The Independence of Axioms of Hypergroup over Group

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September 3, 2018

Overview

Introduction

The concept of **hypergroup over group** was introduced in **2008** by **S. Dalalyan** and consists of a set and of a group with 6 mappings which satisfy 12 axioms. This concept arises when one tries to extend the concept of quotient group in case of any subgroup of the given group.

Definition of hypergroup over group

Let H be a group. A (right) hypergroup over group H is a set M together with a system of structural mappings $\Omega = (\Phi, \Psi, \Xi, \Lambda)$, where

- (Φ) $\quad \Phi : M \times H \to M, \quad \Phi(a, \alpha) := a^{\alpha},$
- (Ψ) $\quad \Psi : M \times H \to H, \quad \Psi(a, \alpha) := {}^{a}\alpha,$
- (Ξ) $\quad \Xi : M \times M \to M, \quad \Xi(a, b) := [a, b],$
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are mappings which satisfy following conditions.

P1) The mapping Ξ is a binary operation on M such that

(i) any equation $[x, a] = b$ with elements $a, b \in M$ has a unique solution in M,

(ii) (M, Ξ) has a left neutral element $o \in M$, i.e. $[o, a] = a$ for any element $a \in M$.

Definition of hypergroup over group

P2) The mapping Φ is a (right) action of the group H on the set M, that is

(i) $(a^{\alpha})^{\beta} = a^{\alpha \cdot \beta}$ for any elements $\alpha, \beta \in H$ and for every $a \in M$,

(ii) $a^{\varepsilon} = a$ for each $a \in M$, where ε is the neutral element of the group H.

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- **(A5)** $\quad (a, b) \cdot ([a, b], c) = {}^{a}(b, c) \cdot (a^{(b,c)}, [b, c]).$

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- **(A5)** $(a, b) \cdot ([a, b], c) = {}^{a}(b, c) \cdot (a^{(b,c)}, [b, c])$.

We denote hypergroup over group by M_H.

Independence of axioms of hypergroup over group

Theorem 1

The system $\{(P1),(P2),(P3),(A1),(A2),(A3),(A4),(A5)\}$ of axioms of hypergroup over group is independent.

The case of faithful action Φ

Definition

The action Φ of the group H on the set M is called *faithful*, if there is no non-trivial element $\alpha \in H$ such that $\Phi(a, \alpha) = a^{\alpha} = a$ for all elements $a \in M$.

Let H be a group, M be a set and $\Omega = (\Phi, \Psi, \Xi, \Lambda)$ be a system of mappings.

Theorem 2

If the conditions $(P1), (P2), (A2), (A4)$ are satisfied, the action Φ is faithful and $\Phi(o, \alpha) = o^{\alpha} = o$ for any element $\alpha \in H$, then the conditions $(P3), (A1), (A3), (A5)$ are satisfied also.

The standard construction of hypergroup over group

Let G be a group and H be a subgroup of G. Consider a right **transversal** of the subgroup H in the group G, i.e. a subset $M \subset G$ such that $|M \cap Ha| = 1$ for every element $a \in G$. Then for every element $g \in G$ there are unique $\alpha \in H$ and $a \in M$ such that $g = \alpha \cdot a$.

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So we can define mappings

- $\Phi : M \times H \to M, \quad \Phi(a, \alpha) = a^{\alpha}$,
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Let M_H and $M'_{H'}$ be hypergroups over group with systems of structural mappings $\Omega = (\Phi, \Psi, \Xi, \Lambda)$ and $\Omega' = (\Phi', \Psi', \Xi', \Lambda')$ respectively. Then the pair (f_0, f_1), consisted of isomorphism $f_0 : H \to H'$ and bijection $f_1 : M \to M'$, is called an isomorphism between M_H and $M'_{H'}$ if:

- $\Phi \circ f_1 = (f_1 \times f_0) \circ \Phi'$,
- $\Psi \circ f_0 = (f_1 \times f_0) \circ \Psi'$,
- $\Xi \circ f_1 = (f_1 \times f_1) \circ \Xi'$,
- $\Lambda \circ f_0 = (f_1 \times f_1) \circ \Lambda'$.

The universality property of standard construction

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For every hypergroup M_H there exists a hypergroup $\overline{M}_{\overline{H}}$ obtained in standard construction and isomorphic to M_H.

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We construct the hypergroup over group $\overline{M}_{\overline{H}}$ as follows. Let $\overline{G} = \{\alpha a \quad : \alpha \in H, a \in M\}$, consider on $\overline{G}$ the operation

$$\alpha a \cdot \beta b = (\alpha \cdot {}^{a}\beta \cdot (a^{\beta}, b))[a^{\beta}, b].$$

This is a group operation, and $\overline{H} = \{(\alpha \cdot (o, o)^{-1})o \quad : \alpha \in H\}$ is a subgroup of $\overline{G}$. The set $\overline{M} = \{\varepsilon a \quad : a \in M\}$ is a right transversal of $\overline{H}$, and the corresponding hypergroup over group (obtained by standard construction) $\overline{M}_{\overline{H}}$ is isomorphic to M_H.

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We call the group $\overline{G}$ the **exact product** of H and M related with the hypergroup over group M_H.

Thank You !!!