Knot-theoretic ternary groups

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Each ternary quasigroup $(A, [\,])$ has left, middle and right **cancellation property**, e.g. for all $x, y, a, b \in A$

$$[xab] = [yab] \Rightarrow x = y \text{ (left-cancellativity).}$$

Ternary groups

An operation $[\,]\colon A^3 \to A$ is **associative** if for all $a, b, c, d, e \in A$

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Generalizing binary groups: Kasner 1904, Dörnte 1929, Lehmer 1932, Post 1940 (n-groups, polyadic groups)

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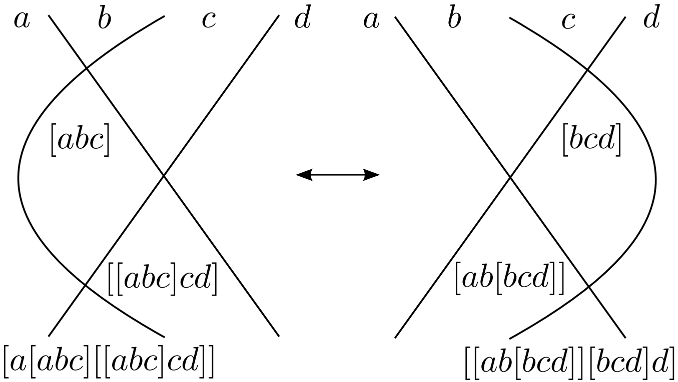
Virtual knot theory (Kauffman 1999)

Third Reidemeister move (flat version)



Niebrzydowski 2014

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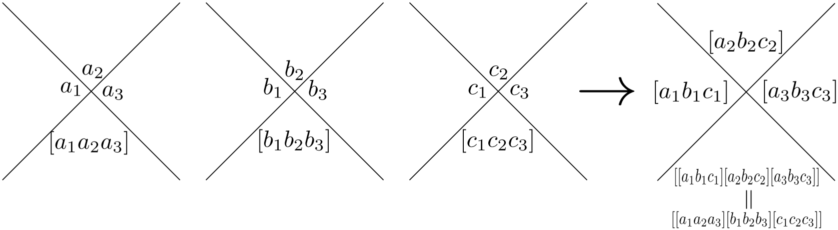
Semi-commutativity $=$ lack of orientation

Theorem (Głazek-Gleichgewicht, 1982)

A ternary group is semi-commutative if and only if it is entropic.

Entropicity



$$[[a_1 a_2 a_3][b_1 b_2 b_3][c_1 c_2 c_3]] = [[a_1 b_1 c_1][a_2 b_2 c_2][a_3 b_3 c_3]].$$

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Theorem (Borowiec-Dudek-Duplij, 2006)

If $(A, [\,])$ *is a ternary group, then*

$$[\bar{a}aa] = [a\bar{a}a] = [aa\bar{a}] = a$$
$$[ba\bar{a}] = [b\bar{a}a] = [a\bar{a}b] = [\bar{a}ab] = b$$
$$\overline{[abc]} = [\bar{c}\bar{b}\bar{a}]$$
$$\bar{\bar{a}} = a$$

for every a, b, $c \in A$.

Knot-theoretic ternary groups

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Theorem

Let $(A, [\,])$ be a ternary group. $(A, [\,])$ is a knot-theoretic ternary group if and only if $(A, [\,])$ is semi-commutative and satisfies

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for every $a, b \in A$.

In particular, in each knot-theoretic group

$$[ccc] = \bar{c},$$

for every $c \in A$.

Example

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Let $k > 1$ be a natural number, $(\mathbb{Z}_k, +)$ be a cyclic group and a be a fixed element of $\mathbb{Z}_k$.

Define on the set $\mathbb{Z}_k$ the ternary operation

$$[xyz] = x - y + z + a \pmod{k}.$$

Then $(\mathbb{Z}_k, [\,])$ is a knot-theoretic ternary group if and only if $2a = 0 \pmod{k}$ in $\mathbb{Z}_k$.

For each even k, there are exactly two knot-theoretic groups constructed in this way:

1. idempotent one for $a = 0$,
2. non-idempotent for a being the element of order 2 in $\mathbb{Z}_k$.

Retracts

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Let $(A, [\,])$ be a ternary groupoid. A binary groupoid $(A, *)$, where $x * y = [xay]$ for some fixed $a \in A$, is a **retract** of $(A, [\,])$ (denoted by $ret_a(A, [\,])$).

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A retract $ret_a(A, [\,])$ of a semi-commutative ternary group is an abelian group with the neutral element $\bar{a}$ and the inverse of $x \in A$ given by $\overline{[axa]}$.

Knot-theoretic ternary group

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Theorem

Each knot-theoretic ternary group $(A, [\,])$ *is determined by an abelian group* $(A, +)$ *and an element* $a \in A$ *of order one or two in* $(A, +)$. *Then for every* $x, y, z \in A$

$$[xyz] = x - y + z + a \quad \text{and} \quad \bar{x} = x + a.$$

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We denote the knot-theoretic ternary group $(A, [\,])$ described above by $\mathcal{T}((A, +), a)$. The group $(A, +)$ is the **associated group** of $(A, [\,]) = \mathcal{T}((A, +), a)$.

$$ret_a(\mathcal{T}((A, +), a)) = (A, +).$$

Isomorphisms of KTTG

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Theorem

Let $(A_1, [\,]_1) = \mathcal{T}((A_1, +_1), a)$ and $(A_2, [\,]_2) = \mathcal{T}((A_2, +_2), b)$ be two knot-theoretic ternary groups. Then the following statements are equivalent:

1. $(A_1, [\,]_1)$ *and* $(A_2, [\,]_2)$ *are isomorphic;*
2. *there exists a group isomorphism* $h\colon (A_1, +_1) \to (A_2, +_2)$ *such that* $b = h(a)$.

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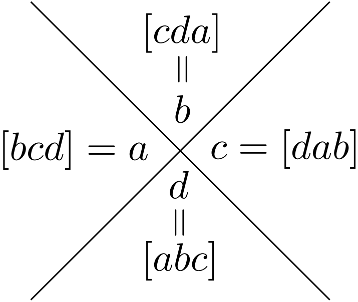
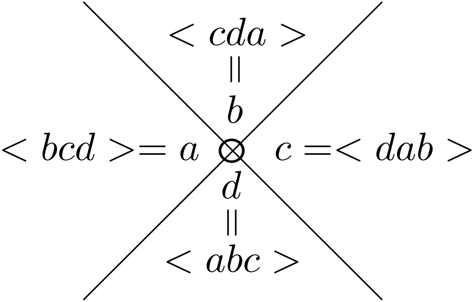
Enumeration of isomorphism classes of knot-theoretic ternary groups (up to 64 elements)

Virtual knot theory





Compatible ternary groups

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Two ternary groups $(A, [\,])$ and $(A, <>)$ are **compatible** if the following condition is satisfied:

$$[ab < bcd >] =< a < abc > [< abc > cd] >,$$

for any $a, b, c, d \in A$.

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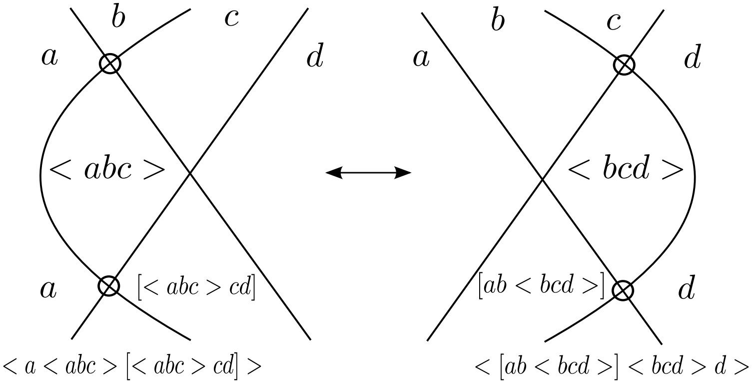
1. Any knot-theoretic ternary group is compatible with itself.
2. For a finite abelian binary group (of an even rank) $(A, +)$ with a neutral element 0 and x an element of order 2: two (non-isomorphic) knot-theoretic ternary groups: $(A, [\,]) = \mathcal{T}((A, +), 0)$ and $(A, < >) = \mathcal{T}((A, +), x)$ are compatible.

Mixed Reidemeister type three move



Application for flat virtual links

Theorem

For a pair of compatible knot-theoretic ternary groups $(A, [\,])$ *and* $(A, <>)$, *and a diagram* D *of a flat virtual link, the number of knot-theoretic ternary group colorings of* D *is not changed by the Reidemeister moves used for flat virtual links.*

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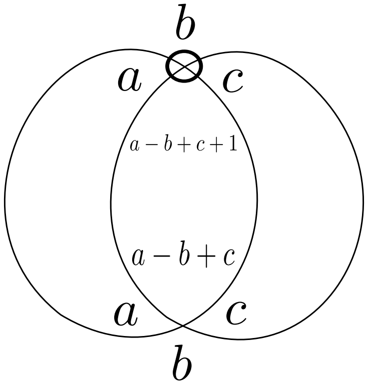
Example: The flat-virtual Hopf link is distinguished from the unlink (two disjoint unknotted loops) by two-element ternary groups.

Example



Thank you for your attention!