Semigroups with a completely simple kernel

Roman S. Gigoń

Department of Mathematics
University of Bielsko-Biala
POLAND
email: rgigon@ath.bielsko.pl

School on Algebra and Ordered Sets
Spindleruv Mlyn

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Outline

Definition

By a **semigroup** we shall mean a non-empty set S together with an associative binary operation.

Denote the set of its **idempotents** by E_S, that is,

$$E_S = \{e \in S : ee = e\}.$$

Definition

Let S be a semigroup. Recall that the relation $\leq$ defined on E_S by

$$e \leq f \Leftrightarrow e = ef = fe$$

is a partial order on E_S (the **natural partial order** on E_S).
We say that an idempotent of S is **primitive** if it is minimal with respect to the natural partial order.

Fact

Let S be a semigroup and let $e \in E_S$. Then eSe is a monoid with identity e.

Definition

A non-empty subset A of a semigroup S is said to be an **ideal** of S if $AS \cup SA \subseteq A$.
A least ideal of S (if exists) is called its **kernel**. This ideal is then denoted by K_S.

Definition

An element a of a semigroup S is said to be **regular** if there is $x \in S$ such that $a = axa, x = xax$. Denote the set of all regular elements of S by $\mathrm{Reg}(S)$.

Definition

Recall that a semigroup S is called:
(a) **simple** if has no proper ideals;
(b) **completely simple** if it is simple and contains a primitive idempotent (in fact, **any** idempotent of a completely simple semigroup is primitive; recall that such a semigroup is a (disjoint) union of groups eSe ($e \in E_S$));
(c) **regular** if $S = \mathrm{Reg}(S)$;
(d) **eventually regular** if every element of S has a regular power;
(e) an **epigroup** if each of its elements belongs to some subgroup of S;
(f) E**-inversive** if for every $a \in S$ there is $x \in S$ such that $ax \in E_S$ (the class of E-inversive semigroups is very large and contains **almost all** classes of semigroups which have been studied in literature).

Fact

Let S be an E-inversive semigroup and let $e \in E_S$. Then the monoid eSe is E-inversive.

Fact

Let A be an ideal of a semigroup S and let $a \in A$. Then the ideal SaS is contained in A.

Lemma

A semigroup S has a completely simple kernel if there is $e \in E_S$ such that eSe is a group. In that case, $K_S = SeS$.

Dowód.

Let $a \in S$. Then eae belongs to the group eSe (with identity e). Hence $e = x_a eae$ for some $x_a \in eSe$. Thus $e \in SaS$ for all $a \in S$, therefore, S has a kernel which contains e. Finally, suppose that $f \leq e$, where $f \in E_S$. Then $f = efe \in eSe$. Thus $e = f$ and so e is a primitive idempotent of S. Consequently, SeS is a completely simple kernel of S. □

By the definition, if a semigroup has a **completely simple** kernel, then it must contain **at least one primitive idempotent.** It turns out that in the class of E-inversive semigroups this necessary condition is also sufficient!
Denote the set of all primitive idempotents of a semigroup S by PE_S.

Definition

A semigroup is called **unipotent** if it has exactly one idempotent.

Theorem

An E-inversive semigroup S has a completely simple kernel if and only if it contains a primitive idempotent.
Moreover, in that case,

$$K_S = \bigcup\{eSe : e \in PE_S\},$$

where each eSe is a group.

Dowód.

Let $e \in PE_S$. Then eSe is a unipotent monoid with identity e. Also, the semigroup eSe is E-inversive, so eSe is a group. Hence S has a completely simple kernel. By the proof of the above lemma, $e \in K_S$, so $eSe \subseteq K_S$, therefore,

$$\bigcup\{eSe : e \in PE_S\} \subseteq K_S.$$

Definition

A semigroup is called **primitive** if $E_S = PE_S$.

Denote the set of all elements of a semigroup S that belong to some subgroup of S by $\mathrm{Gr}(S)$.

Corollary

An E-inversive semigroup S is primitive if and only if $K_S = Reg(S) = Gr(S)$ is a completely simple semigroup. In particular, if S is unipotent, then K_S is a group.

The following result is quite surprising.

Fact

Let an E-inversive semigroup S contains exactly one primitive idempotent e. Then e is a least idempotent of S with respect to the natural partial order on E_S.

Dowód.

In the light of the above corollary, $K_S = eSe$ is a group with identity e. Let $a \in S$. Then $ea, ae \in K_S$. Hence $ea = eae = ae$. Thus $ef = fe$ for all $f \in E_S$, so $(ef)^2 = e(fe)f = e(ef)f = ef = e$ for any $f \in E_S$, as required. □

Some further corollaries are contained in my paper ***E*-inversive semigroups with a completely simple kernel** (Communications in Algebra (online first articles)).

The results presented below are a part of my paper **bands of E-inversive unipotent semigroups** (submitted to Periodica Mathematica Hungarica).

Definition

Recall that a semigroup S is:
(a) **poor** if it is a unipotent semigroup with zero (that is, $E_S = \{0\}$);
(b) a **band** if $S = E_S$;
(c) a **semilattice** if it is a commutative band;
(d) a **rectangular band** if it is a band that meets the identity **xyz = xz** (or equivalently, if it is a **primitive band**).

Definition

Recall that an equivalence relation ρ on a semigroup S is a **congruence** if for all $a, b \in S$, $(a\rho) \cdot (b\rho) \subseteq (ab)\rho$ (the algebraic product of subsets of S).

Definition

Let $\mathcal{C}$ be some fixed class of semigroups. We say that a congruence ρ on a semigroup S is a $\mathcal{C}$**-congruence** if $S/\rho \in \mathcal{C}$ (for example, ρ is a **band** congruence on S if the semigroup S/ρ is a band). It is clear that the least band congruence on an arbitrary semigroup always exists. Denote it by β.

Definition

Let also $\mathcal{B}$ be some variety of bands (for instance, the variety of semilattices). A semigroup S is said to be a $\mathcal{B}$**-band of** $\mathcal{C}$**-semigroups** if there exists a congruence ρ on S such that $S/\rho \in \mathcal{B}$ and every ρ-class of S is a $\mathcal{C}$-semigroup (for example, S is a **band of** E**-inversive unipotent semigroups** if there is a band congruence ρ on S such that every ρ-class is an E-inversive unipotent semigroup).

Fact

If ρ is a band congruence on a semigroup S, then every ρ-class of S is a semigroup (and conversely).

Fact

Any semigroup S which is a rectangular band of unipotent semigroups is primitive.

Theorem

Let a semigroup S be a band of E-inversive unipotent semigroups. Then

$$Reg(S) = Gr(S).$$

Moreover, S is a semilattice of rectangular bands of E-inversive unipotent semigroups.

Dowód.

Indeed, let ρ be the corresponding band congruence on S. As any band is a semilattice of rectangular bands, S is a semilattice I of semigroups S_i ($i \in I$), where each semigroup S_i is a rectangular band of some ρ-classes of S. It is obvious that each semigroup S_i is E-inversive. Furthermore, every idempotent in any semigroup S_i is primitive, so
$\mathrm{Reg}(S_i) = \mathrm{Gr}(S_i)$ for all $i \in I$.
On the other hand, it is clear (and well known) that for any $a \in \mathrm{Reg}(S)$ (say: $a \in S_i$ for some $i \in I$), every inverse of a belongs to S_i. This implies (together with the above) the thesis of the proposition. □

Let ρ be a such band congruence on a semigroup S that every ρ-class of S is an E-inversive unipotent semigroup. For every $e \in E_S$, put:

$$S_e := e\rho, \quad G_e := \mathrm{Reg}(e\rho).$$

Let also $\varphi_e : S_e \to G_e$ be defined for all a in S_e, as follows:

$$a\varphi_e = ea.$$

Then φ_e is a retraction of S_e onto G_e. Also,

$$\varphi = \bigcup_{e \in E_S} \varphi_e$$

is a mapping of S onto

$$\bigcup_{e \in E_S} G_e = \mathrm{Gr}(S) = \mathrm{Reg}(S).$$

Finally, put:

$$\nu := \ker(\varphi).$$

A similar notation and nomenclature will be used for semilattices or rectangular bands of E-inversive unipotent semigroups.

Definition

Recall that a semigroup is called an R-**semigroup** if Reg(S) is a subsemigroup of S.

Theorem

Let an R-semigroup S be the band S/ρ of the E-inversive unipotent semigroups $S_e = e\rho$ $(e \in E_S)$. Then the map φ is a retraction of S onto Reg(S). In particular,

$$S/\nu \cong Reg(S)$$

is a band of groups.

Definition

It is clear that for an arbitrary ideal A of a semigroup S, the relation

$$\rho_A = (A \times A) \cup 1_S,$$

where 1_S is the identity relation on S, is a congruence on S (the so-called *Rees congruence*). We denote the (Rees) quotient semigroup induced by this congruence by S/A. In that case, we also say that S is an **ideal extension** of the semigroup A by the semigroup S/A. In particular, if S/A is a poor semigroup, then S is called a **poor extension** of A. Moreover, every extension of a semigroup S by its ideal A such that A is a retract of A is called **retractive** (or a retract extension).

Let S be a semigroup and let $a \in S$. Put

$$\mathcal{H}_a = \{b \in S : a\mathcal{H}\, b\},$$

where $\mathcal{H}$ is a one of Green's equivalence relations on S. Recall that $\mathcal{H}_e$ is a group for any idempotent e of S.

Fact

A semigroup is completely simple if and only if Green's relation $\mathcal{H}$ is a (least) rectangular band congruence on it.

Fact

Every equivalence class of a rectangular band congruence on an E-inversive semigroup is an E-inversive semigroup (this is my result (2012)).

Further, we have the following important characterization (this theorem extends nearly forty-year-old results from the theory of epigroups).

Theorem

The following conditions concerning an E-inversive semigroup S are equivalent:
(a) S is a rectangular band of unipotent semigroups;
(b) S is primitive and $Reg(S)$ is a retract of S;
(c) S is a retractive poor extension of a completely simple semigroup;
(d) S is a subdirect product of a poor semigroup and a completely simple semigroup.

Dowód.

$(a) \Longrightarrow (b)$. We know that any idempotent of S is primitive. This implies that $\mathrm{Reg}(S)$ is a completely simple kernel of S . Hence $\mathrm{Reg}(S)$ is also a retract of S by the above theorem.
$(b) \Longrightarrow (c)$. This is clear by the above proof.
$(c) \Longrightarrow (d)$. This follows from the fact that $\rho_{\mathrm{Reg}(S)} \cap \nu = 1_S$.
$(d) \Longrightarrow (a)$. Suppose that S is a subdirect product of a poor semigroup A and a completely simple semigroup B (we may assume that $S \subseteq A \times B$). Define the relation ρ on S, as follows ($(a, b), (c, d) \in S$):

$$(a, b)\,\rho\,(c, d) \iff b\,\mathcal{H}\,d.$$

Since $\mathcal{H}$ is a rectangular band congruence on B, ρ is a rectangular band congruence on S. Hence every ρ-class of S is an E-inversive semigroup. Finally, each ρ-class of S is unipotent. □

Note that

$$\mathrm{Reg}(S) = \{(0, b) : b \in B\},$$

where 0 is a zero of A (we have $E_A = \{0\}$), is an ideal of S.

Definition

We say that a semigroup A is *β-fruitful* if every β-class of the least band congruence β on A is an E-inversive semigroup.

Observe that in any β-fruitful semigroup S which is the band S/ρ of unipotent semigroups, we get $\rho = \beta$.

Moreover, the following more general result is also valid (this result seems to be new for all epigroups).

Theorem

The following conditions concerning a β-fruitful semigroup S are equivalent:
(a) *S is a band of unipotent semigroups and $Reg(S)$ is an ideal of S;*
(b) *$Reg(S)$ is a band of groups and a retract ideal of S;*
(c) *S is a retractive poor extension of a band of groups;*
(d) *S is a subdirect product of a poor semigroup and a band of groups.*

Definition

Recall that a poor epigroup is called a **nil semigroup**. Let A be an ideal of a semigroup S. We say that S is a **nil-extension** of A if S/A forms a nil semigroup.

Corollary

The following conditions on an epigroup S are equivalent:

(a) *S is a band of unipotent semigroups and $Reg(S)$ is an ideal of S*;

(b) *$Reg(S)$ is a band of groups and a retract ideal of S*;

(c) *S is a retractive nil-extension of a band of groups*;

(d) *S is a subdirect product of a nil semigroup and a band of groups.*