Convexity and the Kalmbach monad

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- This correspondence generalizes nicely: every monad T on a category $\mathcal{C}$ gives rise to the category of algebras (or Eilenberg-Moore category) $\mathcal{C}^T$.
- So one can consider monads to be a generalization of varieties of algebras.

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- μ is <u>the multiplication of the monad</u>

Getting the variety back

Let (T, η, μ) be a monad on a category $\mathcal{C}$.

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- It is denoted by $\mathcal{C}^T$.

Theorem

For every finitary variety $\mathcal{V}$, $\mathcal{V} \simeq \mathbf{Set}^{T_\mathcal{V}}$.

Algebras over other categories than **Set**

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- Small categories are algebras for a monad on the category of directed multigraphs.

The Kalmbach embedding

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Corollary

Orthomodular lattices do not satisfy any special lattice equation.

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- Let L be a bounded lattice. Let $K(L)$ be the set of all finite chains in L with even number of elements.
- Introduce a partial order on the set $K(L)$ by the following rule:

$$[a_1 < a_2 < \cdots < a_{2n-1} < a_{2n}] \leq [b_1 < b_2 < \cdots < b_{2n-1} < b_{2k}]$$

if and only if for every $1 \leq i \leq n$ there exists $1 \leq j \leq k$ such that $b_{2j-1} \leq a_{2i-1} < a_{2i} \leq b_{2j}$.

- Then $K(L)$ is a bounded lattice.
- Moreover, it is an orthomodular lattice: the orthocomplementation is

$$(\{a_i\}_{i=1}^{2n})' := \{a_i\}_{i=1}^{2n} \Delta \{0, 1\},$$

where Δ denotes the symmetric difference and
- the mapping $\eta_L : L \to K(L)$ given by $\eta_L(x) = \{0, x\}$ for $x > 0$ and $\eta_L(0) = \emptyset$ is a injective morphism of lattices.

The Kalmbach embedding

- K cannot be made to a functor from the category of lattices into the category of orthomodular lattices.
- However, K can be extended to a functor from the category of bounded posets to the category of orthomodular posets;
- for $f : P \to Q$ is **BPos**, $K(f) : K(P) \to K(Q)$ is given by the rule

$$K(f)([a_1 < a_2 < \cdots < a_{2n-1} < a_{2n}]) = \Delta_{i=1}^{2n}\{f(a_i)\}.$$

- [Harding, 2004] K is left adjoint to the forgetful functor U from the category of orthomodular posets **OMP** to the category of bounded posets **BPos**.

- Every adjunction like this

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd}$$

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  between two categories induces a monad on $\mathcal{C}$ and a comonad on $\mathcal{D}$.
- Therefore, the adjunction discovered by Harding induces a monad on **BPos**.

The Kalmbach monad

Definition

The Kalmbach monad (T, η, μ) on the category **BPos** is given as follows

- $T : \mathbf{BPos} \to \mathbf{BPos}$ is the Kalmbach embedding $K : \mathbf{BPos} \to \mathbf{OMP}$ composed with the forgetful functor $U : \mathbf{OMP} \to \mathbf{BPos}$, that means, $T = U \circ K$;
- $\eta_P : P \to T(P)$ is given by

$$\eta_P(x) = \begin{cases} \{0, x\} & x > 0 \\ \emptyset & x = 0 \end{cases}$$

- $\mu_P : T^2(P) \to T(P)$ is given by

$$\mu_P([C_1 < C_2 < \dots < C_{2n-1} < C_{2n}]) = C_1 \Delta C_2 \Delta \dots \Delta C_{2n},$$

where Δ denotes the symmetric difference of sets.

What are algebras for the Kalmbach monad?

Answer: effect algebras

Effect algebras

An effect algebra [Foulis and Bennett, 1994, Kôpka and Chovanec, 1994, Giuntini and Greuling, 1989]

- $(A; +, 0, 1)$
- $+$ is a binary partial operation.
- $0, 1$ are constants.

(E1) If $a + b$ is defined, then $b + a$ is defined and $a + b = b + a$.

(E2) If $a + b$ and $(a + b) + c$ are defined, then $b + c$ and $a + (b + c)$ are defined and $(a + b) + c = a + (b + c)$.

(E3) For every $a \in E$ there is a unique $a' \in E$ such that $a + a'$ exists and $a + a' = 1$.

(E4) If $a + 1$ is defined, then $a = 0$.

Convex effect algebras

Definition

A convex effect algebra is an effect algebra E equipped with a multiplication by real numbers from interval $[0, 1]$ such that, for all $\rho, \psi \in [0, 1]$ and $a, b \in E$,

(C1) $a.1 = a$

(C2) $(a.\rho).\psi = a.(\rho.\psi)$

(C3) If $a + b$ is defined, then $a.\rho + b.\rho$ is defined and $(a + b).\rho = a.\rho + b.\rho$

(C4) If $\rho + \psi < 1$, then $a.\rho + a.\psi$ is defined and $a.(\rho + \psi) = a.\rho + a.\psi$.

Theorem

[Jacobs, 2010] The category of convex effect algebras **ConvEA** *is an Eilenberg-Moore category for a monad on the category of effect algebras* **EA**.

Problem

Is **ConvEA** *a category of algebras for some monad on* **BPos***?*

The $\square$ product on **BPos**

- Let A, B, C be bounded posets. We say that a **BPos**-morphism $H : A \times B \to C$ is a 0-bimorphism if and only if, for all $a \in A$ and $b \in B$, $h(0, b) = h(b, 0) = 0$.

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- Let us write $A \square B$ for the poset $A \times B / \sim$, where $\sim$ is the equivalence on $A \times B$ generated by the relations $(a, 0) \sim (0, b)$, for all $a \in A$ $b \in B$.

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- Let us write $A \square B$ for the poset $A \times B/\sim$, where $\sim$ is the equivalence on $A \times B$ generated by the relations $(a, 0) \sim (0, b)$, for all $a \in A$ $b \in B$.
- All the elements of $A \times B$ that have 0 in first or second coordinate form one of the equivalence classes of $\sim$, all the other elements form singleton equivalence classes.

The $\Box$ product on **BPos**

Clearly, the mapping $\Box : A \times B \to A\Box B$ that takes an element of $A \times B$ to its equivalence class is a 0-bimorphism. Moreover, it is an universal 0-bimorphism in the following sense: for every 0-bimorphism $h : A \times B \to C$, there is a unique morphism of bounded posets $f : A\Box B \to C$ such that

$$\begin{tikzcd} A \times B \arrow[d, "\Box"'] \arrow[dr, "h"] & \\ A\Box B \arrow[r, dotted, "f"'] & C \end{tikzcd}$$

commutes.

Fact

The category $(\mathbf{BPos}, \square, 2)$ *is a monoidal category.*[a]

[a] Here, 2 is a 2-element chain

$[0,1]$ is a monoid

Proposition

The real interval $[0,1]$, *equipped with multiplication of reals is a monoid in the monoidal category* $(\mathbf{BPos}, \square, 2)$.

Every monoid induces a monad

There is a monad (S, μ^S, η^S) on **BPos** associated with $[0, 1]$. Explicitly,

- S : **BPos** $\to$ **BPos** is an endofunctor given by the rule $S(A) = A \square [0, 1]$

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This monad is called the <u>free $[0,1]$-action monad on **BPos**</u>.

Distributive laws [Beck, 1969]

- If S, T are monads on a category, it may happen that $T \circ S$ can be made to a monad.
- The additional data needed to do that is a natural transformation $\lambda\colon ST \to TS$, satisfying certain conditions.

Distributive laws [Beck, 1969]

$$
\begin{tikzcd}
 & S \arrow[dl, "S\eta^T"'] \arrow[dr, "\eta^T S"] & \\
ST \arrow[rr, "\lambda"'] & & TS
\end{tikzcd}
\qquad
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ST \arrow[rr, "\lambda"'] & & TS
\end{tikzcd}
$$

$$
\begin{tikzcd}
SST \arrow[r, "S\lambda"] \arrow[d, "\mu^S T"'] & STS \arrow[r, "\lambda S"] & TSS \arrow[d, "T\mu^S"] \\
ST \arrow[rr, "\lambda"'] & & TS
\end{tikzcd}
\qquad
\begin{tikzcd}
STT \arrow[r, "\lambda T"] \arrow[d, "S\mu^T"'] & TST \arrow[r, "T\lambda"] & TTS \arrow[d, "\mu^T S"] \\
ST \arrow[rr, "\lambda"'] & & TS
\end{tikzcd}
$$

Example

There is a distributive law between

- the 'free abelian group' monad on **Set** and
- the 'free monoid' monad on **Set**.

The composite monad is the 'free ring' monad.

Results

Theorem

There is a distributive law between

- *the Kalmbach monad T on* **BPos** *and*
- *the free $[0,1]$-action monad on* **BPos**.

Results

Theorem

The category of algebras for the composite monad TS is equivalent for the category of effect algebras equipped with multiplication with a scalar, satisfying the following conditions:

(C1) $a.1 = a$

(C2) $(a.\rho).\psi = a.(\rho.\psi)$

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Note: these are not convex effect algebras, the axiom

(C4) If $\rho + \psi < 1$, then $a.\rho + a.\psi$ is defined and $a.(\rho + \psi) = a.\rho + a.\psi$.

is missing.

What next?

Call the algebras on the previous slide <u>weak effect algebras</u>.

Problem

Are convex effect algebras algebras for a monad over weak effect algebras?

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