg, {}^*, 0, 1)$ *be a basic algebra with pseudocomplementation satisfying the identity (M). Then*

$$\mathcal{C}(A) = \mathcal{B}(A) = \mathcal{S}(A) = A^* = A^+.$$

Consequently, the double p-algebra $(A, \vee, \wedge, {}^*, {}^+, 0, 1)$ *is a double Stone algebra.*

$$x \oplus (y \wedge z) = (x \oplus y) \wedge (x \oplus z), \qquad \text{(M)}$$

The condition $C(A) = A^*$ is equivalent to the satisfaction of the identities

$$x \oplus (y \vee z^*) = (x \oplus y) \vee z^* \quad \text{and} \quad x \oplus (y \vee \neg z^*) = (x \oplus y) \vee \neg z^*. \quad (1)$$

Lemma

Let $(A, \oplus, \neg, ^*, 0, 1)$ be an arbitrary basic algebra with pseudocomplementation. If $a \in \mathcal{C}(A)$, then $\boldsymbol{\alpha}_a$ as well as $\boldsymbol{\beta}_a = \boldsymbol{\alpha}_{\neg a}$ is a factor congruence of $(A, \oplus, \neg, ^*, 0, 1)$.

Theorem

Let $(A, \oplus, \neg, ^*, 0, 1)$ *be a basic algebra with pseudocomplementation such that* $\mathcal{C}(A) = A^*$. *The following statements are equivalent:*

(i) $(A, \oplus, \neg, ^*, 0, 1)$ *is a subdirectly irreducible algebra;*

(ii) *the underlying lattice is a chain;*

(iii) $(A, \oplus, \neg, ^*, 0, 1)$ *is a simple algebra.*

Corollary

Every finite basic algebra which satisfies the identity (M) is an MV-algebra.

Corollary

Let $(A, \oplus, \neg, {}^*, 0, 1)$ *be a lattice effect algebra with pseudocomplementation. Then* $\mathcal{C}(A) = A^*$ *iff* $(A, \oplus, \neg, 0, 1)$ *is an MV-algebra.*

THE END. THANK YOU!

Corollary

The variety of pseudocomplemented basic algebras satisfying the identities (1) is a discriminator variety.

Discriminator term ... $t(x, y, z) = (x \vee d(x, y)^*) \wedge (z \vee d(x, y)^{**})$
where $d(x, y) = (x \oslash y) \vee (y \oslash x)$

$t(a, a, c) = (a \vee 1) \wedge (c \vee 0) = c$

$t(a, b, c) = (a \vee 0) \wedge (c \vee 1) = a$ for all $a, b, c \in A$ with $a \neq b$.

$(A, \vee, \wedge, \neg, 0, 1)$ as a subdirect product

Lemma

Let $(A, \oplus, \neg, 0, 1)$ be a basic algebra. Let B is a subalgebra of $(A, \vee, \wedge, \neg, 0, 1)$ which is a Boolean algebra. For every $x \in A \setminus \{0\}$ there exists a maximal ideal M of the Boolean algebra B such that $x \notin (M]$. Consequently, $\bigcap\{(M] \mid M$ is a maximal ideal of $B\} = \{0\}$.

For any ideal I of B, the relation $\alpha_I = \bigcup\{\alpha_a \mid a \in I\}$ is also a congruence of $(A, \vee, \wedge, \neg, 0, 1)$.

For any ideal I of $\mathcal{B}(A)$, we have $(x, y) \in \alpha_I$ iff $x \oslash y, y \oslash x \in (I]$.

Corollary

Let $(A, \oplus, \neg, 0, 1)$ be a basic algebra. Then $\bigcap\{\alpha_M \mid M$ is a maximal ideal of $\mathcal{B}(A)\} = \Delta_A$ and hence the algebra $(A, \vee, \wedge, \neg, 0, 1)$ is a subdirect product of the quotient algebras $(A/\alpha_M, \vee, \wedge, \neg, [0]_{\alpha_M}, [1]_{\alpha_M})$ where M is a maximal ideal of the