# Triples and quadruples in bounded Hilbert algebras and bounded relatively pseudocomplemented posets

**Jan Paseka**
Department of Mathematics and Statistics
Masaryk University
Brno, Czech Republic
paseka@math.muni.cz
Coauthors: Ivan Chajda, Helmut Länger

Supported by the bilateral project 15-34697L New Perspectives on Residuated Posets financed by Austrian Science Fund (FWF) and the Czech Science Foundation (GACR)

**The 56th Summer School on Algebra and Ordered Sets**
**Špindlerův Mlýn, September 3, 2018, Czech Republic**

## Outline

## Outline

## Introduction - triples and quadruples

In this lecture we introduce quadruples consisting of a Boolean algebra $\mathbf{B}$, a Hilbert algebra $\mathbf{D}$, a certain compatibility relation $\mathbf{C}$ between $\mathbf{B}$ and $\mathbf{D}$ and a join-preserving mapping $\varphi$ from $\mathbf{B}$ to a $\vee$-semilattice $\mathrm{Nuc}\,\mathbf{D}$ of nuclei on $\mathbf{D}$ (closure endomorphisms on $\mathbf{D}$) that preserves both the least and the greatest element.

Our goal is to characterize every bounded Hilbert algebra and every relatively pseudocomplemented poset by means of a quadruple.

Notice that characterizing triples were introduced by C. C. Chen and G.Grätzer and intensively studied by T. Katriňák and his collaborators. For modular pseudocomplemented posets similar triples were introduced and investigated by the I. Chajda and R. Halaš.

Also, our results on characterizing Hilbert quadruples should be compared with the results by J. Cīrulis on quasi-decompositions.

## Introduction - triples and quadruples

In this lecture we introduce quadruples consisting of a Boolean algebra $\mathbf{B}$, a Hilbert algebra $\mathbf{D}$, a certain compatibility relation $\mathbf{C}$ between $\mathbf{B}$ and $\mathbf{D}$ and a join-preserving mapping $\varphi$ from $\mathbf{B}$ to a $\vee$-semilattice $\mathrm{Nuc}\,\mathbf{D}$ of nuclei on $\mathbf{D}$ (closure endomorphisms on $\mathbf{D}$) that preserves both the least and the greatest element.

Our goal is to characterize every bounded Hilbert algebra and every relatively pseudocomplemented poset by means of a quadruple.

Notice that characterizing triples were introduced by C. C. Chen and G.Grätzer and intensively studied by T. Katriňák and his collaborators. For modular pseudocomplemented posets similar triples were introduced and investigated by the I. Chajda and R. Halaš.

Also, our results on characterizing Hilbert quadruples should be compared with the results by J. Cīrulis on quasi-decompositions.

## Introduction - triples and quadruples

In this lecture we introduce quadruples consisting of a Boolean algebra $\mathbf{B}$, a Hilbert algebra $\mathbf{D}$, a certain compatibility relation $\mathbf{C}$ between $\mathbf{B}$ and $\mathbf{D}$ and a join-preserving mapping $\varphi$ from $\mathbf{B}$ to a $\vee$-semilattice $\mathrm{Nuc}\,\mathbf{D}$ of nuclei on $\mathbf{D}$ (closure endomorphisms on $\mathbf{D}$) that preserves both the least and the greatest element.

Our goal is to characterize every bounded Hilbert algebra and every relatively pseudocomplemented poset by means of a quadruple.

Notice that characterizing triples were introduced by C. C. Chen and G.Grätzer and intensively studied by T. Katriňák and his collaborators. For modular pseudocomplemented posets similar triples were introduced and investigated by the I. Chajda and R. Halaš.

Also, our results on characterizing Hilbert quadruples should be compared with the results by J. Cīrulis on quasi-decompositions.

## Introduction - triples and quadruples

In this lecture we introduce quadruples consisting of a Boolean algebra $\mathbf{B}$, a Hilbert algebra $\mathbf{D}$, a certain compatibility relation $\mathbf{C}$ between $\mathbf{B}$ and $\mathbf{D}$ and a join-preserving mapping $\varphi$ from $\mathbf{B}$ to a $\vee$-semilattice $\mathrm{Nuc}\,\mathbf{D}$ of nuclei on $\mathbf{D}$ (closure endomorphisms on $\mathbf{D}$) that preserves both the least and the greatest element.

Our goal is to characterize every bounded Hilbert algebra and every relatively pseudocomplemented poset by means of a quadruple.

Notice that characterizing triples were introduced by C. C. Chen and G.Grätzer and intensively studied by T. Katriňák and his collaborators. For modular pseudocomplemented posets similar triples were introduced and investigated by the I. Chajda and R. Halaš.

Also, our results on characterizing Hilbert quadruples should be compared with the results by J. Cīrulis on quasi-decompositions.

## Introduction - triples and quadruples

The usual motivation for various triple constructions has been the desire for reduction of algebras under consideration to simpler, or better known algebraic structures.

In our construction we follow the idea of W. Nemitz for constructing bounded implicative semilattices having a given Boolean algebra for closed algebra, and a given implicative semilattice for dense filter.

As Nemitz we work with an action of the Boolean algebra on dense elements but instead of factorizing the cartesian product of the Boolean algebra with the dense filter we directly specify possible pairs of elements of our Boolean algebra and dense elements.

The reason is that our represented structure does not have meets and congruences do not always induce relatively pseudocomplemented posets, i.e., we need to be more precise in other conditions.

## Introduction - triples and quadruples

The usual motivation for various triple constructions has been the desire for reduction of algebras under consideration to simpler, or better known algebraic structures.

In our construction we follow the idea of W. Nemitz for constructing bounded implicative semilattices having a given Boolean algebra for closed algebra, and a given implicative semilattice for dense filter.

As Nemitz we work with an action of the Boolean algebra on dense elements but instead of factorizing the cartesian product of the Boolean algebra with the dense filter we directly specify possible pairs of elements of our Boolean algebra and dense elements.

The reason is that our represented structure does not have meets and congruences do not always induce relatively pseudocomplemented posets, i.e., we need to be more precise in other conditions.

## Introduction - triples and quadruples

The usual motivation for various triple constructions has been the desire for reduction of algebras under consideration to simpler, or better known algebraic structures.

In our construction we follow the idea of W. Nemitz for constructing bounded implicative semilattices having a given Boolean algebra for closed algebra, and a given implicative semilattice for dense filter.

As Nemitz we work with an action of the Boolean algebra on dense elements but instead of factorizing the cartesian product of the Boolean algebra with the dense filter we directly specify possible pairs of elements of our Boolean algebra and dense elements.

The reason is that our represented structure does not have meets and congruences do not always induce relatively pseudocomplemented posets, i.e., we need to be more precise in other conditions.

## Introduction - triples and quadruples

The usual motivation for various triple constructions has been the desire for reduction of algebras under consideration to simpler, or better known algebraic structures.

In our construction we follow the idea of W. Nemitz for constructing bounded implicative semilattices having a given Boolean algebra for closed algebra, and a given implicative semilattice for dense filter.

As Nemitz we work with an action of the Boolean algebra on dense elements but instead of factorizing the cartesian product of the Boolean algebra with the dense filter we directly specify possible pairs of elements of our Boolean algebra and dense elements.

The reason is that our represented structure does not have meets and congruences do not always induce relatively pseudocomplemented posets, i.e., we need to be more precise in other conditions.

# Outline

## Hilbert algebras

- ***Hilbert algebra*** (Diego): $\mathbf{A} = (A, *, 1)$ may be treated as a poset $(A, \leq)$ with the greatest element 1 equipped with a binary operation $*$ such that

$$x * y = 1 \text{ if and only if } x \leq y,$$
$$x \leq y * x, \quad x * (y * z) \leq (x * y) * (x * z) \quad .$$

  $\mathbf{A}$ will be called *bounded* if it has a smallest element $0$; in this case, for $x \in A$ we put $x^* := x * 0$.

- ***Homomorphism of Hilbert algebras***: a mapping between Hilbert algebras preserving operations $*$ and $1$ (and hence preserving also order).

  The class $\mathscr{H}$ ($\mathscr{H}_0$) of all (bounded) Hilbert algebras considered as algebras of the form $(A, *, 1)$ ($(A, *, 1, 0)$ respectively) is equationally definable.

- Since any Boolean algebra is a bounded Hilbert algebra we will further use the denotion $(B, *, 1, 0)$ for Boolean algebras as well.

## Hilbert algebras

- ***Hilbert algebra*** (Diego): $\mathbf{A} = (A, *, 1)$ may be treated as a poset $(A, \leq)$ with the greatest element 1 equipped with a binary operation $*$ such that

$$x * y = 1 \text{ if and only if } x \leq y,$$
$$x \leq y * x, \quad x * (y * z) \leq (x * y) * (x * z) \ .$$

  $\mathbf{A}$ will be called *bounded* if it has a smallest element $0$; in this case, for $x \in A$ we put $x^* := x * 0$.

- ***Homomorphism of Hilbert algebras***: a mapping between Hilbert algebras preserving operations $*$ and $1$ (and hence preserving also order).

  The class $\mathscr{H}$ ($\mathscr{H}_0$) of all (bounded) Hilbert algebras considered as algebras of the form $(A, *, 1)$ ($(A, *, 1, 0)$ respectively) is equationally definable.

- Since any Boolean algebra is a bounded Hilbert algebra we will further use the denotion $(B, *, 1, 0)$ for Boolean algebras as well.

## Hilbert algebras

- ***Hilbert algebra*** (Diego): $\mathbf{A} = (A, *, 1)$ may be treated as a poset $(A, \leq)$ with the greatest element 1 equipped with a binary operation $*$ such that

$$x * y = 1 \text{ if and only if } x \leq y,$$
$$x \leq y * x, \quad x * (y * z) \leq (x * y) * (x * z) \ .$$

  $\mathbf{A}$ will be called *bounded* if it has a smallest element $0$; in this case, for $x \in A$ we put $x^* := x * 0$.

- ***Homomorphism of Hilbert algebras***: a mapping between Hilbert algebras preserving operations $*$ and $1$ (and hence preserving also order).

  The class $\mathscr{H}$ ($\mathscr{H}_0$) of all (bounded) Hilbert algebras considered as algebras of the form $(A, *, 1)$ ($(A, *, 1, 0)$ respectively) is equationally definable.

- Since any Boolean algebra is a bounded Hilbert algebra we will further use the denotion $(B, *, 1, 0)$ for Boolean algebras as well.

## Hilbert algebras

- ***Hilbert algebra*** (Diego): $\mathbf{A} = (A, *, 1)$ may be treated as a poset $(A, \leq)$ with the greatest element 1 equipped with a binary operation $*$ such that

$$x * y = 1 \text{ if and only if } x \leq y,$$
$$x \leq y * x, \quad x * (y * z) \leq (x * y) * (x * z) \ .$$

  $\mathbf{A}$ will be called *bounded* if it has a smallest element $0$; in this case, for $x \in A$ we put $x^* := x * 0$.

- ***Homomorphism of Hilbert algebras***: a mapping between Hilbert algebras preserving operations $*$ and $1$ (and hence preserving also order).

  The class $\mathscr{H}$ ($\mathscr{H}_0$) of all (bounded) Hilbert algebras considered as algebras of the form $(A, *, 1)$ ($(A, *, 1, 0)$ respectively) is equationally definable.

- Since any Boolean algebra is a bounded Hilbert algebra we will further use the denotion $(B, *, 1, 0)$ for Boolean algebras as well.

## Hilbert algebras

- ***Hilbert algebra*** (Diego): $\mathbf{A} = (A, *, 1)$ may be treated as a poset $(A, \leq)$ with the greatest element 1 equipped with a binary operation $*$ such that

$$x * y = 1 \text{ if and only if } x \leq y,$$
$$x \leq y * x, \quad x * (y * z) \leq (x * y) * (x * z) \ .$$

  $\mathbf{A}$ will be called *bounded* if it has a smallest element $0$; in this case, for $x \in A$ we put $x^* := x * 0$.

- ***Homomorphism of Hilbert algebras***: a mapping between Hilbert algebras preserving operations $*$ and $1$ (and hence preserving also order).

  The class $\mathscr{H}$ ($\mathscr{H}_0$) of all (bounded) Hilbert algebras considered as algebras of the form $(A, *, 1)$ ($(A, *, 1, 0)$ respectively) is equationally definable.

- Since any Boolean algebra is a bounded Hilbert algebra we will further use the denotion $(B, *, 1, 0)$ for Boolean algebras as well.

## Relatively pseudocomplemented posets

- ***Relative pseudocomplement of $a$ with respect to $b$***: is an element $c$ of a poset $\mathbf{A} = (A, \leq)$, in symbols $c = a * b$, such that $c$ is the greatest element $x$ of $\mathbf{A}$ satisfying $z \leq a, x \implies z \leq b$.
- ***Relatively pseudocomplemented poset***: is a poset $\mathbf{A} = (A, \leq)$ such that $a * b$ exists for all $a, b \in A$.

  S. Rudeanu has shown that the class of all relatively pseudocomplemented posets is a proper subclass of the class of all Hilbert algebras. Hence we may work further in the context of Hilbert algebras.
- ***Relatively pseudocomplemented poset***: is a Hilbert algebra $(A, *, 1)$ such that $a * b$ is the greatest element of the set $\{x \in A \mid z \leq a, x \implies z \leq b\}$ for all $a, b \in A$.

  Let $\mathscr{R}$ ($\mathscr{R}_0$) denote the class of all (bounded) relatively pseudocomplemented posets.
- ***Homomorphism of (bounded) relatively pseudocomplemented posets***: is a homomorphism of (bounded) Hilbert algebras.

## Relatively pseudocomplemented posets

- ***Relative pseudocomplement of** $a$ **with respect to** $b$*: is an element $c$ of a poset $\mathbf{A} = (A, \leq)$, in symbols $c = a * b$, such that $c$ is the greatest element $x$ of $\mathbf{A}$ satisfying $z \leq a, x \implies z \leq b$.
- ***Relatively pseudocomplemented poset***: is a poset $\mathbf{A} = (A, \leq)$ such that $a * b$ exists for all $a, b \in A$.

  S. Rudeanu has shown that the class of all relatively pseudocomplemented posets is a proper subclass of the class of all Hilbert algebras. Hence we may work further in the context of Hilbert algebras.
- ***Relatively pseudocomplemented poset***: is a Hilbert algebra $(A, *, 1)$ such that $a * b$ is the greatest element of the set $\{x \in A \mid z \leq a, x \implies z \leq b\}$ for all $a, b \in A$.

  Let $\mathscr{R}$ ($\mathscr{R}_0$) denote the class of all (bounded) relatively pseudocomplemented posets.
- ***Homomorphism of (bounded) relatively pseudocomplemented posets***: is a homomorphism of (bounded) Hilbert algebras.

## Relatively pseudocomplemented posets

- ***Relative pseudocomplement of*** $a$ ***with respect to*** $b$: is an element $c$ of a poset $\mathbf{A} = (A, \leq)$, in symbols $c = a * b$, such that $c$ is the greatest element $x$ of $\mathbf{A}$ satisfying $z \leq a, x \implies z \leq b$.
- ***Relatively pseudocomplemented poset***: is a poset $\mathbf{A} = (A, \leq)$ such that $a * b$ exists for all $a, b \in A$.

S. Rudeanu has shown that the class of all relatively pseudocomplemented posets is a proper subclass of the class of all Hilbert algebras. Hence we may work further in the context of Hilbert algebras.

- ***Relatively pseudocomplemented poset***: is a Hilbert algebra $(A, *, 1)$ such that $a * b$ is the greatest element of the set $\{x \in A \mid z \leq a, x \implies z \leq b\}$ for all $a, b \in A$.

Let $\mathscr{R}$ ($\mathscr{R}_0$) denote the class of all (bounded) relatively pseudocomplemented posets.

- ***Homomorphism of (bounded) relatively pseudocomplemented posets***: is a homomorphism of (bounded) Hilbert algebras.

## Relatively pseudocomplemented posets

- ***Relative pseudocomplement of $a$ with respect to $b$***: is an element $c$ of a poset $\mathbf{A} = (A, \leq)$, in symbols $c = a * b$, such that $c$ is the greatest element $x$ of $\mathbf{A}$ satisfying $z \leq a, x \implies z \leq b$.
- ***Relatively pseudocomplemented poset***: is a poset $\mathbf{A} = (A, \leq)$ such that $a * b$ exists for all $a, b \in A$.

  S. Rudeanu has shown that the class of all relatively pseudocomplemented posets is a proper subclass of the class of all Hilbert algebras. Hence we may work further in the context of Hilbert algebras.
- ***Relatively pseudocomplemented poset***: is a Hilbert algebra $(A, *, 1)$ such that $a * b$ is the greatest element of the set $\{x \in A \mid z \leq a, x \implies z \leq b\}$ for all $a, b \in A$.

  Let $\mathscr{R}$ ($\mathscr{R}_0$) denote the class of all (bounded) relatively pseudocomplemented posets.
- ***Homomorphism of (bounded) relatively pseudocomplemented posets***: is a homomorphism of (bounded) Hilbert algebras.

## Relatively pseudocomplemented posets

- ***Relative pseudocomplement of*** $a$ ***with respect to*** $b$: is an element $c$ of a poset $\mathbf{A} = (A, \leq)$, in symbols $c = a * b$, such that $c$ is the greatest element $x$ of $\mathbf{A}$ satisfying $z \leq a, x \implies z \leq b$.
- ***Relatively pseudocomplemented poset***: is a poset $\mathbf{A} = (A, \leq)$ such that $a * b$ exists for all $a, b \in A$.

  S. Rudeanu has shown that the class of all relatively pseudocomplemented posets is a proper subclass of the class of all Hilbert algebras. Hence we may work further in the context of Hilbert algebras.
- ***Relatively pseudocomplemented poset***: is a Hilbert algebra $(A, *, 1)$ such that $a * b$ is the greatest element of the set $\{x \in A \mid z \leq a, x \implies z \leq b\}$ for all $a, b \in A$.

  Let $\mathscr{R}$ ($\mathscr{R}_0$) denote the class of all (bounded) relatively pseudocomplemented posets.
- ***Homomorphism of (bounded) relatively pseudocomplemented posets***: is a homomorphism of (bounded) Hilbert algebras.

## Relatively pseudocomplemented posets

- ***Relative pseudocomplement of $a$ with respect to $b$***: is an element $c$ of a poset $\mathbf{A} = (A, \leq)$, in symbols $c = a * b$, such that $c$ is the greatest element $x$ of $\mathbf{A}$ satisfying $z \leq a, x \implies z \leq b$.
- ***Relatively pseudocomplemented poset***: is a poset $\mathbf{A} = (A, \leq)$ such that $a * b$ exists for all $a, b \in A$.

  S. Rudeanu has shown that the class of all relatively pseudocomplemented posets is a proper subclass of the class of all Hilbert algebras. Hence we may work further in the context of Hilbert algebras.
- ***Relatively pseudocomplemented poset***: is a Hilbert algebra $(A, *, 1)$ such that $a * b$ is the greatest element of the set $\{x \in A \mid z \leq a, x \implies z \leq b\}$ for all $a, b \in A$.

  Let $\mathscr{R}$ ($\mathscr{R}_0$) denote the class of all (bounded) relatively pseudocomplemented posets.
- ***Homomorphism of (bounded) relatively pseudocomplemented posets***: is a homomorphism of (bounded) Hilbert algebras.

## Relatively pseudocomplemented posets

- ***Relative pseudocomplement of $a$ with respect to $b$***: is an element $c$ of a poset $\mathbf{A} = (A, \leq)$, in symbols $c = a * b$, such that $c$ is the greatest element $x$ of $\mathbf{A}$ satisfying $z \leq a, x \implies z \leq b$.
- ***Relatively pseudocomplemented poset***: is a poset $\mathbf{A} = (A, \leq)$ such that $a * b$ exists for all $a, b \in A$.

  S. Rudeanu has shown that the class of all relatively pseudocomplemented posets is a proper subclass of the class of all Hilbert algebras. Hence we may work further in the context of Hilbert algebras.
- ***Relatively pseudocomplemented poset***: is a Hilbert algebra $(A, *, 1)$ such that $a * b$ is the greatest element of the set $\{x \in A \mid z \leq a, x \implies z \leq b\}$ for all $a, b \in A$.

  Let $\mathscr{R}$ ($\mathscr{R}_0$) denote the class of all (bounded) relatively pseudocomplemented posets.
- ***Homomorphism of (bounded) relatively pseudocomplemented posets***: is a homomorphism of (bounded) Hilbert algebras.

## Implicative semilattices

Obviously, in the case when $\mathbf{A}$ is a meet-semilattice we have that $z \leq a, x \implies z \leq b$ is equivalent to $a \wedge x \leq b$ and hence the concept of relative pseudocomplement in posets is a generalization of the corresponding concept for meet-semilattices.

- ***Implicative semilattice***: is an algebra $(A, \wedge, *, 1)$ of type $(2, 2, 0)$ such that $(A, \wedge)$ is a meet-semilattice and $(A, *, 1)$ is a relatively pseudocomplemented poset.

  Let $\mathscr{IS}$ ($\mathscr{IS}_0$) denote the class of all (bounded) implicative semilattices.

- ***Homomorphism of (bounded) implicative semilattices***: is a homomorphism of (bounded) Hilbert algebras that preserves finite meets.

## Implicative semilattices

Obviously, in the case when $\mathbf{A}$ is a meet-semilattice we have that $z \leq a, x \implies z \leq b$ is equivalent to $a \wedge x \leq b$ and hence the concept of relative pseudocomplement in posets is a generalization of the corresponding concept for meet-semilattices.

- ***Implicative semilattice***: is an algebra $(A, \wedge, *, 1)$ of type $(2, 2, 0)$ such that $(A, \wedge)$ is a meet-semilattice and $(A, *, 1)$ is a relatively pseudocomplemented poset.

  Let $\mathscr{IS}$ ($\mathscr{IS}_0$) denote the class of all (bounded) implicative semilattices.

- ***Homomorphism of (bounded) implicative semilattices***: is a homomorphism of (bounded) Hilbert algebras that preserves finite meets.

## Implicative semilattices

Obviously, in the case when $\mathbf{A}$ is a meet-semilattice we have that $z \leq a, x \implies z \leq b$ is equivalent to $a \wedge x \leq b$ and hence the concept of relative pseudocomplement in posets is a generalization of the corresponding concept for meet-semilattices.

- ***Implicative semilattice***: is an algebra $(A, \wedge, *, 1)$ of type $(2, 2, 0)$ such that $(A, \wedge)$ is a meet-semilattice and $(A, *, 1)$ is a relatively pseudocomplemented poset.

Let $\mathscr{IS}$ ($\mathscr{IS}_0$) denote the class of all (bounded) implicative semilattices.

- ***Homomorphism of (bounded) implicative semilattices***: is a homomorphism of (bounded) Hilbert algebras that preserves finite meets.

## Implicative semilattices

Obviously, in the case when $\mathbf{A}$ is a meet-semilattice we have that $z \leq a, x \implies z \leq b$ is equivalent to $a \wedge x \leq b$ and hence the concept of relative pseudocomplement in posets is a generalization of the corresponding concept for meet-semilattices.

- ***Implicative semilattice***: is an algebra $(A, \wedge, *, 1)$ of type $(2,2,0)$ such that $(A, \wedge)$ is a meet-semilattice and $(A, *, 1)$ is a relatively pseudocomplemented poset.

  Let $\mathscr{IS}$ ($\mathscr{IS}_0$) denote the class of all (bounded) implicative semilattices.

- ***Homomorphism of (bounded) implicative semilattices***: is a homomorphism of (bounded) Hilbert algebras that preserves finite meets.

## Implicative semilattices

Obviously, in the case when $\mathbf{A}$ is a meet-semilattice we have that $z \leq a, x \implies z \leq b$ is equivalent to $a \wedge x \leq b$ and hence the concept of relative pseudocomplement in posets is a generalization of the corresponding concept for meet-semilattices.

- ***Implicative semilattice***: is an algebra $(A, \wedge, *, 1)$ of type $(2, 2, 0)$ such that $(A, \wedge)$ is a meet-semilattice and $(A, *, 1)$ is a relatively pseudocomplemented poset.

  Let $\mathscr{I}\mathscr{S}$ ($\mathscr{I}\mathscr{S}_0$) denote the class of all (bounded) implicative semilattices.

- ***Homomorphism of (bounded) implicative semilattices***: is a homomorphism of (bounded) Hilbert algebras that preserves finite meets.

## Basic results

We will write $\mathrm{U_I}$ for the forgetful functor from $\mathscr{IS}$ to $\mathscr{H}$. Fix $\mathbf{A} \in \mathscr{H}$, a pair $(\mathbf{G}, e)$, where $\mathbf{G}$ is an implicative semilattice and $e$ is an injective homomorphism from $\mathbf{A}$ to $\mathrm{U_I}(\mathbf{G})$ is said to be an *implicative semilattice envelope of* $\mathbf{A}$ if for every $y \in G$ there exists a finite subset $X \subseteq A$ such that $y = \bigwedge e(X)$.

It is well known that there is a functor $\mathrm{S}$ from $\mathscr{H}$ to $\mathscr{IS}$ such that, for any $\mathbf{A} \in \mathscr{H}$, we have an injective homomorphism $e$ from $\mathbf{A}$ to $\mathrm{U_I}(\mathrm{S}(\mathbf{A}))$ and $(\mathrm{S}(\mathbf{A}), e)$ is an implicative semilattice envelope of $\mathbf{A}$.

In what follows we will always assume that $A \subseteq \mathrm{S}(\mathbf{A})$, i.e., $\mathbf{A}$ will be a Hilbert subalgebra of $\mathrm{U_I}(\mathrm{S}(\mathbf{A}))$ and $e$ an inclusion.

### Theorem (Celani and Jansana, 2012)

*There exists a functor* $\mathrm{S}\colon \mathscr{H} \to \mathscr{IS}$ *that maps every Hilbert algebra* $\mathbf{A}$ *to an implicative semilattice* $\mathrm{S}(\mathbf{A})$*, and every homomorphism* $h\colon \mathbf{A} \to \mathbf{B}$ *of Hilbert algebras to a homomorphism* $\mathrm{S}(h)\colon \mathrm{S}(\mathbf{A}) \to \mathrm{S}(\mathbf{B})$ *of implicative semilattices such that* $\mathrm{S}(h)(\bigwedge X) = \bigwedge h(X)$ *for any finite subset* $X \subseteq A$*. The functor* $\mathrm{S}$ *is a left adjoint to* $\mathrm{U_I}$.

## Basic results

We will write $\mathrm{U_I}$ for the forgetful functor from $\mathscr{I}\mathscr{S}$ to $\mathscr{H}$. Fix $\mathbf{A} \in \mathscr{H}$, a pair $(\mathbf{G}, e)$, where $\mathbf{G}$ is an implicative semilattice and $e$ is an injective homomorphism from $\mathbf{A}$ to $\mathrm{U_I}(\mathbf{G})$ is said to be an *implicative semilattice envelope of* $\mathbf{A}$ if for every $y \in G$ there exists a finite subset $X \subseteq A$ such that $y = \bigwedge e(X)$.

It is well known that there is a functor $\mathrm{S}$ from $\mathscr{H}$ to $\mathscr{I}\mathscr{S}$ such that, for any $\mathbf{A} \in \mathscr{H}$, we have an injective homomorphism $e$ from $\mathbf{A}$ to $\mathrm{U_I}(\mathrm{S}(\mathbf{A}))$ and $(\mathrm{S}(\mathbf{A}), e)$ is an implicative semilattice envelope of $\mathbf{A}$.

In what follows we will always assume that $A \subseteq \mathrm{S}(\mathbf{A})$, i.e., $\mathbf{A}$ will be a Hilbert subalgebra of $\mathrm{U_I}(\mathrm{S}(\mathbf{A}))$ and $e$ an inclusion.

### Theorem (Celani and Jansana, 2012)

*There exists a functor $\mathrm{S}\colon \mathscr{H} \to \mathscr{I}\mathscr{S}$ that maps every Hilbert algebra $\mathbf{A}$ to an implicative semilattice $\mathrm{S}(\mathbf{A})$, and every homomorphism $h\colon \mathbf{A} \to \mathbf{B}$ of Hilbert algebras to a homomorphism $\mathrm{S}(h)\colon \mathrm{S}(\mathbf{A}) \to \mathrm{S}(\mathbf{B})$ of implicative semilattices such that $\mathrm{S}(h)(\bigwedge X) = \bigwedge h(X)$ for any finite subset $X \subseteq A$. The functor $\mathrm{S}$ is a left adjoint to $\mathrm{U_I}$.*

## Basic results

We will write $\mathrm{U_I}$ for the forgetful functor from $\mathscr{IS}$ to $\mathscr{H}$. Fix $\mathbf{A} \in \mathscr{H}$, a pair $(\mathbf{G}, e)$, where $\mathbf{G}$ is an implicative semilattice and $e$ is an injective homomorphism from $\mathbf{A}$ to $\mathrm{U_I}(\mathbf{G})$ is said to be an *implicative semilattice envelope of* $\mathbf{A}$ if for every $y \in G$ there exists a finite subset $X \subseteq A$ such that $y = \bigwedge e(X)$.

It is well known that there is a functor $\mathrm{S}$ from $\mathscr{H}$ to $\mathscr{IS}$ such that, for any $\mathbf{A} \in \mathscr{H}$, we have an injective homomorphism $e$ from $\mathbf{A}$ to $\mathrm{U_I}(\mathrm{S}(\mathbf{A}))$ and $(\mathrm{S}(\mathbf{A}), e)$ is an implicative semilattice envelope of $\mathbf{A}$.

In what follows we will always assume that $A \subseteq \mathrm{S}(\mathbf{A})$, i.e., $\mathbf{A}$ will be a Hilbert subalgebra of $\mathrm{U_I}(\mathrm{S}(\mathbf{A}))$ and $e$ an inclusion.

### Theorem (Celani and Jansana, 2012)

*There exists a functor $\mathrm{S}\colon \mathscr{H} \to \mathscr{IS}$ that maps every Hilbert algebra $\mathbf{A}$ to an implicative semilattice $\mathrm{S}(\mathbf{A})$, and every homomorphism $h\colon \mathbf{A} \to \mathbf{B}$ of Hilbert algebras to a homomorphism $\mathrm{S}(h)\colon \mathrm{S}(\mathbf{A}) \to \mathrm{S}(\mathbf{B})$ of implicative semilattices such that $\mathrm{S}(h)(\bigwedge X) = \bigwedge h(X)$ for any finite subset $X \subseteq A$. The functor $\mathrm{S}$ is a left adjoint to $\mathrm{U_I}$.*

## Basic results

We will write $\mathrm{U_I}$ for the forgetful functor from $\mathscr{IS}$ to $\mathscr{H}$. Fix $\mathbf{A} \in \mathscr{H}$, a pair $(\mathbf{G}, e)$, where $\mathbf{G}$ is an implicative semilattice and $e$ is an injective homomorphism from $\mathbf{A}$ to $\mathrm{U_I}(\mathbf{G})$ is said to be an *implicative semilattice envelope of* $\mathbf{A}$ if for every $y \in G$ there exists a finite subset $X \subseteq A$ such that $y = \bigwedge e(X)$.

It is well known that there is a functor $\mathrm{S}$ from $\mathscr{H}$ to $\mathscr{IS}$ such that, for any $\mathbf{A} \in \mathscr{H}$, we have an injective homomorphism $e$ from $\mathbf{A}$ to $\mathrm{U_I}(\mathrm{S}(\mathbf{A}))$ and $(\mathrm{S}(\mathbf{A}), e)$ is an implicative semilattice envelope of $\mathbf{A}$.

In what follows we will always assume that $A \subseteq \mathrm{S}(\mathbf{A})$, i.e., $\mathbf{A}$ will be a Hilbert subalgebra of $\mathrm{U_I}(\mathrm{S}(\mathbf{A}))$ and $e$ an inclusion.

### Theorem (Celani and Jansana, 2012)

*There exists a functor $\mathrm{S}\colon \mathscr{H} \to \mathscr{IS}$ that maps every Hilbert algebra $\mathbf{A}$ to an implicative semilattice $\mathrm{S}(\mathbf{A})$, and every homomorphism $h\colon \mathbf{A} \to \mathbf{B}$ of Hilbert algebras to a homomorphism $\mathrm{S}(h)\colon \mathrm{S}(\mathbf{A}) \to \mathrm{S}(\mathbf{B})$ of implicative semilattices such that $\mathrm{S}(h)(\bigwedge X) = \bigwedge h(X)$ for any finite subset $X \subseteq A$. The functor $\mathrm{S}$ is a left adjoint to $\mathrm{U_I}$.*

## Basic results

We will write $\mathrm{U_I}$ for the forgetful functor from $\mathscr{IS}$ to $\mathscr{H}$. Fix $\mathbf{A} \in \mathscr{H}$, a pair $(\mathbf{G}, e)$, where $\mathbf{G}$ is an implicative semilattice and $e$ is an injective homomorphism from $\mathbf{A}$ to $\mathrm{U_I}(\mathbf{G})$ is said to be an *implicative semilattice envelope of* $\mathbf{A}$ if for every $y \in G$ there exists a finite subset $X \subseteq A$ such that $y = \bigwedge e(X)$.

It is well known that there is a functor $\mathrm{S}$ from $\mathscr{H}$ to $\mathscr{IS}$ such that, for any $\mathbf{A} \in \mathscr{H}$, we have an injective homomorphism $e$ from $\mathbf{A}$ to $\mathrm{U_I}(\mathrm{S}(\mathbf{A}))$ and $(\mathrm{S}(\mathbf{A}), e)$ is an implicative semilattice envelope of $\mathbf{A}$.

In what follows we will always assume that $A \subseteq \mathrm{S}(\mathbf{A})$, i.e., $\mathbf{A}$ will be a Hilbert subalgebra of $\mathrm{U_I}(\mathrm{S}(\mathbf{A}))$ and $e$ an inclusion.

## Theorem (Celani and Jansana, 2012)

*There exists a functor* $\mathrm{S}\colon \mathscr{H} \to \mathscr{IS}$ *that maps every Hilbert algebra* $\mathbf{A}$ *to an implicative semilattice* $\mathrm{S}(\mathbf{A})$, *and every homomorphism* $h\colon \mathbf{A} \to \mathbf{B}$ *of Hilbert algebras to a homomorphism* $\mathrm{S}(h)\colon \mathrm{S}(\mathbf{A}) \to \mathrm{S}(\mathbf{B})$ *of implicative semilattices such that* $\mathrm{S}(h)(\bigwedge X) = \bigwedge h(X)$ *for any finite subset* $X \subseteq A$. *The functor* $\mathrm{S}$ *is a left adjoint to* $\mathrm{U_I}$.

## Compatibility relation

Let $\mathbf{A} = (A, *, 1)$ be a Hilbert algebra. The *compatibility relation* $\mathrm{C}$ on Hilbert algebras was introduced by E. L. Marsden.

We will use an equivalent definition due to J. Cīrulis : elements $a, b \in A$ are said to be *compatible* (in symbols, $a \,\mathrm{C}\, b$) if they have a lower bound $c$ such that $a \leq b * c$.

This lower bound $c$ is necessary a meet of elements $a$ and $b$; we call also a meet arising in this way *compatible*.

A subset of $A$ is its *relative subsemilattice* if it is closed under existing compatible meets.

To emphasize that the meet of $a$ and $b$ is compatible, it will be written as $a \curlywedge b$.

$(A, *, 1)$ is an implicative semilattice if and only if all meets in $A$ exist and are compatible.

If $(\mathbf{G}, e)$ is an implicative semilattice envelope of $\mathbf{A}$ then $e(a \curlywedge b) = e(a) \wedge e(b)$ for all $a, b \in A$ such that $a \,\mathrm{C}\, b$.

## Compatibility relation

Let $\mathbf{A} = (A, *, 1)$ be a Hilbert algebra. The *compatibility relation* $\mathrm{C}$ on Hilbert algebras was introduced by E. L. Marsden.

We will use an equivalent definition due to J. Cīrulis : elements $a, b \in A$ are said to be *compatible* (in symbols, $a \,\mathrm{C}\, b$) if they have a lower bound $c$ such that $a \leq b * c$.

This lower bound $c$ is necessary a meet of elements $a$ and $b$; we call also a meet arising in this way *compatible*.

A subset of $A$ is its *relative subsemilattice* if it is closed under existing compatible meets.

To emphasize that the meet of $a$ and $b$ is compatible, it will be written as $a \curlywedge b$.

$(A, *, 1)$ is an implicative semilattice if and only if all meets in $A$ exist and are compatible.

If $(\mathbf{G}, e)$ is an implicative semilattice envelope of $\mathbf{A}$ then $e(a \curlywedge b) = e(a) \wedge e(b)$ for all $a, b \in A$ such that $a \,\mathrm{C}\, b$.

## Compatibility relation

Let $\mathbf{A} = (A, *, 1)$ be a Hilbert algebra. The *compatibility relation* $\mathrm{C}$ on Hilbert algebras was introduced by E. L. Marsden.

We will use an equivalent definition due to J. Cīrulis : elements $a, b \in A$ are said to be *compatible* (in symbols, $a\,\mathrm{C}\,b$) if they have a lower bound $c$ such that $a \leq b * c$.

This lower bound $c$ is necessary a meet of elements $a$ and $b$; we call also a meet arising in this way *compatible*.

A subset of $A$ is its *relative subsemilattice* if it is closed under existing compatible meets.

To emphasize that the meet of $a$ and $b$ is compatible, it will be written as $a \curlywedge b$.

$(A, *, 1)$ is an implicative semilattice if and only if all meets in $A$ exist and are compatible.

If $(\mathbf{G}, e)$ is an implicative semilattice envelope of $\mathbf{A}$ then $e(a \curlywedge b) = e(a) \wedge e(b)$ for all $a, b \in A$ such that $a\,\mathrm{C}\,b$.

## Compatibility relation

Let $\mathbf{A} = (A, *, 1)$ be a Hilbert algebra. The *compatibility relation* $\mathrm{C}$ on Hilbert algebras was introduced by E. L. Marsden.

We will use an equivalent definition due to J. Cīrulis : elements $a, b \in A$ are said to be *compatible* (in symbols, $a \,\mathrm{C}\, b$) if they have a lower bound $c$ such that $a \leq b * c$.

This lower bound $c$ is necessary a meet of elements $a$ and $b$; we call also a meet arising in this way *compatible*.

A subset of $A$ is its *relative subsemilattice* if it is closed under existing compatible meets.

To emphasize that the meet of $a$ and $b$ is compatible, it will be written as $a \curlywedge b$.

$(A, *, 1)$ is an implicative semilattice if and only if all meets in $A$ exist and are compatible.

If $(\mathbf{G}, e)$ is an implicative semilattice envelope of $\mathbf{A}$ then $e(a \curlywedge b) = e(a) \wedge e(b)$ for all $a, b \in A$ such that $a \,\mathrm{C}\, b$.

## Compatibility relation

Let $\mathbf{A} = (A, *, 1)$ be a Hilbert algebra. The *compatibility relation* $\mathrm{C}$ on Hilbert algebras was introduced by E. L. Marsden.

We will use an equivalent definition due to J. Cīrulis : elements $a, b \in A$ are said to be *compatible* (in symbols, $a \mathrel{\mathrm{C}} b$) if they have a lower bound $c$ such that $a \leq b * c$.

This lower bound $c$ is necessary a meet of elements $a$ and $b$; we call also a meet arising in this way *compatible*.

A subset of $A$ is its *relative subsemilattice* if it is closed under existing compatible meets.

To emphasize that the meet of $a$ and $b$ is compatible, it will be written as $a \curlywedge b$.

$(A, *, 1)$ is an implicative semilattice if and only if all meets in $A$ exist and are compatible.

If $(\mathbf{G}, e)$ is an implicative semilattice envelope of $\mathbf{A}$ then $e(a \curlywedge b) = e(a) \wedge e(b)$ for all $a, b \in A$ such that $a \mathrel{\mathrm{C}} b$.

## Compatibility relation

Let $\mathbf{A} = (A, *, 1)$ be a Hilbert algebra. The *compatibility relation* $\mathrm{C}$ on Hilbert algebras was introduced by E. L. Marsden.

We will use an equivalent definition due to J. Cīrulis : elements $a, b \in A$ are said to be *compatible* (in symbols, $a \,\mathrm{C}\, b$) if they have a lower bound $c$ such that $a \leq b * c$.

This lower bound $c$ is necessary a meet of elements $a$ and $b$; we call also a meet arising in this way *compatible*.

A subset of $A$ is its *relative subsemilattice* if it is closed under existing compatible meets.

To emphasize that the meet of $a$ and $b$ is compatible, it will be written as $a \curlywedge b$.

$(A, *, 1)$ is an implicative semilattice if and only if all meets in $A$ exist and are compatible.

If $(\mathbf{G}, e)$ is an implicative semilattice envelope of $\mathbf{A}$ then $e(a \curlywedge b) = e(a) \wedge e(b)$ for all $a, b \in A$ such that $a \,\mathrm{C}\, b$.

## Compatibility relation

Let $\mathbf{A} = (A, *, 1)$ be a Hilbert algebra. The *compatibility relation* $\mathrm{C}$ on Hilbert algebras was introduced by E. L. Marsden.

We will use an equivalent definition due to J. Cīrulis : elements $a, b \in A$ are said to be *compatible* (in symbols, $a \,\mathrm{C}\, b$) if they have a lower bound $c$ such that $a \leq b * c$.

This lower bound $c$ is necessary a meet of elements $a$ and $b$; we call also a meet arising in this way *compatible*.

A subset of $A$ is its *relative subsemilattice* if it is closed under existing compatible meets.

To emphasize that the meet of $a$ and $b$ is compatible, it will be written as $a \curlywedge b$.

$(A, *, 1)$ is an implicative semilattice if and only if all meets in $A$ exist and are compatible.

If $(\mathbf{G}, e)$ is an implicative semilattice envelope of $\mathbf{A}$ then $e(a \curlywedge b) = e(a) \wedge e(b)$ for all $a, b \in A$ such that $a \,\mathrm{C}\, b$.

## Compatibility relation

Let $\mathbf{A} = (A, *, 1)$ be a Hilbert algebra. The *compatibility relation* $\mathrm{C}$ on Hilbert algebras was introduced by E. L. Marsden.

We will use an equivalent definition due to J. Cīrulis : elements $a, b \in A$ are said to be *compatible* (in symbols, $a \,\mathrm{C}\, b$) if they have a lower bound $c$ such that $a \leq b * c$.

This lower bound $c$ is necessary a meet of elements $a$ and $b$; we call also a meet arising in this way *compatible*.

A subset of $A$ is its *relative subsemilattice* if it is closed under existing compatible meets.

To emphasize that the meet of $a$ and $b$ is compatible, it will be written as $a \curlywedge b$.

$(A, *, 1)$ is an implicative semilattice if and only if all meets in $A$ exist and are compatible.

If $(\mathbf{G}, e)$ is an implicative semilattice envelope of $\mathbf{A}$ then $e(a \curlywedge b) = e(a) \wedge e(b)$ for all $a, b \in A$ such that $a \,\mathrm{C}\, b$.

## Outline

## Congruences

### Congruences

Let $\mathrm{Con}\,\mathbf{A}$ denote the set of all congruences on a Hilbert algebra $\mathbf{A}$.

A *congruence* $\theta$ on a relatively pseudocomplemented poset $\mathbf{A} = (A, *, 1)$ will be the congruence on the Hilbert algebra $\mathbf{A}$.

In particular, all results on congruences valid for Hilbert algebras will be also true for relatively pseudocomplemented posets.

An equivalence relation $\theta$ on a poset $(A, \leq)$ is called *convex* if $a, b, c \in A$, $a \leq b \leq c$ and $(a, c) \in \theta$ together imply $(a, b) \in \theta$, i.e. if every class of $\theta$ is a convex subset of $(A, \leq)$.

### Theorem

*Let $\mathbf{A} = (A, *, 1)$ be a Hilbert algebra and $\theta \in \mathrm{Con}\,\mathbf{A}$. Then $\theta$ is convex and any congruence class of $\theta$ is up-directed.*

# Congruences

## Congruences

Let $\mathrm{Con}\,\mathbf{A}$ denote the set of all congruences on a Hilbert algebra $\mathbf{A}$.

A *congruence* $\theta$ on a relatively pseudocomplemented poset $\mathbf{A} = (A, *, 1)$ will be the congruence on the Hilbert algebra $\mathbf{A}$.

In particular, all results on congruences valid for Hilbert algebras will be also true for relatively pseudocomplemented posets.

An equivalence relation $\theta$ on a poset $(A, \leq)$ is called *convex* if $a, b, c \in A$, $a \leq b \leq c$ and $(a, c) \in \theta$ together imply $(a, b) \in \theta$, i.e. if every class of $\theta$ is a convex subset of $(A, \leq)$.

## Theorem

*Let $\mathbf{A} = (A, *, 1)$ be a Hilbert algebra and $\theta \in \mathrm{Con}\,\mathbf{A}$. Then $\theta$ is convex and any congruence class of $\theta$ is up-directed.*

## Congruences

### Congruences

Let $\operatorname{Con}\mathbf{A}$ denote the set of all congruences on a Hilbert algebra $\mathbf{A}$.

A *congruence* $\theta$ on a relatively pseudocomplemented poset $\mathbf{A} = (A, *, 1)$ will be the congruence on the Hilbert algebra $\mathbf{A}$.

In particular, all results on congruences valid for Hilbert algebras will be also true for relatively pseudocomplemented posets.

An equivalence relation $\theta$ on a poset $(A, \leq)$ is called *convex* if $a, b, c \in A$, $a \leq b \leq c$ and $(a, c) \in \theta$ together imply $(a, b) \in \theta$, i.e. if every class of $\theta$ is a convex subset of $(A, \leq)$.

### Theorem

*Let $\mathbf{A} = (A, *, 1)$ be a Hilbert algebra and $\theta \in \operatorname{Con}\mathbf{A}$. Then $\theta$ is convex and any congruence class of $\theta$ is up-directed.*

## Congruences

### Congruences

Let $\mathrm{Con}\,\mathbf{A}$ denote the set of all congruences on a Hilbert algebra $\mathbf{A}$.

A *congruence* $\theta$ on a relatively pseudocomplemented poset $\mathbf{A} = (A, *, 1)$ will be the congruence on the Hilbert algebra $\mathbf{A}$.

In particular, all results on congruences valid for Hilbert algebras will be also true for relatively pseudocomplemented posets.

An equivalence relation $\theta$ on a poset $(A, \leq)$ is called *convex* if $a, b, c \in A$, $a \leq b \leq c$ and $(a, c) \in \theta$ together imply $(a, b) \in \theta$, i.e. if every class of $\theta$ is a convex subset of $(A, \leq)$.

### Theorem

*Let $\mathbf{A} = (A, *, 1)$ be a Hilbert algebra and $\theta \in \mathrm{Con}\,\mathbf{A}$. Then $\theta$ is convex and any congruence class of $\theta$ is up-directed.*

## Congruences

### Congruences

Let $\mathrm{Con}\,\mathbf{A}$ denote the set of all congruences on a Hilbert algebra $\mathbf{A}$.

A *congruence* $\theta$ on a relatively pseudocomplemented poset $\mathbf{A} = (A, *, 1)$ will be the congruence on the Hilbert algebra $\mathbf{A}$.

In particular, all results on congruences valid for Hilbert algebras will be also true for relatively pseudocomplemented posets.

An equivalence relation $\theta$ on a poset $(A, \leq)$ is called *convex* if $a, b, c \in A$, $a \leq b \leq c$ and $(a, c) \in \theta$ together imply $(a, b) \in \theta$, i.e. if every class of $\theta$ is a convex subset of $(A, \leq)$.

### Theorem

*Let $\mathbf{A} = (A, *, 1)$ be a Hilbert algebra and $\theta \in \mathrm{Con}\,\mathbf{A}$. Then $\theta$ is convex and any congruence class of $\theta$ is up-directed.*

## Congruences

### Congruences

Let $\mathrm{Con}\,\mathbf{A}$ denote the set of all congruences on a Hilbert algebra $\mathbf{A}$.

A *congruence* $\theta$ on a relatively pseudocomplemented poset $\mathbf{A} = (A, *, 1)$ will be the congruence on the Hilbert algebra $\mathbf{A}$.

In particular, all results on congruences valid for Hilbert algebras will be also true for relatively pseudocomplemented posets.

An equivalence relation $\theta$ on a poset $(A, \leq)$ is called *convex* if $a, b, c \in A$, $a \leq b \leq c$ and $(a, c) \in \theta$ together imply $(a, b) \in \theta$, i.e. if every class of $\theta$ is a convex subset of $(A, \leq)$.

### Theorem

*Let $\mathbf{A} = (A, *, 1)$ be a Hilbert algebra and $\theta \in \mathrm{Con}\,\mathbf{A}$. Then $\theta$ is convex and any congruence class of $\theta$ is up-directed.*

## Nuclei

### Definition

Let $\mathbf{A} = (A, *, 1)$ be a Hilbert algebra. A *nucleus on* $\mathbf{A}$ is a map $j\colon A \to A$ such that for any $a, b \in A$:

(i) $a \leq b$ implies $j(a) \leq j(b)$;
(ii) $a \leq j(a)$;
(iii) $(j \circ j)(a) = j(a)$;
(iv) $j(j(a) * j(b)) = j(a * b)$.

We put $A_j = \{a \in A \mid j(a) = a\}$ and define a binary operation $*_j$ on $A_j$ by $a *_j b = j(a * b)$. Clearly, $A_j$ is a subposet of $A$.

## Nuclei

### Definition

Let $\mathbf{A} = (A, *, 1)$ be a Hilbert algebra. A *nucleus on* $\mathbf{A}$ is a map $j\colon A \to A$ such that for any $a, b \in A$:

(i) $a \leq b$ implies $j(a) \leq j(b)$;

(ii) $a \leq j(a)$;

(iii) $(j \circ j)(a) = j(a)$;

(iv) $j(j(a) * j(b)) = j(a * b)$.

We put $A_j = \{a \in A \mid j(a) = a\}$ and define a binary operation $*_j$ on $A_j$ by $a *_j b = j(a * b)$. Clearly, $A_j$ is a subposet of $A$.

## Nuclei

### Definition

Let $\mathbf{A} = (A, *, 1)$ be a Hilbert algebra. A *nucleus on* $\mathbf{A}$ is a map $j\colon A \to A$ such that for any $a, b \in A$:

(i) $a \leq b$ implies $j(a) \leq j(b)$;

(ii) $a \leq j(a)$;

(iii) $(j \circ j)(a) = j(a)$;

(iv) $j(j(a) * j(b)) = j(a * b)$.

We put $A_j = \{a \in A \mid j(a) = a\}$ and define a binary operation $*_j$ on $A_j$ by $a *_j b = j(a * b)$. Clearly, $A_j$ is a subposet of $A$.

## Nuclei

### Definition

Let $\mathbf{A} = (A, *, 1)$ be a Hilbert algebra. A *nucleus on* $\mathbf{A}$ is a map $j\colon A \to A$ such that for any $a, b \in A$:

(i) $a \leq b$ implies $j(a) \leq j(b)$;

(ii) $a \leq j(a)$;

(iii) $(j \circ j)(a) = j(a)$;

(iv) $j(j(a) * j(b)) = j(a * b)$.

We put $A_j = \{a \in A \mid j(a) = a\}$ and define a binary operation $*_j$ on $A_j$ by $a *_j b = j(a * b)$. Clearly, $A_j$ is a subposet of $A$.

## Nuclei

### Definition

Let $\mathbf{A} = (A, *, 1)$ be a Hilbert algebra. A *nucleus on* $\mathbf{A}$ is a map $j\colon A \to A$ such that for any $a, b \in A$:

(i) $a \leq b$ implies $j(a) \leq j(b)$;

(ii) $a \leq j(a)$;

(iii) $(j \circ j)(a) = j(a)$;

(iv) $j(j(a) * j(b)) = j(a * b)$.

We put $A_j = \{a \in A \mid j(a) = a\}$ and define a binary operation $*_j$ on $A_j$ by $a *_j b = j(a * b)$. Clearly, $A_j$ is a subposet of $A$.

## Nuclei

### Definition

Let $\mathbf{A} = (A, *, 1)$ be a Hilbert algebra. A *nucleus on* $\mathbf{A}$ is a map $j\colon A \to A$ such that for any $a, b \in A$:

(i) $a \leq b$ implies $j(a) \leq j(b)$;

(ii) $a \leq j(a)$;

(iii) $(j \circ j)(a) = j(a)$;

(iv) $j(j(a) * j(b)) = j(a * b)$.

We put $A_j = \{a \in A \mid j(a) = a\}$ and define a binary operation $*_j$ on $A_j$ by $a *_j b = j(a * b)$. Clearly, $A_j$ is a subposet of $A$.

## Nuclei

### Definition

Let $\mathbf{A} = (A, *, 1)$ be a Hilbert algebra. A *nucleus on* $\mathbf{A}$ is a map $j\colon A \to A$ such that for any $a, b \in A$:

(i) $a \leq b$ implies $j(a) \leq j(b)$;

(ii) $a \leq j(a)$;

(iii) $(j \circ j)(a) = j(a)$;

(iv) $j(j(a) * j(b)) = j(a * b)$.

We put $A_j = \{a \in A \mid j(a) = a\}$ and define a binary operation $*_j$ on $A_j$ by $a *_j b = j(a * b)$. Clearly, $A_j$ is a subposet of $A$.

Notice that any closure operator on $\mathbf{A}$ which is an endomorphism of $\mathbf{A}$ is a nucleus on $\mathbf{A}$.

The identity mapping $\mathrm{id}_A$ and the unit mapping $\mathbf{1}\colon x \mapsto 1$ are examples of nuclei.

## Nuclei

### Definition

Let $\mathbf{A} = (A, *, 1)$ be a Hilbert algebra. A *nucleus on* $\mathbf{A}$ is a map $j\colon A \to A$ such that for any $a, b \in A$:

(i) $a \leq b$ implies $j(a) \leq j(b)$;

(ii) $a \leq j(a)$;

(iii) $(j \circ j)(a) = j(a)$;

(iv) $j(j(a) * j(b)) = j(a * b)$.

We put $A_j = \{a \in A \mid j(a) = a\}$ and define a binary operation $*_j$ on $A_j$ by $a *_j b = j(a * b)$. Clearly, $A_j$ is a subposet of $A$.

Notice that any closure operator on $\mathbf{A}$ which is an endomorphism of $\mathbf{A}$ is a nucleus on $\mathbf{A}$.

The identity mapping $\mathrm{id}_A$ and the unit mapping $\mathbf{1}\colon x \mapsto 1$ are examples of nuclei.

## Nuclei

### Definition

Let $\mathbf{A} = (A, *, 1)$ be a Hilbert algebra. A *nucleus on* $\mathbf{A}$ is a map $j \colon A \to A$ such that for any $a, b \in A$:

(i) $a \leq b$ implies $j(a) \leq j(b)$;

(ii) $a \leq j(a)$;

(iii) $(j \circ j)(a) = j(a)$;

(iv) $j(j(a) * j(b)) = j(a * b)$.

We put $A_j = \{a \in A \mid j(a) = a\}$ and define a binary operation $*_j$ on $A_j$ by $a *_j b = j(a * b)$. Clearly, $A_j$ is a subposet of $A$.

Notice that any closure operator on $\mathbf{A}$ which is an endomorphism of $\mathbf{A}$ is a nucleus on $\mathbf{A}$.

The identity mapping $\mathrm{id}_A$ and the unit mapping $\mathbf{1} \colon x \mapsto 1$ are examples of nuclei.

## Nuclei

### Proposition

*Let $\mathbf{A} = (A, *, 1)$ be a Hilbert algebra and let $j$ be a nucleus on $\mathbf{A}$. Then*

(i) $j\colon A \to A$ *is an endomorphism of* $\mathbf{A}$ *and* $*_j$ *is the restriction of* $*$ *to* $A_j$.

(ii) *The algebra* $\mathbf{A}_j = (A_j, *_j, 1)$ *is a Hilbert algebra which is a subalgebra isomorphic to a quotient algebra of* $\mathbf{A}$.

(iii) *The quotient homomorphism* $j\colon A \to A_j$ *is a retract with respect to the inclusion homomorphism* $i\colon A_j \to A$.

## Nuclei

### Proposition

*Let* $\mathbf{A} = (A, *, 1)$ *be a Hilbert algebra and let* $j$ *be a nucleus on* $\mathbf{A}$. *Then*

(i) $j\colon A \to A$ *is an endomorphism of* $\mathbf{A}$ *and* $*_j$ *is the restriction of* $*$ *to* $A_j$.

(ii) *The algebra* $\mathbf{A}_j = (A_j, *_j, 1)$ *is a Hilbert algebra which is a subalgebra isomorphic to a quotient algebra of* $\mathbf{A}$.

(iii) *The quotient homomorphism* $j\colon A \to A_j$ *is a retract with respect to the inclusion homomorphism* $i\colon A_j \to A$.

## Nuclei

### Proposition

*Let $\mathbf{A} = (A, *, 1)$ be a Hilbert algebra and let $j$ be a nucleus on $\mathbf{A}$. Then*

(i) $j \colon A \to A$ *is an endomorphism of* $\mathbf{A}$ *and* $*_j$ *is the restriction of* $*$ *to* $A_j$*.*

(ii) *The algebra* $\mathbf{A}_j = (A_j, *_j, 1)$ *is a Hilbert algebra which is a subalgebra isomorphic to a quotient algebra of* $\mathbf{A}$*.*

(iii) *The quotient homomorphism* $j \colon A \to A_j$ *is a retract with respect to the inclusion homomorphism* $i \colon A_j \to A$*.*

## Nuclei

### Proposition

*Let $\mathbf{A} = (A, *, 1)$ be a Hilbert algebra and let $j$ be a nucleus on $\mathbf{A}$. Then*

(i) *$j\colon A \to A$ is an endomorphism of $\mathbf{A}$ and $*_j$ is the restriction of $*$ to $A_j$.*

(ii) *The algebra $\mathbf{A}_j = (A_j, *_j, 1)$ is a Hilbert algebra which is a subalgebra isomorphic to a quotient algebra of $\mathbf{A}$.*

(iii) *The quotient homomorphism $j\colon A \to A_j$ is a retract with respect to the inclusion homomorphism $i\colon A_j \to A$.*

## Nuclei and congruences

**Theorem**

*Let* $\mathbf{A} = (A, *, 1)$ *be a Hilbert algebra.*

(i) *Let* $j$ *be a nucleus on* $\mathbf{A}$*. Then the relation* $\theta_j$ *on* $A$ *defined by*

$$a\ \theta_j\ b \quad \text{if and only if} \quad j(a) = j(b),$$

$a, b \in A$*, is a congruence on* $\mathbf{A}$*. Moreover, any congruence class* $\overline{a}_{\theta_j}$ *has the greatest element* $j(a)$*.*

(ii) *Let* $\theta$ *be a congruence on* $\mathbf{A}$ *such that any congruence class* $\overline{a}_\theta$ *has a greatest element* $\widehat{a}_\theta$*. Then the mapping* $j_\theta : A \to A$ *defined by*

$$j_\theta(a) = \widehat{a}_\theta$$

*for all* $a \in A$ *is a nucleus on* $\mathbf{A}$*.*

## Nuclei and congruences

### Theorem

*Let $\mathbf{A} = (A, *, 1)$ be a Hilbert algebra.*

(i) *Let $j$ be a nucleus on $\mathbf{A}$. Then the relation $\theta_j$ on $A$ defined by*

$$a \, \theta_j \, b \quad \text{if and only if} \quad j(a) = j(b),$$

*$a, b \in A$, is a congruence on $\mathbf{A}$. Moreover, any congruence class $\overline{a}_{\theta_j}$ has the greatest element $j(a)$.*

(ii) *Let $\theta$ be a congruence on $\mathbf{A}$ such that any congruence class $\overline{a}_\theta$ has a greatest element $\widehat{a}_\theta$. Then the mapping $j_\theta : A \to A$ defined by*

$$j_\theta(a) = \widehat{a}_\theta$$

*for all $a \in A$ is a nucleus on $\mathbf{A}$.*

## Nuclei and congruences

**Theorem**

*Let* $\mathbf{A} = (A, *, 1)$ *be a Hilbert algebra.*

(i) *Let* $j$ *be a nucleus on* $\mathbf{A}$*. Then the relation* $\theta_j$ *on* $A$ *defined by*

$$a \,\theta_j\, b \quad \text{if and only if} \quad j(a) = j(b),$$

$a, b \in A$*, is a congruence on* $\mathbf{A}$*. Moreover, any congruence class* $\overline{a}_{\theta_j}$ *has the greatest element* $j(a)$*.*

(ii) *Let* $\theta$ *be a congruence on* $\mathbf{A}$ *such that any congruence class* $\overline{a}_\theta$ *has a greatest element* $\widehat{a}_\theta$*. Then the mapping* $j_\theta : A \to A$ *defined by*

$$j_\theta(a) = \widehat{a}_\theta$$

*for all* $a \in A$ *is a nucleus on* $\mathbf{A}$*.*

## Nuclei and congruences

Let $\mathbf{A} = (A, *, 1)$ be a Hilbert algebra.

We will denote by $\operatorname{Con}_n \mathbf{A}$ the set of congruences $\theta$ on $\mathbf{A}$ such that any congruence class of $\theta$ has a greatest element.

Similarly, we will denote by $\operatorname{Nuc} \mathbf{A}$ the set of nuclei $j$ on $\mathbf{A}$.

Both sets can be ordered, $\operatorname{Con}_n \mathbf{A}$ via inclusion of relations and $\operatorname{Nuc} \mathbf{A}$ as a subposet of $A^A$.

Due to J. Cīrulis we know that the set $\operatorname{Nuc} \mathbf{A}$ is closed under composition $\circ$ and pointwise defined meets.

The algebra $(\operatorname{Nuc} \mathbf{A}, \circ, \wedge, \mathrm{id}_A, \mathbf{1})$ is a bounded distributive lattice, in which $\circ$ acts as join and the natural ordering may be defined pointwise.

**Theorem**

*Let* $\mathbf{A} = (A, *, 1) \in \mathscr{R}$ *and* $j \in \operatorname{Nuc} \mathbf{A}$. *Then* $\mathbf{A}_j = (A_j, *_j, 1) \in \mathscr{R}$.

## Nuclei and congruences

Let $\mathbf{A} = (A, *, 1)$ be a Hilbert algebra.

We will denote by $\operatorname{Con}_{\mathrm{n}} \mathbf{A}$ the set of congruences $\theta$ on $\mathbf{A}$ such that any congruence class of $\theta$ has a greatest element.

Similarly, we will denote by $\operatorname{Nuc} \mathbf{A}$ the set of nuclei $j$ on $\mathbf{A}$.

Both sets can be ordered, $\operatorname{Con}_{\mathrm{n}} \mathbf{A}$ via inclusion of relations and $\operatorname{Nuc} \mathbf{A}$ as a subposet of $A^A$.

Due to J. Cīrulis we know that the set $\operatorname{Nuc} \mathbf{A}$ is closed under composition $\circ$ and pointwise defined meets.

The algebra $(\operatorname{Nuc} \mathbf{A}, \circ, \wedge, \mathrm{id}_A, \mathbf{1})$ is a bounded distributive lattice, in which $\circ$ acts as join and the natural ordering may be defined pointwise.

**Theorem**

*Let $\mathbf{A} = (A, *, 1) \in \mathscr{R}$ and $j \in \operatorname{Nuc} \mathbf{A}$. Then $\mathbf{A}_j = (A_j, *_j, 1) \in \mathscr{R}$.*

## Nuclei and congruences

Let $\mathbf{A} = (A, *, 1)$ be a Hilbert algebra.

We will denote by $\operatorname{Con}_{\mathrm{n}} \mathbf{A}$ the set of congruences $\theta$ on $\mathbf{A}$ such that any congruence class of $\theta$ has a greatest element.

Similarly, we will denote by $\operatorname{Nuc} \mathbf{A}$ the set of nuclei $j$ on $\mathbf{A}$.

Both sets can be ordered, $\operatorname{Con}_{\mathrm{n}} \mathbf{A}$ via inclusion of relations and $\operatorname{Nuc} \mathbf{A}$ as a subposet of $A^A$.

Due to J. Cīrulis we know that the set $\operatorname{Nuc} \mathbf{A}$ is closed under composition $\circ$ and pointwise defined meets.

The algebra $(\operatorname{Nuc} \mathbf{A}, \circ, \wedge, \mathrm{id}_A, \mathbf{1})$ is a bounded distributive lattice, in which $\circ$ acts as join and the natural ordering may be defined pointwise.

**Theorem**

*Let* $\mathbf{A} = (A, *, 1) \in \mathscr{R}$ *and* $j \in \operatorname{Nuc} \mathbf{A}$. *Then* $\mathbf{A}_j = (A_j, *_j, 1) \in \mathscr{R}$.

## Nuclei and congruences

Let $\mathbf{A} = (A, *, 1)$ be a Hilbert algebra.

We will denote by $\operatorname{Conn}\mathbf{A}$ the set of congruences $\theta$ on $\mathbf{A}$ such that any congruence class of $\theta$ has a greatest element.

Similarly, we will denote by $\operatorname{Nuc}\mathbf{A}$ the set of nuclei $j$ on $\mathbf{A}$.

Both sets can be ordered, $\operatorname{Conn}\mathbf{A}$ via inclusion of relations and $\operatorname{Nuc}\mathbf{A}$ as a subposet of $A^A$.

Due to J. Cīrulis we know that the set $\operatorname{Nuc}\mathbf{A}$ is closed under composition $\circ$ and pointwise defined meets.

The algebra $(\operatorname{Nuc}\mathbf{A}, \circ, \wedge, \mathrm{id}_A, \mathbf{1})$ is a bounded distributive lattice, in which $\circ$ acts as join and the natural ordering may be defined pointwise.

**Theorem**

*Let* $\mathbf{A} = (A, *, 1) \in \mathscr{R}$ *and* $j \in \operatorname{Nuc}\mathbf{A}$. *Then* $\mathbf{A}_j = (A_j, *_j, 1) \in \mathscr{R}$.

## Nuclei and congruences

Let $\mathbf{A} = (A, *, 1)$ be a Hilbert algebra.

We will denote by $\operatorname{Con}_n \mathbf{A}$ the set of congruences $\theta$ on $\mathbf{A}$ such that any congruence class of $\theta$ has a greatest element.

Similarly, we will denote by $\operatorname{Nuc} \mathbf{A}$ the set of nuclei $j$ on $\mathbf{A}$.

Both sets can be ordered, $\operatorname{Con}_n \mathbf{A}$ via inclusion of relations and $\operatorname{Nuc} \mathbf{A}$ as a subposet of $A^A$.

Due to J. Cīrulis we know that the set $\operatorname{Nuc} \mathbf{A}$ is closed under composition $\circ$ and pointwise defined meets.

The algebra $(\operatorname{Nuc} \mathbf{A}, \circ, \wedge, \mathrm{id}_A, \mathbf{1})$ is a bounded distributive lattice, in which $\circ$ acts as join and the natural ordering may be defined pointwise.

**Theorem**

*Let $\mathbf{A} = (A, *, 1) \in \mathscr{R}$ and $j \in \operatorname{Nuc} \mathbf{A}$. Then $\mathbf{A}_j = (A_j, *_j, 1) \in \mathscr{R}$.*

## Nuclei and congruences

Let $\mathbf{A} = (A, *, 1)$ be a Hilbert algebra.

We will denote by $\operatorname{Conn}\mathbf{A}$ the set of congruences $\theta$ on $\mathbf{A}$ such that any congruence class of $\theta$ has a greatest element.

Similarly, we will denote by $\operatorname{Nuc}\mathbf{A}$ the set of nuclei $j$ on $\mathbf{A}$.

Both sets can be ordered, $\operatorname{Conn}\mathbf{A}$ via inclusion of relations and $\operatorname{Nuc}\mathbf{A}$ as a subposet of $A^A$.

Due to J. Cīrulis we know that the set $\operatorname{Nuc}\mathbf{A}$ is closed under composition $\circ$ and pointwise defined meets.

The algebra $(\operatorname{Nuc}\mathbf{A}, \circ, \wedge, \mathrm{id}_A, \mathbf{1})$ is a bounded distributive lattice, in which $\circ$ acts as join and the natural ordering may be defined pointwise.

**Theorem**

*Let $\mathbf{A} = (A, *, 1) \in \mathscr{R}$ and $j \in \operatorname{Nuc}\mathbf{A}$. Then $\mathbf{A}_j = (A_j, *_j, 1) \in \mathscr{R}$.*

## Nuclei and congruences

Let $\mathbf{A} = (A, *, 1)$ be a Hilbert algebra.

We will denote by $\operatorname{Conn}\mathbf{A}$ the set of congruences $\theta$ on $\mathbf{A}$ such that any congruence class of $\theta$ has a greatest element.

Similarly, we will denote by $\operatorname{Nuc}\mathbf{A}$ the set of nuclei $j$ on $\mathbf{A}$.

Both sets can be ordered, $\operatorname{Conn}\mathbf{A}$ via inclusion of relations and $\operatorname{Nuc}\mathbf{A}$ as a subposet of $A^A$.

Due to J. Cīrulis we know that the set $\operatorname{Nuc}\mathbf{A}$ is closed under composition $\circ$ and pointwise defined meets.

The algebra $(\operatorname{Nuc}\mathbf{A}, \circ, \wedge, \mathrm{id}_A, \mathbf{1})$ is a bounded distributive lattice, in which $\circ$ acts as join and the natural ordering may be defined pointwise.

**Theorem**

*Let $\mathbf{A} = (A, *, 1) \in \mathscr{R}$ and $j \in \operatorname{Nuc}\mathbf{A}$. Then $\mathbf{A}_j = (A_j, *_j, 1) \in \mathscr{R}$.*

## Nuclei and congruences

Let $\mathbf{A} = (A, *, 1)$ be a Hilbert algebra.

We will denote by $\mathrm{Conn}\,\mathbf{A}$ the set of congruences $\theta$ on $\mathbf{A}$ such that any congruence class of $\theta$ has a greatest element.

Similarly, we will denote by $\mathrm{Nuc}\,\mathbf{A}$ the set of nuclei $j$ on $\mathbf{A}$.

Both sets can be ordered, $\mathrm{Conn}\,\mathbf{A}$ via inclusion of relations and $\mathrm{Nuc}\,\mathbf{A}$ as a subposet of $A^A$.

Due to J. Cīrulis we know that the set $\mathrm{Nuc}\,\mathbf{A}$ is closed under composition $\circ$ and pointwise defined meets.

The algebra $(\mathrm{Nuc}\,\mathbf{A}, \circ, \wedge, \mathrm{id}_A, \mathbf{1})$ is a bounded distributive lattice, in which $\circ$ acts as join and the natural ordering may be defined pointwise.

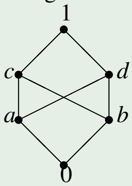
**Theorem**

*Let* $\mathbf{A} = (A, *, 1) \in \mathscr{R}$ *and* $j \in \mathrm{Nuc}\,\mathbf{A}$. *Then* $\mathbf{A}_j = (A_j, *_j, 1) \in \mathscr{R}$.

## Nuclei and congruences

### Example

The poset $\mathbf{A} = (A, \leq)$ with the Hasse diagram




is relatively pseudocomplemented and the operation table of $*$ looks as follows:

| $*$ | $0$ | $a$ | $b$ | $c$ | $d$ | $1$ |
|---|---|---|---|---|---|---|
| $0$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $a$ | $b$ | $1$ | $b$ | $1$ | $1$ | $1$ |
| $b$ | $a$ | $a$ | $1$ | $1$ | $1$ | $1$ |
| $c$ | $0$ | $a$ | $b$ | $1$ | $d$ | $1$ |
| $d$ | $0$ | $a$ | $b$ | $c$ | $1$ | $1$ |
| $1$ | $0$ | $a$ | $b$ | $c$ | $d$ | $1$ |

## Nuclei and congruences

### Example

We have $\Phi := \{0,b\}^2 \cup \{a,c,d,1\}^2 \in \operatorname{Con}\mathbf{A}$. Hence $(A/\Phi, *', \overline{1}_\Phi)$ is a relatively pseudocomplemented poset.

### Remark

Unfortunately, contrary to the class of Hilbert algebras, the class of relatively pseudocomplemented posets is not closed under substructures. Consider e.g. our relatively pseudocomplemented poset $\mathbf{A} = (A, *, 1)$ from previous example and its subset $S = \{0, c, d, 1\}$. It is immediate that $S$ is closed under $*$. However, this $*$ is not a relative pseudocomplementation in $\mathbf{S} = (S, *, 1)$. Namely, the relative pseudocomplement of $c$ with respect to $0$ in $\mathbf{S}$ is $d$ but $d$ differs from $0 = c * 0$.




| $*$ | $0$ | $c$ | $d$ | $1$ |
|---|---|---|---|---|
| $0$ | $1$ | $1$ | $1$ | $1$ |
| $c$ | $0$ | $1$ | $d$ | $1$ |
| $d$ | $0$ | $c$ | $1$ | $1$ |
| $1$ | $0$ | $c$ | $d$ | $1$ |

## Nuclei and congruences

### Example

We have $\Phi := \{0,b\}^2 \cup \{a,c,d,1\}^2 \in \operatorname{Con}\mathbf{A}$. Hence $(A/\Phi, *', \overline{1}_\Phi)$ is a relatively pseudocomplemented poset.

### Remark

Unfortunately, contrary to the class of Hilbert algebras, the class of relatively pseudocomplemented posets is not closed under substructures. Consider e.g. our relatively pseudocomplemented poset $\mathbf{A} = (A, *, 1)$ from previous example and its subset $S = \{0, c, d, 1\}$. It is immediate that $S$ is closed under $*$. However, this $*$ is not a relative pseudocomplementation in $\mathbf{S} = (S, *, 1)$. Namely, the relative pseudocomplement of $c$ with respect to 0 in $\mathbf{S}$ is $d$ but $d$ differs from $0 = c * 0$.




| $*$ | $0$ | $c$ | $d$ | $1$ |
|---|---|---|---|---|
| $0$ | $1$ | $1$ | $1$ | $1$ |
| $c$ | $0$ | $1$ | $d$ | $1$ |
| $d$ | $0$ | $c$ | $1$ | $1$ |
| $1$ | $0$ | $c$ | $d$ | $1$ |

## Nuclei and congruences

### Example

We have $\Phi := \{0,b\}^2 \cup \{a,c,d,1\}^2 \in \operatorname{Con}\mathbf{A}$. Hence $(A/\Phi, *', \overline{1}_\Phi)$ is a relatively pseudocomplemented poset.

### Remark

Unfortunately, contrary to the class of Hilbert algebras, the class of relatively pseudocomplemented posets is not closed under substructures. Consider e.g. our relatively pseudocomplemented poset $\mathbf{A} = (A, *, 1)$ from previous example and its subset $S = \{0, c, d, 1\}$. It is immediate that $S$ is closed under $*$. However, this $*$ is not a relative pseudocomplementation in $\mathbf{S} = (S, *, 1)$. Namely, the relative pseudocomplement of $c$ with respect to $0$ in $\mathbf{S}$ is $d$ but $d$ differs from $0 = c * 0$.




| $*$ | $0$ | $c$ | $d$ | $1$ |
|---|---|---|---|---|
| $0$ | $1$ | $1$ | $1$ | $1$ |
| $c$ | $0$ | $1$ | $d$ | $1$ |
| $d$ | $0$ | $c$ | $1$ | $1$ |
| $1$ | $0$ | $c$ | $d$ | $1$ |

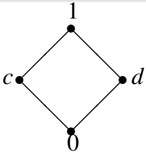
## Nuclei and congruences

### Example

We have $\Phi := \{0,b\}^2 \cup \{a,c,d,1\}^2 \in \operatorname{Con}\mathbf{A}$. Hence $(A/\Phi, *', \overline{1}_\Phi)$ is a relatively pseudocomplemented poset.

### Remark

Unfortunately, contrary to the class of Hilbert algebras, the class of relatively pseudocomplemented posets is not closed under substructures. Consider e.g. our relatively pseudocomplemented poset $\mathbf{A} = (A, *, 1)$ from previous example and its subset $S = \{0, c, d, 1\}$. It is immediate that $S$ is closed under $*$. However, this $*$ is not a relative pseudocomplementation in $\mathbf{S} = (S, *, 1)$. Namely, the relative pseudocomplement of $c$ with respect to $0$ in $\mathbf{S}$ is $d$ but $d$ differs from $0 = c * 0$.




| $*$ | $0$ | $c$ | $d$ | $1$ |
|---|---|---|---|---|
| $0$ | $1$ | $1$ | $1$ | $1$ |
| $c$ | $0$ | $1$ | $d$ | $1$ |
| $d$ | $0$ | $c$ | $1$ | $1$ |
| $1$ | $0$ | $c$ | $d$ | $1$ |

## Implicative filters

An implicative filter $F$ of a Hilbert algebra $\mathbf{A}$ is a non-empty subset of $A$ satisfying $y, y*z \in F$ implies $z \in F$.

Denote by $\mathrm{Fil}\,\mathbf{A}$ the set of all implicative filters of $\mathbf{A}$ and, for any subset $B$ of $A$, put $\theta_B := \{(x,y) \in A^2 \mid x*y, y*x \in B\}$.

**Theorem (Diego 1965,1966)**

*Let $\mathbf{A} = (A, *, 1)$ be a Hilbert algebra. Then the mappings $\Phi \mapsto \overline{1}_\Phi$ and $F \mapsto \theta_F$ are mutually inverse isomorphisms between the lattices $(\mathrm{Con}\,\mathbf{A}, \subseteq)$ and $(\mathrm{Fil}\,\mathbf{A}, \subseteq)$.*

## Implicative filters

An implicative filter $F$ of a Hilbert algebra $\mathbf{A}$ is a non-empty subset of $A$ satisfying $y, y * z \in F$ implies $z \in F$.

Denote by $\mathrm{Fil}\,\mathbf{A}$ the set of all implicative filters of $\mathbf{A}$ and, for any subset $B$ of $A$, put $\theta_B := \{(x,y) \in A^2 \mid x * y, y * x \in B\}$.

**Theorem (Diego 1965,1966)**

*Let $\mathbf{A} = (A, *, 1)$ be a Hilbert algebra. Then the mappings $\Phi \mapsto \overline{1}_\Phi$ and $F \mapsto \theta_F$ are mutually inverse isomorphisms between the lattices $(\mathrm{Con}\,\mathbf{A}, \subseteq)$ and $(\mathrm{Fil}\,\mathbf{A}, \subseteq)$.*

## Implicative filters

An implicative filter $F$ of a Hilbert algebra $\mathbf{A}$ is a non-empty subset of $A$ satisfying $y, y * z \in F$ implies $z \in F$.

Denote by $\mathrm{Fil}\,\mathbf{A}$ the set of all implicative filters of $\mathbf{A}$ and, for any subset $B$ of $A$, put $\theta_B := \{(x,y) \in A^2 \mid x * y, y * x \in B\}$.

**Theorem (Diego 1965,1966)**

*Let $\mathbf{A} = (A, *, 1)$ be a Hilbert algebra. Then the mappings $\Phi \mapsto \overline{1}_\Phi$ and $F \mapsto \theta_F$ are mutually inverse isomorphisms between the lattices $(\mathrm{Con}\,\mathbf{A}, \subseteq)$ and $(\mathrm{Fil}\,\mathbf{A}, \subseteq)$.*

## Implicative filters

An implicative filter $F$ of a Hilbert algebra $\mathbf{A}$ is a non-empty subset of $A$ satisfying $y, y*z \in F$ implies $z \in F$.

Denote by $\mathrm{Fil}\,\mathbf{A}$ the set of all implicative filters of $\mathbf{A}$ and, for any subset $B$ of $A$, put $\theta_B := \{(x,y) \in A^2 \mid x*y, y*x \in B\}$.

**Theorem (Diego 1965,1966)**

*Let $\mathbf{A} = (A, *, 1)$ be a Hilbert algebra. Then the mappings $\Phi \mapsto \overline{1}_\Phi$ and $F \mapsto \theta_F$ are mutually inverse isomorphisms between the lattices $(\mathrm{Con}\,\mathbf{A}, \subseteq)$ and $(\mathrm{Fil}\,\mathbf{A}, \subseteq)$.*

## Open nuclei

Let $\mathbf{A} = (A, *, 1)$ be a Hilbert algebra, $F \subseteq A$ an implicative filter of $\mathbf{A}$ and $a \in A$.

$\mathbf{F} = (F, *, 1)$ is a Hilbert subalgebra of $\mathbf{A}$.

We define a mapping $u_a^{\mathbf{F}} : F \to F$ as follows:

$$u_a^{\mathbf{F}}(x) = a * x$$

for all $x \in F$.

Since $F$ is an upper set in $\mathbf{A}$ our definition is correct.

It is well know that $u_a^{\mathbf{F}}$ is an endomorphism of $\mathbf{F}$ and a closure operator on $\mathbf{F}$.

We then define the following congruence $\theta_a^{\mathbf{F}}$ on $\mathbf{F}$:

$$x \; \theta_a^{\mathbf{F}} \; y \text{ if and only if } u_a^{\mathbf{F}}(x) = u_a^{\mathbf{F}}(y) \qquad (\clubsuit)$$

for all $x, y \in F$. If $A = F$ we put $\theta_a = \theta_a^{\mathbf{A}}$.

## Open nuclei

Let $\mathbf{A} = (A, *, 1)$ be a Hilbert algebra, $F \subseteq A$ an implicative filter of $\mathbf{A}$ and $a \in A$.

$\mathbf{F} = (F, *, 1)$ is a Hilbert subalgebra of $\mathbf{A}$.

We define a mapping $u_a^{\mathbf{F}} : F \to F$ as follows:

$$u_a^{\mathbf{F}}(x) = a * x$$

for all $x \in F$.

Since $F$ is an upper set in $\mathbf{A}$ our definition is correct.

It is well know that $u_a^{\mathbf{F}}$ is an endomorphism of $\mathbf{F}$ and a closure operator on $\mathbf{F}$.

We then define the following congruence $\theta_a^{\mathbf{F}}$ on $\mathbf{F}$:

$$x \, \theta_a^{\mathbf{F}} \, y \text{ if and only if } u_a^{\mathbf{F}}(x) = u_a^{\mathbf{F}}(y) \qquad (\clubsuit)$$

for all $x, y \in F$. If $A = F$ we put $\theta_a = \theta_a^{\mathbf{A}}$.

## Open nuclei

Let $\mathbf{A} = (A, *, 1)$ be a Hilbert algebra, $F \subseteq A$ an implicative filter of $\mathbf{A}$ and $a \in A$.

$\mathbf{F} = (F, *, 1)$ is a Hilbert subalgebra of $\mathbf{A}$.

We define a mapping $u_a^{\mathbf{F}} : F \to F$ as follows:

$$u_a^{\mathbf{F}}(x) = a * x$$

for all $x \in F$.

Since $F$ is an upper set in $\mathbf{A}$ our definition is correct.

It is well know that $u_a^{\mathbf{F}}$ is an endomorphism of $\mathbf{F}$ and a closure operator on $\mathbf{F}$.

We then define the following congruence $\theta_a^{\mathbf{F}}$ on $\mathbf{F}$:

$$x \ \theta_a^{\mathbf{F}} \ y \text{ if and only if } u_a^{\mathbf{F}}(x) = u_a^{\mathbf{F}}(y) \qquad (\clubsuit)$$

for all $x, y \in F$. If $A = F$ we put $\theta_a = \theta_a^{\mathbf{A}}$.

## Open nuclei

Let $\mathbf{A} = (A, *, 1)$ be a Hilbert algebra, $F \subseteq A$ an implicative filter of $\mathbf{A}$ and $a \in A$.

$\mathbf{F} = (F, *, 1)$ is a Hilbert subalgebra of $\mathbf{A}$.

We define a mapping $u_a^{\mathbf{F}} \colon F \to F$ as follows:

$$u_a^{\mathbf{F}}(x) = a * x$$

for all $x \in F$.

Since $F$ is an upper set in $\mathbf{A}$ our definition is correct.

It is well know that $u_a^{\mathbf{F}}$ is an endomorphism of $\mathbf{F}$ and a closure operator on $\mathbf{F}$.

We then define the following congruence $\theta_a^{\mathbf{F}}$ on $\mathbf{F}$:

$$x \, \theta_a^{\mathbf{F}} \, y \text{ if and only if } u_a^{\mathbf{F}}(x) = u_a^{\mathbf{F}}(y) \qquad (\clubsuit)$$

for all $x, y \in F$. If $A = F$ we put $\theta_a = \theta_a^{\mathbf{A}}$.

## Open nuclei

Let $\mathbf{A} = (A, *, 1)$ be a Hilbert algebra, $F \subseteq A$ an implicative filter of $\mathbf{A}$ and $a \in A$.

$\mathbf{F} = (F, *, 1)$ is a Hilbert subalgebra of $\mathbf{A}$.

We define a mapping $u_a^{\mathbf{F}} \colon F \to F$ as follows:

$$u_a^{\mathbf{F}}(x) = a * x$$

for all $x \in F$.

Since $F$ is an upper set in $\mathbf{A}$ our definition is correct.

It is well know that $u_a^{\mathbf{F}}$ is an endomorphism of $\mathbf{F}$ and a closure operator on $\mathbf{F}$.

We then define the following congruence $\theta_a^{\mathbf{F}}$ on $\mathbf{F}$:

$$x \, \theta_a^{\mathbf{F}} \, y \text{ if and only if } u_a^{\mathbf{F}}(x) = u_a^{\mathbf{F}}(y) \qquad (\clubsuit)$$

for all $x, y \in F$. If $A = F$ we put $\theta_a = \theta_a^{\mathbf{A}}$.

## Open nuclei

Let $\mathbf{A} = (A, *, 1)$ be a Hilbert algebra, $F \subseteq A$ an implicative filter of $\mathbf{A}$ and $a \in A$.

$\mathbf{F} = (F, *, 1)$ is a Hilbert subalgebra of $\mathbf{A}$.

We define a mapping $u_a^{\mathbf{F}} \colon F \to F$ as follows:

$$u_a^{\mathbf{F}}(x) = a * x$$

for all $x \in F$.

Since $F$ is an upper set in $\mathbf{A}$ our definition is correct.

It is well know that $u_a^{\mathbf{F}}$ is an endomorphism of $\mathbf{F}$ and a closure operator on $\mathbf{F}$.

We then define the following congruence $\theta_a^{\mathbf{F}}$ on $\mathbf{F}$:

$$x \, \theta_a^{\mathbf{F}} \, y \text{ if and only if } u_a^{\mathbf{F}}(x) = u_a^{\mathbf{F}}(y) \qquad (\clubsuit)$$

for all $x, y \in F$. If $A = F$ we put $\theta_a = \theta_a^{\mathbf{A}}$.

## Open nuclei

Let $\mathbf{A} = (A, *, 1)$ be a Hilbert algebra, $F \subseteq A$ an implicative filter of $\mathbf{A}$ and $a \in A$.

$\mathbf{F} = (F, *, 1)$ is a Hilbert subalgebra of $\mathbf{A}$.

We define a mapping $u_a^{\mathbf{F}} \colon F \to F$ as follows:

$$u_a^{\mathbf{F}}(x) = a * x$$

for all $x \in F$.

Since $F$ is an upper set in $\mathbf{A}$ our definition is correct.

It is well know that $u_a^{\mathbf{F}}$ is an endomorphism of $\mathbf{F}$ and a closure operator on $\mathbf{F}$.

We then define the following congruence $\theta_a^{\mathbf{F}}$ on $\mathbf{F}$:

$$x \, \theta_a^{\mathbf{F}} \, y \text{ if and only if } u_a^{\mathbf{F}}(x) = u_a^{\mathbf{F}}(y) \qquad (\clubsuit)$$

for all $x, y \in F$. If $A = F$ we put $\theta_a = \theta_a^{\mathbf{A}}$.

## Glivenko equivalence

Let $\mathbf{A} = (A, *, 1, 0)$ be a bounded Hilbert algebra.
We define a mapping $\gamma_0^{\mathbf{A}} : A \to A$ as follows:

$$\gamma_0^{\mathbf{A}}(x) = (x * 0) * 0$$

for all $x \in A$.
We then define the so-called *Glivenko equivalence* $\Gamma_0^{\mathbf{A}}$ on $A$ by

$$(x, y) \in \Gamma_0^{\mathbf{A}} \text{ if and only if } x * 0 = y * 0 \text{ if and only if } \gamma_0^{\mathbf{A}}(x) = \gamma_0^{\mathbf{A}}(y).$$

**Theorem (S. Rudeanu)**

*Let $\mathbf{A} = (A, *, 1)$ be a bounded Hilbert algebra with smallest element $0$. Then $\gamma_0^{\mathbf{A}} : A \to A$ is an endomorphism of $\mathbf{A}$ such that the identity $x * \gamma_0^{\mathbf{A}}(y) = \gamma_0^{\mathbf{A}}(x) * \gamma_0^{\mathbf{A}}(y)$ is satisfied, $\gamma_0^{\mathbf{A}} \in \mathrm{Nuc}\,\mathbf{A}$, $\Gamma_0^{\mathbf{A}} \in \mathrm{Conn}\,\mathbf{A}$ and $(A/\Gamma_0^{\mathbf{A}}, *', 1', 0')$ is a Boolean algebra.*

## Glivenko equivalence

Let $\mathbf{A} = (A, *, 1, 0)$ be a bounded Hilbert algebra.

We define a mapping $\gamma_0^{\mathbf{A}} : A \to A$ as follows:

$$\gamma_0^{\mathbf{A}}(x) = (x * 0) * 0$$

for all $x \in A$.

We then define the so-called *Glivenko equivalence* $\Gamma_0^{\mathbf{A}}$ on $A$ by

$$(x, y) \in \Gamma_0^{\mathbf{A}} \text{ if and only if } x * 0 = y * 0 \text{ if and only if } \gamma_0^{\mathbf{A}}(x) = \gamma_0^{\mathbf{A}}(y).$$

**Theorem (S. Rudeanu)**

*Let $\mathbf{A} = (A, *, 1)$ be a bounded Hilbert algebra with smallest element $0$. Then $\gamma_0^{\mathbf{A}} : A \to A$ is an endomorphism of $\mathbf{A}$ such that the identity $x * \gamma_0^{\mathbf{A}}(y) = \gamma_0^{\mathbf{A}}(x) * \gamma_0^{\mathbf{A}}(y)$ is satisfied, $\gamma_0^{\mathbf{A}} \in \mathrm{Nuc}\,\mathbf{A}$, $\Gamma_0^{\mathbf{A}} \in \mathrm{Conn}\,\mathbf{A}$ and $(A/\Gamma_0^{\mathbf{A}}, *', 1', 0')$ is a Boolean algebra.*

## Glivenko equivalence

Let $\mathbf{A} = (A, *, 1, 0)$ be a bounded Hilbert algebra.
We define a mapping $\gamma_0^{\mathbf{A}} : A \to A$ as follows:

$$\gamma_0^{\mathbf{A}}(x) = (x * 0) * 0$$

for all $x \in A$.

We then define the so-called *Glivenko equivalence* $\Gamma_0^{\mathbf{A}}$ on $A$ by

$$(x, y) \in \Gamma_0^{\mathbf{A}} \text{ if and only if } x * 0 = y * 0 \text{ if and only if } \gamma_0^{\mathbf{A}}(x) = \gamma_0^{\mathbf{A}}(y).$$

**Theorem (S. Rudeanu)**

*Let $\mathbf{A} = (A, *, 1)$ be a bounded Hilbert algebra with smallest element $0$. Then $\gamma_0^{\mathbf{A}} : A \to A$ is an endomorphism of $\mathbf{A}$ such that the identity $x * \gamma_0^{\mathbf{A}}(y) = \gamma_0^{\mathbf{A}}(x) * \gamma_0^{\mathbf{A}}(y)$ is satisfied, $\gamma_0^{\mathbf{A}} \in \mathrm{Nuc}\,\mathbf{A}$, $\Gamma_0^{\mathbf{A}} \in \mathrm{Conn}\,\mathbf{A}$ and $(A/\Gamma_0^{\mathbf{A}}, *', 1', 0')$ is a Boolean algebra.*

## Glivenko equivalence

Let $\mathbf{A} = (A, *, 1, 0)$ be a bounded Hilbert algebra.
We define a mapping $\gamma_0^{\mathbf{A}} : A \to A$ as follows:

$$\gamma_0^{\mathbf{A}}(x) = (x * 0) * 0$$

for all $x \in A$.
We then define the so-called *Glivenko equivalence* $\Gamma_0^{\mathbf{A}}$ on $A$ by

$$(x, y) \in \Gamma_0^{\mathbf{A}} \text{ if and only if } x * 0 = y * 0 \text{ if and only if } \gamma_0^{\mathbf{A}}(x) = \gamma_0^{\mathbf{A}}(y).$$

**Theorem (S. Rudeanu)**

*Let $\mathbf{A} = (A, *, 1)$ be a bounded Hilbert algebra with smallest element $0$. Then $\gamma_0^{\mathbf{A}} : A \to A$ is an endomorphism of $\mathbf{A}$ such that the identity $x * \gamma_0^{\mathbf{A}}(y) = \gamma_0^{\mathbf{A}}(x) * \gamma_0^{\mathbf{A}}(y)$ is satisfied, $\gamma_0^{\mathbf{A}} \in \operatorname{Nuc} \mathbf{A}$, $\Gamma_0^{\mathbf{A}} \in \operatorname{Conn} \mathbf{A}$ and $(A/\Gamma_0^{\mathbf{A}}, *', 1', 0')$ is a Boolean algebra.*

## Glivenko equivalence

Let $\mathbf{A} = (A, *, 1, 0)$ be a bounded Hilbert algebra.
We define a mapping $\gamma_0^{\mathbf{A}} : A \to A$ as follows:

$$\gamma_0^{\mathbf{A}}(x) = (x * 0) * 0$$

for all $x \in A$.
We then define the so-called *Glivenko equivalence* $\Gamma_0^{\mathbf{A}}$ on $A$ by

$$(x, y) \in \Gamma_0^{\mathbf{A}} \text{ if and only if } x * 0 = y * 0 \text{ if and only if } \gamma_0^{\mathbf{A}}(x) = \gamma_0^{\mathbf{A}}(y).$$

**Theorem (S. Rudeanu)**

*Let $\mathbf{A} = (A, *, 1)$ be a bounded Hilbert algebra with smallest element $0$. Then $\gamma_0^{\mathbf{A}} : A \to A$ is an endomorphism of $\mathbf{A}$ such that the identity $x * \gamma_0^{\mathbf{A}}(y) = \gamma_0^{\mathbf{A}}(x) * \gamma_0^{\mathbf{A}}(y)$ is satisfied, $\gamma_0^{\mathbf{A}} \in \mathrm{Nuc}\,\mathbf{A}$, $\Gamma_0^{\mathbf{A}} \in \mathrm{Conn}\,\mathbf{A}$ and $(A/\Gamma_0^{\mathbf{A}}, *', 1', 0')$ is a Boolean algebra.*

## Outline

## Characterizing quadruples

Let $\mathbf{A} = (A, *, 1, 0)$ be a bounded Hilbert algebra. In what follows, we shall denote, by $x^* = x * 0$ and by $B(\mathbf{A})$ the Boolean algebra $(B_0(\mathbf{A}), *, 1, 0)$.

For $a \in B(\mathbf{A})$, put

$$F_a := \{x \in A \mid x^{**} = a\}.$$

The sets $F_a$ are equivalence classes of the Glivenko equivalence on $\mathbf{A}$.

The greatest element of $F_a$ is $a$.

We say that an element $x \in A$ is dense if $x^* = 0$, i.e., if and only if $x \in F_1$.

We denote by $D(\mathbf{A})$ the set of all dense elements of $\mathbf{A}$.

Note that $B(\mathbf{A}) \cap D(\mathbf{A}) = \{1\}$, $D(\mathbf{A})$ is an implicative filter on $\mathbf{A}$ and $(D(\mathbf{A}), *, 1)$ is a Hilbert subalgebra of $\mathbf{A}$.

## Characterizing quadruples

Let $\mathbf{A} = (A, *, 1, 0)$ be a bounded Hilbert algebra. In what follows, we shall denote, by $x^* = x * 0$ and by $B(\mathbf{A})$ the Boolean algebra $(B_0(\mathbf{A}), *, 1, 0)$.

For $a \in B(\mathbf{A})$, put

$$F_a := \{x \in A \mid x^{**} = a\}.$$

The sets $F_a$ are equivalence classes of the Glivenko equivalence on $\mathbf{A}$.

The greatest element of $F_a$ is $a$.

We say that an element $x \in A$ is dense if $x^* = 0$, i.e., if and only if $x \in F_1$.

We denote by $D(\mathbf{A})$ the set of all dense elements of $\mathbf{A}$.

Note that $B(\mathbf{A}) \cap D(\mathbf{A}) = \{1\}$, $D(\mathbf{A})$ is an implicative filter on $\mathbf{A}$ and $(D(\mathbf{A}), *, 1)$ is a Hilbert subalgebra of $\mathbf{A}$.

## Characterizing quadruples

Let $\mathbf{A} = (A, *, 1, 0)$ be a bounded Hilbert algebra. In what follows, we shall denote, by $x^* = x * 0$ and by $B(\mathbf{A})$ the Boolean algebra $(B_0(\mathbf{A}), *, 1, 0)$.

For $a \in B(\mathbf{A})$, put

$$F_a := \{x \in A \mid x^{**} = a\}.$$

The sets $F_a$ are equivalence classes of the Glivenko equivalence on $\mathbf{A}$.

The greatest element of $F_a$ is $a$.

We say that an element $x \in A$ is *dense* if $x^* = 0$, i.e., if and only if $x \in F_1$.

We denote by $D(\mathbf{A})$ the set of all dense elements of $\mathbf{A}$.

Note that $B(\mathbf{A}) \cap D(\mathbf{A}) = \{1\}$, $D(\mathbf{A})$ is an implicative filter on $\mathbf{A}$ and $(D(\mathbf{A}), *, 1)$ is a Hilbert subalgebra of $\mathbf{A}$.

## Characterizing quadruples

Let $\mathbf{A} = (A, *, 1, 0)$ be a bounded Hilbert algebra. In what follows, we shall denote, by $x^* = x * 0$ and by $B(\mathbf{A})$ the Boolean algebra $(B_0(\mathbf{A}), *, 1, 0)$.

For $a \in B(\mathbf{A})$, put

$$F_a := \{x \in A \mid x^{**} = a\}.$$

The sets $F_a$ are equivalence classes of the Glivenko equivalence on $\mathbf{A}$.

The greatest element of $F_a$ is $a$.

## Characterizing quadruples

Let $\mathbf{A} = (A, *, 1, 0)$ be a bounded Hilbert algebra. In what follows, we shall denote, by $x^* = x * 0$ and by $B(\mathbf{A})$ the Boolean algebra $(B_0(\mathbf{A}), *, 1, 0)$.

For $a \in B(\mathbf{A})$, put

$$F_a := \{x \in A \mid x^{**} = a\}.$$

The sets $F_a$ are equivalence classes of the Glivenko equivalence on $\mathbf{A}$.

The greatest element of $F_a$ is $a$.

We say that an element $x \in A$ is *dense* if $x^* = 0$, i.e., if and only if $x \in F_1$.

We denote by $D(\mathbf{A})$ the set of all dense elements of $\mathbf{A}$.

Note that $B(\mathbf{A}) \cap D(\mathbf{A}) = \{1\}$; $D(\mathbf{A})$ is an implicative filter on $\mathbf{A}$ and $(D(\mathbf{A}), *, 1)$ is a Hilbert subalgebra of $\mathbf{A}$.

## Characterizing quadruples

Let $\mathbf{A} = (A, *, 1, 0)$ be a bounded Hilbert algebra. In what follows, we shall denote, by $x^* = x * 0$ and by $B(\mathbf{A})$ the Boolean algebra $(B_0(\mathbf{A}), *, 1, 0)$.

For $a \in B(\mathbf{A})$, put

$$F_a := \{x \in A \mid x^{**} = a\}.$$

The sets $F_a$ are equivalence classes of the Glivenko equivalence on $\mathbf{A}$.

The greatest element of $F_a$ is $a$.

We say that an element $x \in A$ is *dense* if $x^* = 0$, i.e., if and only if $x \in F_1$.

We denote by $D(\mathbf{A})$ the set of all dense elements of $\mathbf{A}$.

Note that $B(\mathbf{A}) \cap D(\mathbf{A}) = \{1\}$, $D(\mathbf{A})$ is an implicative filter on $\mathbf{A}$ and $(D(\mathbf{A}), *, 1)$ is a Hilbert subalgebra of $\mathbf{A}$.

## Characterizing quadruples

Let $\mathbf{A} = (A, *, 1, 0)$ be a bounded Hilbert algebra. In what follows, we shall denote, by $x^* = x * 0$ and by $B(\mathbf{A})$ the Boolean algebra $(B_0(\mathbf{A}), *, 1, 0)$.

For $a \in B(\mathbf{A})$, put

$$F_a := \{x \in A \mid x^{**} = a\}.$$

The sets $F_a$ are equivalence classes of the Glivenko equivalence on $\mathbf{A}$.

The greatest element of $F_a$ is $a$.

We say that an element $x \in A$ is *dense* if $x^* = 0$, i.e., if and only if $x \in F_1$.

We denote by $D(\mathbf{A})$ the set of all dense elements of $\mathbf{A}$.

Note that $B(\mathbf{A}) \cap D(\mathbf{A}) = \{1\}$, $D(\mathbf{A})$ is an implicative filter on $\mathbf{A}$ and $(D(\mathbf{A}), *, 1)$ is a Hilbert subalgebra of $\mathbf{A}$.

## Characterizing quadruples

Let $\mathbf{A} = (A, *, 1, 0)$ be a bounded Hilbert algebra. In what follows, we shall denote, by $x^* = x * 0$ and by $B(\mathbf{A})$ the Boolean algebra $(B_0(\mathbf{A}), *, 1, 0)$.

For $a \in B(\mathbf{A})$, put

$$F_a := \{x \in A \mid x^{**} = a\}.$$

The sets $F_a$ are equivalence classes of the Glivenko equivalence on $\mathbf{A}$.

The greatest element of $F_a$ is $a$.

We say that an element $x \in A$ is *dense* if $x^* = 0$, i.e., if and only if $x \in F_1$.

We denote by $D(\mathbf{A})$ the set of all dense elements of $\mathbf{A}$.

Note that $B(\mathbf{A}) \cap D(\mathbf{A}) = \{1\}$, $D(\mathbf{A})$ is an implicative filter on $\mathbf{A}$ and $(D(\mathbf{A}), *, 1)$ is a Hilbert subalgebra of $\mathbf{A}$.

## Characterizing quadruples

Let $\mathbf{A} = (A, *, 1, 0)$ be a bounded Hilbert algebra. In what follows, we shall denote, by $x^* = x * 0$ and by $B(\mathbf{A})$ the Boolean algebra $(B_0(\mathbf{A}), *, 1, 0)$.

For $a \in B(\mathbf{A})$, put

$$F_a := \{x \in A \mid x^{**} = a\}.$$

The sets $F_a$ are equivalence classes of the Glivenko equivalence on $\mathbf{A}$.

The greatest element of $F_a$ is $a$.

We say that an element $x \in A$ is *dense* if $x^* = 0$, i.e., if and only if $x \in F_1$.

We denote by $D(\mathbf{A})$ the set of all dense elements of $\mathbf{A}$.

Note that $B(\mathbf{A}) \cap D(\mathbf{A}) = \{1\}$, $D(\mathbf{A})$ is an implicative filter on $\mathbf{A}$ and $(D(\mathbf{A}), *, 1)$ is a Hilbert subalgebra of $\mathbf{A}$.

## Characterizing quadruples

Let us consider the mapping $h_a\colon F_a \to D(\mathbf{A})$ given by $x \mapsto a * x$, $x \in F_a$.

Every $x \in F_a$ is uniquely determined by $a * x$, i.e., $x = a \curlywedge (a * x)$.

Hence $h_a$ is an injective isotone mapping.

Denote by $D_a$ the set $h_a(F_a)$.

Then $h_a$ is an order isomorphism between $F_a$ and $D_a$ and

$$a * h_a(x) = h_a(h_a(x)) = h_a(x).$$

## Characterizing quadruples

Let us consider the mapping $h_a\colon F_a \to D(\mathbf{A})$ given by $x \mapsto a * x$, $x \in F_a$.

Every $x \in F_a$ is uniquely determined by $a * x$, i.e., $x = a \curlywedge (a * x)$.

Hence $h_a$ is an injective isotone mapping.

Denote by $D_a$ the set $h_a(F_a)$.

Then $h_a$ is an order isomorphism between $F_a$ and $D_a$ and

$$a * h_a(x) = h_a(h_a(x)) = h_a(x).$$

## Characterizing quadruples

Let us consider the mapping $h_a\colon F_a \to D(\mathbf{A})$ given by $x \mapsto a * x$, $x \in F_a$.

Every $x \in F_a$ is uniquely determined by $a * x$, i.e., $x = a \curlywedge (a * x)$.

Hence $h_a$ is an injective isotone mapping.

Denote by $D_a$ the set $h_a(F_a)$.

Then $h_a$ is an order isomorphism between $F_a$ and $D_a$ and

$$a * h_a(x) = h_a(h_a(x)) = h_a(x).$$

## Characterizing quadruples

Let us consider the mapping $h_a \colon F_a \to D(\mathbf{A})$ given by $x \mapsto a * x$, $x \in F_a$.

Every $x \in F_a$ is uniquely determined by $a * x$, i.e., $x = a \curlywedge (a * x)$.

Hence $h_a$ is an injective isotone mapping.

Denote by $D_a$ the set $h_a(F_a)$.

Then $h_a$ is an order isomorphism between $F_a$ and $D_a$ and

$$a * h_a(x) = h_a(h_a(x)) = h_a(x).$$

## Characterizing quadruples

Let us consider the mapping $h_a\colon F_a \to D(\mathbf{A})$ given by $x \mapsto a * x$, $x \in F_a$.

Every $x \in F_a$ is uniquely determined by $a * x$, i.e., $x = a \curlywedge (a * x)$.

Hence $h_a$ is an injective isotone mapping.

Denote by $D_a$ the set $h_a(F_a)$.

Then $h_a$ is an order isomorphism between $F_a$ and $D_a$ and

$$a * h_a(x) = h_a(h_a(x)) = h_a(x).$$

## Characterizing quadruples

Let us consider the mapping $h_a\colon F_a \to D(\mathbf{A})$ given by $x \mapsto a * x$, $x \in F_a$.

Every $x \in F_a$ is uniquely determined by $a * x$, i.e., $x = a \curlywedge (a * x)$.

Hence $h_a$ is an injective isotone mapping.

Denote by $D_a$ the set $h_a(F_a)$.

Then $h_a$ is an order isomorphism between $F_a$ and $D_a$ and

$$a * h_a(x) = h_a(h_a(x)) = h_a(x).$$

$$\begin{aligned} D_a &= \{d \in D(\mathbf{A}) \mid d \wedge a \text{ exists in } \mathbf{A} \text{ and } a^* \le d = a * (d \wedge a)\} \\ &= \{d \in D(\mathbf{A}) \mid a\,C\,d \text{ and } a * d = d\}. \end{aligned}$$

## Characterizing quadruples

Let us consider the mapping $h_a\colon F_a \to D(\mathbf{A})$ given by $x \mapsto a * x$, $x \in F_a$.

Every $x \in F_a$ is uniquely determined by $a * x$, i.e., $x = a \curlywedge (a * x)$.

Hence $h_a$ is an injective isotone mapping.

Denote by $D_a$ the set $h_a(F_a)$.

Then $h_a$ is an order isomorphism between $F_a$ and $D_a$ and

$$a * h_a(x) = h_a(h_a(x)) = h_a(x).$$

$$\begin{aligned} D_a &= \{d \in D(\mathbf{A}) \mid d \wedge a \text{ exists in } \mathbf{A} \text{ and } a^* \leq d = a * (d \wedge a)\} \\ &= \{d \in D(\mathbf{A}) \mid a\, C\, d \text{ and } a{*}d = d\}. \end{aligned}$$

Any $x \in A$ is completely determined by a pair $(a, d)$, where $a = x^{**} \in B(\mathbf{A})$ and $d = a * x = a * d \in D_a$, and every such pair $(a, d)$, $a \in B(\mathbf{A})$, $d \in D_a$ determines exactly one element $x$ of $F_a$ such that $x = d \curlywedge a$.

## Characterizing quadruples

Let us consider the mapping $h_a\colon F_a \to D(\mathbf{A})$ given by $x \mapsto a*x$, $x \in F_a$.

Every $x \in F_a$ is uniquely determined by $a*x$, i.e., $x = a \curlywedge (a*x)$.

Hence $h_a$ is an injective isotone mapping.

Denote by $D_a$ the set $h_a(F_a)$.

Then $h_a$ is an order isomorphism between $F_a$ and $D_a$ and

$$a*h_a(x) = h_a(h_a(x)) = h_a(x).$$

$$\begin{aligned} D_a &= \{d \in D(\mathbf{A}) \mid d \wedge a \text{ exists in } \mathbf{A} \text{ and } a^* \le d = a*(d \wedge a)\} \\ &= \{d \in D(\mathbf{A}) \mid a\,C\,d \text{ and } a*d = d\}. \end{aligned}$$

Any $x \in A$ is completely determined by a pair $(a,d)$, where $a = x^{**} \in B(\mathbf{A})$ and $d = a*x = a*d \in D_a$, and every such pair $(a,d)$, $a \in B(\mathbf{A})$, $d \in D_a$ determines exactly one element $x$ of $F_a$ such that $x = d \curlywedge a$.

## Characterizing quadruples

Let us consider the mapping $h_a\colon F_a \to D(\mathbf{A})$ given by $x \mapsto a * x$, $x \in F_a$.

Every $x \in F_a$ is uniquely determined by $a * x$, i.e., $x = a \mathbin{\curlywedge} (a * x)$.

Hence $h_a$ is an injective isotone mapping.

Denote by $D_a$ the set $h_a(F_a)$.

Then $h_a$ is an order isomorphism between $F_a$ and $D_a$ and

$$a * h_a(x) = h_a(h_a(x)) = h_a(x).$$

$$\begin{aligned} D_a &= \{d \in D(\mathbf{A}) \mid d \wedge a \text{ exists in } \mathbf{A} \text{ and } a^* \le d = a * (d \wedge a)\} \\ &= \{d \in D(\mathbf{A}) \mid a\, C\, d \text{ and } a * d = d\}. \end{aligned}$$

Any $x \in A$ is completely determined by a pair $(a, d)$, where $a = x^{**} \in B(\mathbf{A})$ and $d = a * x = a * d \in D_a$, and every such pair $(a, d)$, $a \in B(\mathbf{A})$, $d \in D_a$ determines exactly one element $x$ of $F_a$ such that $x = d \mathbin{\curlywedge} a$.

## Characterizing quadruples

We have compatibility relation $C(\mathbf{A}) \subseteq C \cap (B(\mathbf{A}) \times D(\mathbf{A}))$ and $C(\mathbf{A}) = \bigcup_{a \in B(\mathbf{A})} \{a\} \times D_a = \{(a,d) \in C \cap (B(\mathbf{A}) \times D(\mathbf{A})) \mid u_a^{D(\mathbf{A})}(d) = d\}$.
It holds $(\{1\} \times D(\mathbf{A})) \cup (B(\mathbf{A}) \times \{1\}) \subseteq C(\mathbf{A})$.

For any $a \in B(\mathbf{A})$ we have that $u_a^{D(\mathbf{A})} = a * (-)$ is a nucleus on $D(\mathbf{A})$ and we obtain a join-preserving mapping $\overline{\varphi} : B(\mathbf{A}) \to \mathrm{Nuc}\,D(\mathbf{A})$ defined by $\overline{\varphi}(a) = u_{a^*}^{D(\mathbf{A})}$ such that $\overline{\varphi}(0) = u_1^{D(\mathbf{A})} = \mathrm{id}_A$ and $\overline{\varphi}(1) = u_0^{D(\mathbf{A})} = 1$.

The compatibility relation $C(\mathbf{A})$ has the following property:

$$(a,d), (b,e) \in C(\mathbf{A}) \implies (a * b, \overline{\varphi}(a^*)(d * e)) \in C(\mathbf{A}).$$

## Characterizing quadruples

We have compatibility relation $C(\mathbf{A}) \subseteq C \cap (B(\mathbf{A}) \times D(\mathbf{A}))$ and $C(\mathbf{A}) = \bigcup_{a \in B(\mathbf{A})} \{a\} \times D_a = \{(a,d) \in C \cap (B(\mathbf{A}) \times D(\mathbf{A})) \mid u_a^{D(\mathbf{A})}(d) = d\}$.

It holds $(\{1\} \times D(\mathbf{A})) \cup (B(\mathbf{A}) \times \{1\}) \subseteq C(\mathbf{A})$.

## Characterizing quadruples

We have compatibility relation $C(\mathbf{A}) \subseteq C \cap (B(\mathbf{A}) \times D(\mathbf{A}))$ and $C(\mathbf{A}) = \bigcup_{a \in B(\mathbf{A})} \{a\} \times D_a = \{(a,d) \in C \cap (B(\mathbf{A}) \times D(\mathbf{A})) \mid u_a^{D(\mathbf{A})}(d) = d\}$.
It holds $(\{1\} \times D(\mathbf{A})) \cup (B(\mathbf{A}) \times \{1\}) \subseteq C(\mathbf{A})$.

For any $a \in B(\mathbf{A})$ we have that $u_a^{D(\mathbf{A})} = a * (-)$ is a nucleus on $D(\mathbf{A})$ and we obtain a join-preserving mapping $\overline{\varphi}\colon B(\mathbf{A}) \to \mathrm{Nuc}\, D(\mathbf{A})$ defined by $\overline{\varphi}(a) = u_{a^*}^{D(\mathbf{A})}$ such that $\overline{\varphi}(0) = u_1^{D(\mathbf{A})} = \mathrm{id}_A$ and $\overline{\varphi}(1) = u_0^{D(\mathbf{A})} = 1$.

The compatibility relation $C(\mathbf{A})$ has the following property:

$$(a,d),(b,e) \in C(\mathbf{A}) \implies (a * b, \varphi(a^*)(d * e)) \in C(\mathbf{A}).$$

## Characterizing quadruples

We have compatibility relation $C(\mathbf{A}) \subseteq C \cap (B(\mathbf{A}) \times D(\mathbf{A}))$ and $C(\mathbf{A}) = \bigcup_{a \in B(\mathbf{A})} \{a\} \times D_a = \{(a,d) \in C \cap (B(\mathbf{A}) \times D(\mathbf{A})) \mid u_a^{D(\mathbf{A})}(d) = d\}$.
It holds $(\{1\} \times D(\mathbf{A})) \cup (B(\mathbf{A}) \times \{1\}) \subseteq C(\mathbf{A})$.

For any $a \in B(\mathbf{A})$ we have that $u_a^{D(\mathbf{A})} = a * (-)$ is a nucleus on $D(\mathbf{A})$ and we obtain a join-preserving mapping $\overline{\varphi} \colon B(\mathbf{A}) \to \operatorname{Nuc} D(\mathbf{A})$ defined by $\overline{\varphi}(a) = u_{a^*}^{D(\mathbf{A})}$ such that $\overline{\varphi}(0) = u_1^{D(\mathbf{A})} = \mathrm{id}_A$ and $\overline{\varphi}(1) = u_0^{D(\mathbf{A})} = \mathbf{1}$.

The compatibility relation $C(\mathbf{A})$ has the following property

## Characterizing quadruples

We have compatibility relation $C(\mathbf{A}) \subseteq C \cap (B(\mathbf{A}) \times D(\mathbf{A}))$ and
$C(\mathbf{A}) = \bigcup_{a \in B(\mathbf{A})} \{a\} \times D_a = \{(a,d) \in C \cap (B(\mathbf{A}) \times D(\mathbf{A})) \mid u_a^{D(\mathbf{A})}(d) = d\}$.
It holds $(\{1\} \times D(\mathbf{A})) \cup (B(\mathbf{A}) \times \{1\}) \subseteq C(\mathbf{A})$.

For any $a \in B(\mathbf{A})$ we have that $u_a^{D(\mathbf{A})} = a * (-)$ is a nucleus on $D(\mathbf{A})$ and we obtain a join-preserving mapping $\overline{\varphi} \colon B(\mathbf{A}) \to \mathrm{Nuc}\, D(\mathbf{A})$ defined by $\overline{\varphi}(a) = u_{a^*}^{D(\mathbf{A})}$ such that $\overline{\varphi}(0) = u_1^{D(\mathbf{A})} = \mathrm{id}_A$ and $\overline{\varphi}(1) = u_0^{D(\mathbf{A})} = \mathbf{1}$.

The compatibility relation $C(\mathbf{A})$ has the following property:

$$(a,d),(b,e) \in C(\mathbf{A}) \implies (a*b, \overline{\varphi}(a^*)(d*e)) \in C(\mathbf{A}).$$

## Characterizing quadruples

We have compatibility relation $C(\mathbf{A}) \subseteq C \cap (B(\mathbf{A}) \times D(\mathbf{A}))$ and $C(\mathbf{A}) = \bigcup_{a \in B(\mathbf{A})} \{a\} \times D_a = \{(a,d) \in C \cap (B(\mathbf{A}) \times D(\mathbf{A})) \mid u_a^{D(\mathbf{A})}(d) = d\}$.
It holds $(\{1\} \times D(\mathbf{A})) \cup (B(\mathbf{A}) \times \{1\}) \subseteq C(\mathbf{A})$.

For any $a \in B(\mathbf{A})$ we have that $u_a^{D(\mathbf{A})} = a * (-)$ is a nucleus on $D(\mathbf{A})$ and we obtain a join-preserving mapping $\overline{\varphi}\colon B(\mathbf{A}) \to \mathrm{Nuc}\, D(\mathbf{A})$ defined by $\overline{\varphi}(a) = u_{a^*}^{D(\mathbf{A})}$ such that $\overline{\varphi}(0) = u_1^{D(\mathbf{A})} = \mathrm{id}_A$ and $\overline{\varphi}(1) = u_0^{D(\mathbf{A})} = \mathbf{1}$.

The compatibility relation $C(\mathbf{A})$ has the following property:

$$(a,d),(b,e) \in C(\mathbf{A}) \implies (a*b, \overline{\varphi}(a^*)(d*e)) \in C(\mathbf{A}).$$

## Characterizing quadruples

We have compatibility relation $C(\mathbf{A}) \subseteq C \cap (B(\mathbf{A}) \times D(\mathbf{A}))$ and
$C(\mathbf{A}) = \bigcup_{a \in B(\mathbf{A})} \{a\} \times D_a = \{(a,d) \in C \cap (B(\mathbf{A}) \times D(\mathbf{A})) \mid u_a^{D(\mathbf{A})}(d) = d\}$.
It holds $(\{1\} \times D(\mathbf{A})) \cup (B(\mathbf{A}) \times \{1\}) \subseteq C(\mathbf{A})$.

For any $a \in B(\mathbf{A})$ we have that $u_a^{D(\mathbf{A})} = a * (-)$ is a nucleus on $D(\mathbf{A})$ and we obtain a join-preserving mapping $\overline{\varphi}\colon B(\mathbf{A}) \to \mathrm{Nuc}\, D(\mathbf{A})$ defined by $\overline{\varphi}(a) = u_{a^*}^{D(\mathbf{A})}$ such that $\overline{\varphi}(0) = u_1^{D(\mathbf{A})} = \mathrm{id}_A$ and $\overline{\varphi}(1) = u_0^{D(\mathbf{A})} = \mathbf{1}$.

The compatibility relation $C(\mathbf{A})$ has the following property:

$$(a,d),(b,e) \in C(\mathbf{A}) \implies (a*b, \overline{\varphi}(a^*)(d*e)) \in C(\mathbf{A}).$$

## Characterizing quadruples

### Definition

A *characterizing quadruple* is a quadruple $(\mathbf{B}, \mathbf{D}, \mathbf{C}, \varphi)$, where

(i) $\mathbf{B} = (B, *_{\mathbf{B}}, 1, 0)$ is a Boolean algebra;

(ii) $\mathbf{D} = (D, *_{\mathbf{D}}, 1)$ is a Hilbert algebra;

(iii) $\varphi : \mathbf{B} \to \operatorname{Nuc}\mathbf{D}$ is a join-preserving mapping that preserves both the least and the greatest element.

(iv) $\mathbf{C} \subseteq B \times D$ such that

$$(a,d),(b,e) \in \mathbf{C} \implies (a *_{\mathbf{B}} b, \varphi(a^*)(d *_{\mathbf{D}} e)) \in \mathbf{C}, \varphi(a^*)(d) = d$$

and

$$(\{1\} \times \mathbf{D}) \cup (\mathbf{B} \times \{1\}) \subseteq \mathbf{C}.$$

## Characterizing quadruples

### Definition

A *characterizing quadruple* is a quadruple $(\mathbf{B}, \mathbf{D}, \mathbf{C}, \varphi)$, where

(i) $\mathbf{B} = (B, *_{\mathbf{B}}, 1, 0)$ is a Boolean algebra;

(ii) $\mathbf{D} = (D, *_{\mathbf{D}}, 1)$ is a Hilbert algebra;

(iii) $\varphi\colon \mathbf{B} \to \operatorname{Nuc}\mathbf{D}$ is a join-preserving mapping that preserves both the least and the greatest element.

(iv) $\mathbf{C} \subseteq B \times D$ such that

$$(a, d), (b, e) \in \mathbf{C} \implies (a *_{\mathbf{B}} b, \varphi(a^*)(d *_{\mathbf{D}} e)) \in \mathbf{C}, \varphi(a^*)(d) = d$$

and

$$(\{1\} \times \mathbf{D}) \cup (\mathbf{B} \times \{1\}) \subseteq \mathbf{C}.$$

## Characterizing quadruples

### Definition

A *characterizing quadruple* is a quadruple $(\mathbf{B}, \mathbf{D}, \mathbf{C}, \varphi)$, where

(i) $\mathbf{B} = (B, *_{\mathbf{B}}, 1, 0)$ is a Boolean algebra;

(ii) $\mathbf{D} = (D, *_{\mathbf{D}}, 1)$ is a Hilbert algebra;

(iii) $\varphi \colon \mathbf{B} \to \operatorname{Nuc} \mathbf{D}$ is a join-preserving mapping that preserves both the least and the greatest element.

(iv) $\mathbf{C} \subseteq B \times D$ such that

$$(a,d),(b,e) \in \mathbf{C} \implies (a *_{\mathbf{B}} b, \varphi(a^*)(d *_{\mathbf{D}} e)) \in \mathbf{C}, \varphi(a^*)(d) = d$$

and

$$(\{1\} \times \mathbf{D}) \cup (\mathbf{B} \times \{1\}) \subseteq \mathbf{C}.$$

## Characterizing quadruples

### Definition

A *characterizing quadruple* is a quadruple $(\mathbf{B}, \mathbf{D}, \mathbf{C}, \varphi)$, where

(i) $\mathbf{B} = (B, *_{\mathbf{B}}, 1, 0)$ is a Boolean algebra;

(ii) $\mathbf{D} = (D, *_{\mathbf{D}}, 1)$ is a Hilbert algebra;

(iii) $\varphi\colon \mathbf{B} \to \operatorname{Nuc}\mathbf{D}$ is a join-preserving mapping that preserves both the least and the greatest element.

(iv) $\mathbf{C} \subseteq B \times D$ such that

$$(a,d),(b,e) \in \mathbf{C} \implies (a *_{\mathbf{B}} b, \varphi(a^*)(d *_{\mathbf{D}} e)) \in \mathbf{C}, \varphi(a^*)(d) = d$$

and

$$(\{1\} \times \mathbf{D}) \cup (\mathbf{B} \times \{1\}) \subseteq \mathbf{C}.$$

## Characterizing quadruples

**Definition**

A *characterizing quadruple* is a quadruple $(\mathbf{B}, \mathbf{D}, \mathbf{C}, \varphi)$, where

(i) $\mathbf{B} = (B, *_{\mathbf{B}}, 1, 0)$ is a Boolean algebra;

(ii) $\mathbf{D} = (D, *_{\mathbf{D}}, 1)$ is a Hilbert algebra;

(iii) $\varphi \colon \mathbf{B} \to \operatorname{Nuc}\mathbf{D}$ is a join-preserving mapping that preserves both the least and the greatest element.

(iv) $\mathbf{C} \subseteq B \times D$ such that

$$(a,d),(b,e) \in \mathbf{C} \implies (a *_{\mathbf{B}} b, \varphi(a^*)(d *_{\mathbf{D}} e)) \in \mathbf{C}, \varphi(a^*)(d) = d$$

and

$$(\{1\} \times \mathbf{D}) \cup (\mathbf{B} \times \{1\}) \subseteq \mathbf{C}.$$

## Characterizing quadruples

Note that if $\mathbf{A}$ is a bounded implicative semilattice then our definition of a quadruple will coincide with that of a triple as defined by Nemitz or by Chajda, Halaš and Kühr.

The relation $C(\mathbf{A}) = \{(a,d) \in B(\mathbf{A}) \times D(\mathbf{A}) \mid u_a^{D(\mathbf{A})}(d) = d\}$ is uniquely determined (hence there is no need for a quadruple).

**Theorem**

*Let $\mathbf{A} = (A, *, 1, 0)$ be a bounded Hilbert algebra. Then the quadruple $\mathbf{Q} = (B(\mathbf{A}), D(\mathbf{A}), C(\mathbf{A}), \overline{\varphi})$ is a characterizing quadruple.*

## Characterizing quadruples

Note that if $\mathbf{A}$ is a bounded implicative semilattice then our definition of a quadruple will coincide with that of a triple as defined by Nemitz or by Chajda, Halaš and Kühr.

The relation $C(\mathbf{A}) = \{(a,d) \in B(\mathbf{A}) \times D(\mathbf{A}) \mid u_a^{D(\mathbf{A})}(d) = d\}$ is uniquely determined (hence there is no need for a quadruple).

**Theorem**

*Let $\mathbf{A} = (A, *, 1, 0)$ be a bounded Hilbert algebra. Then the quadruple $\mathbf{Q} = (B(\mathbf{A}), D(\mathbf{A}), C(\mathbf{A}), \overline{\varphi})$ is a characterizing quadruple.*

## Characterizing quadruples

Note that if $\mathbf{A}$ is a bounded implicative semilattice then our definition of a quadruple will coincide with that of a triple as defined by Nemitz or by Chajda, Halaš and Kühr.

The relation $C(\mathbf{A}) = \{(a,d) \in B(\mathbf{A}) \times D(\mathbf{A}) \mid u_a^{D(\mathbf{A})}(d) = d\}$ is uniquely determined (hence there is no need for a quadruple).

**Theorem**

*Let $\mathbf{A} = (A, *, 1, 0)$ be a bounded Hilbert algebra. Then the quadruple $\mathbf{Q} = (B(\mathbf{A}), D(\mathbf{A}), C(\mathbf{A}), \overline{\varphi})$ is a characterizing quadruple.*

## Characterizing quadruples

Note that if $\mathbf{A}$ is a bounded implicative semilattice then our definition of a quadruple will coincide with that of a triple as defined by Nemitz or by Chajda, Halaš and Kühr.

The relation $C(\mathbf{A}) = \{(a,d) \in B(\mathbf{A}) \times D(\mathbf{A}) \mid u_a^{D(\mathbf{A})}(d) = d\}$ is uniquely determined (hence there is no need for a quadruple).

### Theorem

*Let* $\mathbf{A} = (A, *, 1, 0)$ *be a bounded Hilbert algebra. Then the quadruple* $\mathbf{Q} = (B(\mathbf{A}), D(\mathbf{A}), C(\mathbf{A}), \overline{\varphi})$ *is a characterizing quadruple.*

## Representation theorem for Hilbert algebras

### Definition

We say that two characterizing quadruples $(\mathbf{B}_1, \mathbf{D}_1, \mathbf{C}_1, \varphi_1)$ and $(\mathbf{B}_2, \mathbf{D}_2, \mathbf{C}_2, \varphi_2)$ are *isomorphic* if there is an isomorphism $f$ of Boolean algebras from the Boolean algebra $\mathbf{B}_1$ to the Boolean algebra $\mathbf{B}_2$ and an isomorphism $g$ of Hilbert algebras from the Hilbert algebra $\mathbf{D}_1$ to the Hilbert algebra $\mathbf{D}_2$ such that $(f \times g)(\mathbf{C}_1) = \mathbf{C}_2$ and the following diagram

$$\begin{tikzcd} \mathbf{B}_1 \arrow[r, "\varphi_1"'] \arrow[d, "f"'] & \operatorname{Nuc}\mathbf{D}_1 \arrow[d, "\overline{g}"] \\ \mathbf{B}_2 \arrow[r, "\varphi_2"'] & \operatorname{Nuc}\mathbf{D}_2. \end{tikzcd}$$

commutes, where $\overline{g}$ is the isomorphism of $\operatorname{Nuc}\mathbf{D}_1$ to $\operatorname{Nuc}\mathbf{D}_2$ assigning to each $j \in \operatorname{Nuc}\mathbf{D}_1$ the nucleus $g \circ j \circ g^{-1}$ on $\mathbf{D}_2$. Note that, for all $a \in B_1$ and all $d \in D_2$, $\varphi_2(f(a))(d) = g(\varphi_1(a)(g^{-1}(d)))$.

## Representation theorem for Hilbert algebras

**Theorem**

*Two bounded Hilbert algebras are isomorphic if and only if their associated characterizing quadruples are isomorphic.*

## Representation theorem for Hilbert algebras

**Theorem**

*Two bounded Hilbert algebras are isomorphic if and only if their associated characterizing quadruples are isomorphic.*

Sketch of the proof:

- Suppose that the characterizing quadruples $(B(\mathbf{A}_1), D(\mathbf{A}_1), C(\mathbf{A}_1), \overline{\varphi}_1)$ and $(B(\mathbf{A}_2), D(\mathbf{A}_2), D(\mathbf{A}_2), \overline{\varphi}_2)$ are isomorphic.
- Then we obtain isomorphic characterizing quadruples $\mathbf{Q}_1 = (B(\mathbf{A}_1), \mathrm{S}(D(\mathbf{A}_1)), B(\mathbf{A}_1) \times_{\psi_1} \mathrm{S}(D(\mathbf{A}_1)), \overline{\psi}_1)$ and $\mathbf{Q}_2 = (B(\mathbf{A}_2), \mathrm{S}(D(\mathbf{A}_2)), B(\mathbf{A}_2) \times_{\psi_2} \mathrm{S}(D(\mathbf{A}_2)), \overline{\psi}_2)$ where $\overline{\psi_i}(a_i) = \mathrm{S}(u_{a_i^*}^{D(\mathbf{A}_i)}) = u_{a_i^*}^{D(\mathrm{S}(\mathbf{A}_i))}$, $a_i \in A_i$ and $B(\mathbf{A}_1) \times_{\psi_1} \mathrm{S}(D(\mathbf{A}_1)) = \{(a_i, d_i) \in B(\mathbf{A}_i) \times \mathrm{S}(D(\mathbf{A}_i)) \mid u_{a_i}^{\mathrm{S}(D(\mathbf{A}_i))}(d_i) = d_i\}$, $i = 1, 2$ and we use the fact that the corresponding implicative envelopes are isomorphic.

## Representation theorem for Hilbert algebras

**Theorem**

*Two bounded Hilbert algebras are isomorphic if and only if their associated characterizing quadruples are isomorphic.*

Sketch of the proof:

- Suppose that the characterizing quadruples $(B(\mathbf{A}_1), D(\mathbf{A}_1), C(\mathbf{A}_1), \overline{\varphi}_1)$ and $(B(\mathbf{A}_2), D(\mathbf{A}_2), D(\mathbf{A}_2), \overline{\varphi}_2)$ are isomorphic.
- Then we obtain isomorphic characterizing quadruples $\mathbf{Q}_1 = (B(\mathbf{A}_1), \mathrm{S}(D(\mathbf{A}_1)), B(\mathbf{A}_1) \times_{\psi_1} \mathrm{S}(D(\mathbf{A}_1)), \overline{\psi}_1)$ and $\mathbf{Q}_2 = (B(\mathbf{A}_2), \mathrm{S}(D(\mathbf{A}_2)), B(\mathbf{A}_2) \times_{\psi_2} \mathrm{S}(D(\mathbf{A}_2)), \overline{\psi}_2)$ where $\overline{\psi}_i(a_i) = \mathrm{S}(u_{a_i^*}^{D(\mathbf{A}_i)}) = u_{a_i^*}^{D(\mathrm{S}(\mathbf{A}_i))}$, $a_i \in A_i$ and $B(\mathbf{A}_1) \times_{\psi_1} \mathrm{S}(D(\mathbf{A}_1)) = \{(a_i, d_i) \in B(\mathbf{A}_i) \times \mathrm{S}(D(\mathbf{A}_i)) \mid u_{a_i}^{\mathrm{S}(D(\mathbf{A}_i))}(d_i) = d_i\}$, $i = 1, 2$ and we use the fact that the corresponding implicative envelopes are isomorphic.

# Representation theorem for Hilbert algebras

**Theorem**

*Two bounded Hilbert algebras are isomorphic if and only if their associated characterizing quadruples are isomorphic.*

Sketch of the proof:

- Suppose that the characterizing quadruples $(B(\mathbf{A}_1), D(\mathbf{A}_1), C(\mathbf{A}_1), \overline{\varphi}_1)$ and $(B(\mathbf{A}_2), D(\mathbf{A}_2), D(\mathbf{A}_2), \overline{\varphi}_2)$ are isomorphic.
- Then we obtain isomorphic characterizing quadruples $\mathbf{Q}_1 = (B(\mathbf{A}_1), \mathrm{S}(D(\mathbf{A}_1)), B(\mathbf{A}_1) \times_{\psi_1} \mathrm{S}(D(\mathbf{A}_1)), \overline{\psi}_1)$ and $\mathbf{Q}_2 = (B(\mathbf{A}_2), \mathrm{S}(D(\mathbf{A}_2)), B(\mathbf{A}_2) \times_{\psi_2} \mathrm{S}(D(\mathbf{A}_2)), \overline{\psi}_2)$ where $\overline{\psi}_i(a_i) = \mathrm{S}(u_{a_i^*}^{D(\mathbf{A}_i)}) = u_{a_i^*}^{D(\mathrm{S}(\mathbf{A}_i))}$, $a_i \in A_i$ and $B(\mathbf{A}_1) \times_{\psi_1} \mathrm{S}(D(\mathbf{A}_1)) = \{(a_i, d_i) \in B(\mathbf{A}_i) \times \mathrm{S}(D(\mathbf{A}_i)) \mid u_{a_i}^{\mathrm{S}(D(\mathbf{A}_i))}(d_i) = d_i\}$, $i = 1, 2$ and we use the fact that the corresponding implicative envelopes are isomorphic.

# Representation theorem for Hilbert algebras

**Theorem**

*Two bounded Hilbert algebras are isomorphic if and only if their associated characterizing quadruples are isomorphic.*

Sketch of the proof:

- Suppose that the characterizing quadruples $(B(\mathbf{A}_1), D(\mathbf{A}_1), C(\mathbf{A}_1), \overline{\varphi}_1)$ and $(B(\mathbf{A}_2),\, D(\mathbf{A}_2),\, D(\mathbf{A}_2), \overline{\varphi}_2)$ are isomorphic.
- Then we obtain isomorphic characterizing quadruples $\mathbf{Q}_1 = (B(\mathbf{A}_1), \mathrm{S}(D(\mathbf{A}_1)),\, B(\mathbf{A}_1) \times_{\psi_1} \mathrm{S}(D(\mathbf{A}_1)), \overline{\psi}_1)$ and $\mathbf{Q}_2 = (B(\mathbf{A}_2), \mathrm{S}(D(\mathbf{A}_2)), B(\mathbf{A}_2) \times_{\psi_2} \mathrm{S}(D(\mathbf{A}_2)), \overline{\psi}_2)$ where $\overline{\psi}_i(a_i) = \mathrm{S}(u_{a_i^*}^{D(\mathbf{A}_i)}) = u_{a_i^*}^{D(\mathrm{S}(\mathbf{A}_i))}$, $a_i \in A_i$ and $B(\mathbf{A}_1) \times_{\psi_1} \mathrm{S}(D(\mathbf{A}_1)) = \{(a_i, d_i) \in B(\mathbf{A}_i) \times \mathrm{S}(D(\mathbf{A}_i)) \mid u_{a_i}^{\mathrm{S}(D(\mathbf{A}_i))}(d_i) = d_i\}$, $i = 1, 2$ and we use the fact that the corresponding implicative envelopes are isomorphic.

## Representation theorem for Hilbert algebras

### Theorem

*For any characterizing quadruple* $\mathbf{Q} = (\mathbf{B}, \mathbf{D}, \mathbf{C}, \varphi)$ *there exists a bounded Hilbert algebra* $\mathbf{A_Q}$ *whose associated characterizing quadruple* $\mathscr{Q}(\mathbf{A_Q}) = (B(\mathbf{A_Q}), D(\mathbf{A_Q}), C(\mathbf{A_Q}), \overline{\varphi}_{\mathbf{Q}})$, *is isomorphic to* $\mathbf{Q}$.

# Representation theorem for Hilbert algebras

**Theorem**

*For any characterizing quadruple* $\mathbf{Q} = (\mathbf{B}, \mathbf{D}, \mathbf{C}, \varphi)$ *there exists a bounded Hilbert algebra* $\mathbf{A_Q}$ *whose associated characterizing quadruple* $\mathscr{Q}(\mathbf{A_Q}) = (B(\mathbf{A_Q}), D(\mathbf{A_Q}), C(\mathbf{A_Q}), \overline{\varphi}_{\mathbf{Q}})$, *is isomorphic to* $\mathbf{Q}$.

Sketch of the proof:

- We start with characterizing quadruple $\mathbf{Q} = (\mathbf{B}, \mathbf{D}, \mathbf{C}, \varphi)$ and construct a new characterizing quadruple $\widetilde{\mathbf{Q}} = (\mathbf{B}, \widetilde{\mathbf{D}}, \widetilde{\mathbf{C}}, \widetilde{\varphi})$ defined as follows:
- We put $\widetilde{\mathbf{D}} = \mathrm{S}(\mathbf{D})$, $\widetilde{\mathbf{C}} = \mathbf{B} \times_{\varphi} \widetilde{\mathbf{D}} = \{(a, d) \in B \times \mathrm{S}(\mathbf{D}) \mid \widetilde{\varphi}(a^*)(d) = d\}$ and $\widetilde{\varphi}(a) = \mathrm{S}(\varphi(a))$ for all $a \in B$.
- Since for implicative semilattices the theorem is valid we obtain that $\widetilde{\mathbf{Q}}$ is isomorphic to $\mathscr{Q}(\mathbf{A}_{\widetilde{\mathbf{Q}}})$ and then we restrict to $B(\mathbf{A_Q})$ and $D(\mathbf{A_Q})$.
- $$(a, d) *_{\mathbf{A_Q}} (b, e) = (a *_{\mathbf{B}} b, \varphi(a^*)(d *_{\mathbf{D}} e)).$$

## Representation theorem for Hilbert algebras

### Theorem

*For any characterizing quadruple* $\mathbf{Q} = (\mathbf{B}, \mathbf{D}, \mathbf{C}, \varphi)$ *there exists a bounded Hilbert algebra* $\mathbf{A_Q}$ *whose associated characterizing quadruple* $\mathscr{Q}(\mathbf{A_Q}) = (B(\mathbf{A_Q}), D(\mathbf{A_Q}), C(\mathbf{A_Q}), \overline{\varphi}_{\mathbf{Q}})$*, is isomorphic to* $\mathbf{Q}$*.*

Sketch of the proof:

- We start with characterizing quadruple $\mathbf{Q} = (\mathbf{B}, \mathbf{D}, \mathbf{C}, \varphi)$ and construct a new characterizing quadruple $\widetilde{\mathbf{Q}} = (\mathbf{B}, \widetilde{\mathbf{D}}, \widetilde{\mathbf{C}}, \widetilde{\varphi})$ defined as follows:
- We put $\widetilde{\mathbf{D}} = \mathrm{S}(\mathbf{D})$, $\widetilde{\mathbf{C}} = \mathbf{B} \times_{\varphi} \widetilde{\mathbf{D}} = \{(a,d) \in B \times \mathrm{S}(\mathbf{D}) \mid \widetilde{\varphi}(a^*)(d) = d\}$ and $\widetilde{\varphi}(a) = \mathrm{S}(\varphi(a))$ for all $a \in B$.
- Since for implicative semilattices the theorem is valid we obtain that $\widetilde{\mathbf{Q}}$ is isomorphic to $\mathscr{Q}(\mathbf{A}_{\widetilde{\mathbf{Q}}})$ and then we restrict to $B(\mathbf{A_Q})$ and $D(\mathbf{A_Q})$.
- $$(a,d) *_{\mathbf{A_Q}} (b,e) = (a *_{\mathbf{B}} b, \varphi(a^*)(d *_{\mathbf{D}} e)).$$

## Representation theorem for Hilbert algebras

### Theorem

*For any characterizing quadruple* $\mathbf{Q} = (\mathbf{B}, \mathbf{D}, \mathbf{C}, \varphi)$ *there exists a bounded Hilbert algebra* $\mathbf{A_Q}$ *whose associated characterizing quadruple* $\mathscr{Q}(\mathbf{A_Q}) = (B(\mathbf{A_Q}), D(\mathbf{A_Q}), C(\mathbf{A_Q}), \overline{\varphi}_{\mathbf{Q}})$*, is isomorphic to* $\mathbf{Q}$*.*

Sketch of the proof:

- We start with characterizing quadruple $\mathbf{Q} = (\mathbf{B}, \mathbf{D}, \mathbf{C}, \varphi)$ and construct a new characterizing quadruple $\widetilde{\mathbf{Q}} = (\mathbf{B}, \widetilde{\mathbf{D}}, \widetilde{\mathbf{C}}, \widetilde{\varphi})$ defined as follows:
- We put $\widetilde{\mathbf{D}} = \mathrm{S}(\mathbf{D})$, $\widetilde{\mathbf{C}} = \mathbf{B} \times_{\varphi} \widetilde{\mathbf{D}} = \{(a,d) \in B \times \mathrm{S}(\mathbf{D}) \mid \widetilde{\varphi}(a^*)(d) = d\}$ and $\widetilde{\varphi}(a) = \mathrm{S}(\varphi(a))$ for all $a \in B$.
- Since for implicative semilattices the theorem is valid we obtain that $\widetilde{\mathbf{Q}}$ is isomorphic to $\mathscr{Q}(\mathbf{A}_{\widetilde{\mathbf{Q}}})$ and then we restrict to $B(\mathbf{A_Q})$ and $D(\mathbf{A_Q})$.
- $$(a,d) *_{\mathbf{A_Q}} (b,e) = (a *_{\mathbf{B}} b, \varphi(a^*)(d *_{\mathbf{D}} e)).$$

## Representation theorem for Hilbert algebras

### Theorem

*For any characterizing quadruple* $\mathbf{Q} = (\mathbf{B}, \mathbf{D}, \mathbf{C}, \varphi)$ *there exists a bounded Hilbert algebra* $\mathbf{A_Q}$ *whose associated characterizing quadruple* $\mathscr{Q}(\mathbf{A_Q}) = (B(\mathbf{A_Q}), D(\mathbf{A_Q}), C(\mathbf{A_Q}), \overline{\varphi}_{\mathbf{Q}})$, *is isomorphic to* $\mathbf{Q}$.

Sketch of the proof:

- We start with characterizing quadruple $\mathbf{Q} = (\mathbf{B}, \mathbf{D}, \mathbf{C}, \varphi)$ and construct a new characterizing quadruple $\widetilde{\mathbf{Q}} = (\mathbf{B}, \widetilde{\mathbf{D}}, \widetilde{\mathbf{C}}, \widetilde{\varphi})$ defined as follows:
- We put $\widetilde{\mathbf{D}} = \mathrm{S}(\mathbf{D})$, $\widetilde{\mathbf{C}} = \mathbf{B} \times_{\varphi} \widetilde{\mathbf{D}} = \{(a,d) \in B \times \mathrm{S}(\mathbf{D}) \mid \widetilde{\varphi}(a^*)(d) = d\}$ and $\widetilde{\varphi}(a) = \mathrm{S}(\varphi(a))$ for all $a \in B$.
- Since for implicative semilattices the theorem is valid we obtain that $\widetilde{\mathbf{Q}}$ is isomorphic to $\mathscr{Q}(\mathbf{A}_{\widetilde{\mathbf{Q}}})$ and then we restrict to $B(\mathbf{A_Q})$ and $D(\mathbf{A_Q})$.
- $$(a,d) *_{\mathbf{A_Q}} (b,e) = (a *_{\mathbf{B}} b, \varphi(a^*)(d *_{\mathbf{D}} e)).$$

## Representation theorem for Hilbert algebras

**Theorem**

*For any characterizing quadruple* $\mathbf{Q} = (\mathbf{B}, \mathbf{D}, \mathbf{C}, \varphi)$ *there exists a bounded Hilbert algebra* $\mathbf{A_Q}$ *whose associated characterizing quadruple* $\mathcal{Q}(\mathbf{A_Q}) = (B(\mathbf{A_Q}), D(\mathbf{A_Q}), C(\mathbf{A_Q}), \overline{\varphi}_{\mathbf{Q}})$, *is isomorphic to* $\mathbf{Q}$*.*

Sketch of the proof:

- We start with characterizing quadruple $\mathbf{Q} = (\mathbf{B}, \mathbf{D}, \mathbf{C}, \varphi)$ and construct a new characterizing quadruple $\widetilde{\mathbf{Q}} = (\mathbf{B}, \widetilde{\mathbf{D}}, \widetilde{\mathbf{C}}, \widetilde{\varphi})$ defined as follows:
- We put $\widetilde{\mathbf{D}} = \mathrm{S}(\mathbf{D})$, $\widetilde{\mathbf{C}} = \mathbf{B} \times_{\varphi} \widetilde{\mathbf{D}} = \{(a,d) \in B \times \mathrm{S}(\mathbf{D}) \mid \widetilde{\varphi}(a^*)(d) = d\}$ and $\widetilde{\varphi}(a) = \mathrm{S}(\varphi(a))$ for all $a \in B$.
- Since for implicative semilattices the theorem is valid we obtain that $\widetilde{\mathbf{Q}}$ is isomorphic to $\mathcal{Q}(\mathbf{A}_{\widetilde{\mathbf{Q}}})$ and then we restrict to $B(\mathbf{A_Q})$ and $D(\mathbf{A_Q})$.
- $$(a,d) *_{\mathbf{A_Q}} (b,e) = (a *_{\mathbf{B}} b, \varphi(a^*)(d *_{\mathbf{D}} e)).$$

## Representation theorem for Hilbert algebras

### Theorem

*For any characterizing quadruple* $\mathbf{Q} = (\mathbf{B}, \mathbf{D}, \mathbf{C}, \varphi)$ *there exists a bounded Hilbert algebra* $\mathbf{A_Q}$ *whose associated characterizing quadruple* $\mathcal{Q}(\mathbf{A_Q}) = (B(\mathbf{A_Q}), D(\mathbf{A_Q}), C(\mathbf{A_Q}), \overline{\varphi}_{\mathbf{Q}})$, *is isomorphic to* $\mathbf{Q}$.

Sketch of the proof:

- We start with characterizing quadruple $\mathbf{Q} = (\mathbf{B}, \mathbf{D}, \mathbf{C}, \varphi)$ and construct a new characterizing quadruple $\widetilde{\mathbf{Q}} = (\mathbf{B}, \widetilde{\mathbf{D}}, \widetilde{\mathbf{C}}, \widetilde{\varphi})$ defined as follows:
- We put $\widetilde{\mathbf{D}} = \mathrm{S}(\mathbf{D})$, $\widetilde{\mathbf{C}} = \mathbf{B} \times_{\varphi} \widetilde{\mathbf{D}} = \{(a,d) \in B \times \mathrm{S}(\mathbf{D}) \mid \widetilde{\varphi}(a^*)(d) = d\}$ and $\widetilde{\varphi}(a) = \mathrm{S}(\varphi(a))$ for all $a \in B$.
- Since for implicative semilattices the theorem is valid we obtain that $\widetilde{\mathbf{Q}}$ is isomorphic to $\mathcal{Q}(\mathbf{A}_{\widetilde{\mathbf{Q}}})$ and then we restrict to $B(\mathbf{A_Q})$ and $D(\mathbf{A_Q})$.
- $$(a,d) *_{\mathbf{A_Q}} (b,e) = (a *_{\mathbf{B}} b, \varphi(a^*)(d *_{\mathbf{D}} e)).$$

## Representation theorem for Hilbert algebras

### Example

Let $\mathbf{B} = (B, *_{\mathbf{B}}, 1, 0)$ be a Boolean algebra and $\mathbf{D} = (D, *_{\mathbf{D}}, 1)$ be a Hilbert algebra.

Let $\varphi \colon \mathbf{B} \to \operatorname{Nuc}\mathbf{D}$ be a join-preserving mapping that preserves both the least and the greatest element.

(a) We put $\mathbf{C} = (\{1\} \times \mathbf{D}) \cup (\mathbf{B} \times \{1\})$.

Clearly, $(a,d),(b,e) \in \mathbf{C}$ implies $(a *_{\mathbf{B}} b, \overline{\varphi}(a^*)(d *_{\mathbf{D}} e)) \in \mathbf{C}$ and $\overline{\varphi}(a^*)(d) = d$.

Then $\mathbf{Q} = (\mathbf{B}, \mathbf{D}, \mathbf{C}, \varphi)$ is a characterizing quadruple and the corresponding Hilbert algebra will be $(\mathbf{A_Q}, *_{\mathbf{A_Q}}, (1,1), (0,1))$, $\mathbf{A_Q} = \mathbf{C}$.

Evidently, this is the smallest Hilbert algebra with prescribed Boolean algebra of closed elements isomorphic to $\mathbf{B}$ and prescribed Hilbert algebra of dense elements isomorphic to $\mathbf{D}$ and the construction does not depend on $\varphi$.

## Representation theorem for Hilbert algebras

### Example

Let $\mathbf{B} = (B, *_{\mathbf{B}}, 1, 0)$ be a Boolean algebra and $\mathbf{D} = (D, *_{\mathbf{D}}, 1)$ be a Hilbert algebra.

Let $\varphi\colon \mathbf{B} \to \operatorname{Nuc}\mathbf{D}$ be a join-preserving mapping that preserves both the least and the greatest element.

(a) We put $\mathbf{C} = (\{1\} \times \mathbf{D}) \cup (\mathbf{B} \times \{1\})$.

Clearly, $(a,d),(b,e) \in \mathbf{C}$ implies $(a *_{\mathbf{B}} b, \overline{\varphi}(a^*)(d *_{\mathbf{D}} e)) \in \mathbf{C}$ and $\overline{\varphi}(a^*)(d) = d$.

Then $\mathbf{Q} = (\mathbf{B}, \mathbf{D}, \mathbf{C}, \varphi)$ is a characterizing quadruple and the corresponding Hilbert algebra will be $(\mathbf{A_Q}, *_{\mathbf{A_Q}}, (1,1), (0,1))$, $\mathbf{A_Q} = \mathbf{C}$.

Evidently, this is the smallest Hilbert algebra with prescribed Boolean algebra of closed elements isomorphic to $\mathbf{B}$ and prescribed Hilbert algebra of dense elements isomorphic to $\mathbf{D}$ and the construction does not depend on $\varphi$.

## Representation theorem for Hilbert algebras

### Example

Let $\mathbf{B} = (B, *_{\mathbf{B}}, 1, 0)$ be a Boolean algebra and $\mathbf{D} = (D, *_{\mathbf{D}}, 1)$ be a Hilbert algebra.

Let $\varphi\colon \mathbf{B} \to \operatorname{Nuc}\mathbf{D}$ be a join-preserving mapping that preserves both the least and the greatest element.

(a) We put $\mathbf{C} = (\{1\} \times \mathbf{D}) \cup (\mathbf{B} \times \{1\})$.

Clearly, $(a,d),(b,e) \in \mathbf{C}$ implies $(a *_{\mathbf{B}} b, \overline{\varphi}(a^*)(d *_{\mathbf{D}} e)) \in \mathbf{C}$ and $\overline{\varphi}(a^*)(d) = d$.

Then $\mathbf{Q} = (\mathbf{B}, \mathbf{D}, \mathbf{C}, \varphi)$ is a characterizing quadruple and the corresponding Hilbert algebra will be $(\mathbf{A_Q}, *_{\mathbf{A_Q}}, (1,1), (0,1))$, $\mathbf{A_Q} = \mathbf{C}$.

Evidently, this is the smallest Hilbert algebra with prescribed Boolean algebra of closed elements isomorphic to $\mathbf{B}$ and prescribed Hilbert algebra of dense elements isomorphic to $\mathbf{D}$ and the construction does not depend on $\varphi$.

## Representation theorem for Hilbert algebras

### Example

Let $\mathbf{B} = (B, *_{\mathbf{B}}, 1, 0)$ be a Boolean algebra and $\mathbf{D} = (D, *_{\mathbf{D}}, 1)$ be a Hilbert algebra.

Let $\varphi\colon \mathbf{B} \to \operatorname{Nuc} \mathbf{D}$ be a join-preserving mapping that preserves both the least and the greatest element.

(a) We put $\mathbf{C} = (\{1\} \times \mathbf{D}) \cup (\mathbf{B} \times \{1\})$.

Clearly, $(a, d), (b, e) \in \mathbf{C}$ implies $(a *_{\mathbf{B}} b, \overline{\varphi}(a^*)(d *_{\mathbf{D}} e)) \in \mathbf{C}$ and $\overline{\varphi}(a^*)(d) = d$.

Then $\mathbf{Q} = (\mathbf{B}, \mathbf{D}, \mathbf{C}, \varphi)$ is a characterizing quadruple and the corresponding Hilbert algebra will be $(\mathbf{A_Q}, *_{\mathbf{A_Q}}, (1, 1), (0, 1))$, $\mathbf{A_Q} = \mathbf{C}$.

Evidently, this is the smallest Hilbert algebra with prescribed Boolean algebra of closed elements isomorphic to $\mathbf{B}$ and prescribed Hilbert algebra of dense elements isomorphic to $\mathbf{D}$ and the construction does not depend on $\varphi$.

## Representation theorem for Hilbert algebras

### Example

Let $\mathbf{B} = (B, *_{\mathbf{B}}, 1, 0)$ be a Boolean algebra and $\mathbf{D} = (D, *_{\mathbf{D}}, 1)$ be a Hilbert algebra.

Let $\varphi\colon \mathbf{B} \to \operatorname{Nuc}\mathbf{D}$ be a join-preserving mapping that preserves both the least and the greatest element.

(a) We put $\mathbf{C} = (\{1\} \times \mathbf{D}) \cup (\mathbf{B} \times \{1\})$.

Clearly, $(a, d), (b, e) \in \mathbf{C}$ implies $(a *_{\mathbf{B}} b, \overline{\varphi}(a^*)(d *_{\mathbf{D}} e)) \in \mathbf{C}$ and $\overline{\varphi}(a^*)(d) = d$.

Then $\mathbf{Q} = (\mathbf{B}, \mathbf{D}, \mathbf{C}, \varphi)$ is a characterizing quadruple and the corresponding Hilbert algebra will be $(\mathbf{A_Q}, *_{\mathbf{A_Q}}, (1,1), (0,1))$, $\mathbf{A_Q} = \mathbf{C}$.

Evidently, this is the smallest Hilbert algebra with prescribed Boolean algebra of closed elements isomorphic to $\mathbf{B}$ and prescribed Hilbert algebra of dense elements isomorphic to $\mathbf{D}$ and the construction does not depend on $\varphi$.

## Representation theorem for Hilbert algebras

### Example

Let $\mathbf{B} = (B, *_{\mathbf{B}}, 1, 0)$ be a Boolean algebra and $\mathbf{D} = (D, *_{\mathbf{D}}, 1)$ be a Hilbert algebra.

Let $\varphi\colon \mathbf{B} \to \operatorname{Nuc}\mathbf{D}$ be a join-preserving mapping that preserves both the least and the greatest element.

(a) We put $\mathbf{C} = (\{1\} \times \mathbf{D}) \cup (\mathbf{B} \times \{1\})$.

Clearly, $(a,d),(b,e) \in \mathbf{C}$ implies $(a *_{\mathbf{B}} b, \overline{\varphi}(a^*)(d *_{\mathbf{D}} e)) \in \mathbf{C}$ and $\overline{\varphi}(a^*)(d) = d$.

Then $\mathbf{Q} = (\mathbf{B}, \mathbf{D}, \mathbf{C}, \varphi)$ is a characterizing quadruple and the corresponding Hilbert algebra will be $(\mathbf{A_Q}, *_{\mathbf{A_Q}}, (1,1), (0,1))$, $\mathbf{A_Q} = \mathbf{C}$.

Evidently, this is the smallest Hilbert algebra with prescribed Boolean algebra of closed elements isomorphic to $\mathbf{B}$ and prescribed Hilbert algebra of dense elements isomorphic to $\mathbf{D}$ and the construction does not depend on $\varphi$.

# Representation theorem for Hilbert algebras

## Example

Let $\mathbf{B} = (B, *_{\mathbf{B}}, 1, 0)$ be a Boolean algebra and $\mathbf{D} = (D, *_{\mathbf{D}}, 1)$ be a Hilbert algebra.

Let $\varphi\colon \mathbf{B} \to \operatorname{Nuc}\mathbf{D}$ be a join-preserving mapping that preserves both the least and the greatest element.

(a) We put $\mathbf{C} = (\{1\} \times \mathbf{D}) \cup (\mathbf{B} \times \{1\})$.

Clearly, $(a,d),(b,e) \in \mathbf{C}$ implies $(a *_{\mathbf{B}} b, \overline{\varphi}(a^*)(d *_{\mathbf{D}} e)) \in \mathbf{C}$ and $\overline{\varphi}(a^*)(d) = d$.

Then $\mathbf{Q} = (\mathbf{B}, \mathbf{D}, \mathbf{C}, \varphi)$ is a characterizing quadruple and the corresponding Hilbert algebra will be $(\mathbf{A_Q}, *_{\mathbf{A_Q}}, (1,1), (0,1))$, $\mathbf{A_Q} = \mathbf{C}$.

Evidently, this is the smallest Hilbert algebra with prescribed Boolean algebra of closed elements isomorphic to $\mathbf{B}$ and prescribed Hilbert algebra of dense elements isomorphic to $\mathbf{D}$ and the construction does not depend on $\varphi$.

## Representation theorem for Hilbert algebras

### Example

(b) We put $\mathbf{C} = \{(a,d) \in B \times D \mid \varphi(a^*)(d) = d\}$. One can easily check that $\mathbf{Q} = (\mathbf{B}, \mathbf{D}, \mathbf{C}, \varphi)$ is a characterizing quadruple.

Now, the corresponding Hilbert algebra $(\mathbf{A_Q}, *_{\mathbf{A_Q}}, (1,1), (0,1))$ is the greatest Hilbert algebra with prescribed Boolean algebra of closed elements isomorphic to $\mathbf{B}$, prescribed Hilbert algebra of dense elements isomorphic to $\mathbf{D}$ and chosen action of $\mathbf{B}$ on $\mathbf{D}$.

Thank you for your attention.