Ideals in universal algebra: a survey

Paolo Aglianò

DIISM
Università di Siena
agliano@live.com

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Chapter 1

Introduction

1.1 Algebras

In this note we will make full use of the main concepts of general algebra; for a textbook exposition we direct the reader to [24] and [10]. Let A be a set. An **operation of arity** n **on** A is a function from A^n to A where A^n is the cartesian product of n copies of A. An operation of arity 0 on a nonempty set has a unique value; we identify the operation with the only value in its range an we call it **a constant**.

An **algebra** $\mathbf{A}$ is a pair $\langle A, F\rangle$ where A is a nonempty set and $F = (f_i : i \in I)$ is a family of operations on A. The set A is the **universe** of $\mathbf{A}$; the operations in F are the **basic operations** of $\mathbf{A}$.

A **type** τ is a family $(f_i)_{i\in I}$ of symbols that are understood to represent operations, together with a function $\rho : (f_i)_{i\in I} \longrightarrow \mathbb{N}$, which intuitively represents the arity of the operation symbols. An algebra $\mathbf{A} = \langle A, F\rangle$ is **of type** τ if for each $f \in \tau$ there is an operation on A, denoted by $f^{\mathbf{A}}$, whose arity is $\rho(f_i)$. When the context is clear we will confuse the operation on $\mathbf{A}$ with the operation symbol of which it is the realization. Two algebras are **similar** if they have the same type.

Let $\mathbf{A} = \langle A, F\rangle$ be an algebra and $B \subseteq A$. Then $\mathbf{B} = \langle B, F\rangle$ is a **subalgebra** of $\mathbf{A}$ if for all $f \in F$ (of arity n) and for all $b_1, \ldots, b_n \in B$

$$f(b_1, \ldots, b_n) \in B.$$

Note that it is implicit from the definition that $\mathbf{A}$ and $\mathbf{B}$ are similar. Observe that the intersection of any family of subalgebras of $\mathbf{A}$ it is still a subalgebra;

therefore we can define the **subalgebra generated by X in $\mathbf{A}$** as

$$\mathrm{Sub}_{\mathbf{A}}(X) = \bigcap\{\mathbf{B} : \mathbf{B} \text{ is a subalgebra of } \mathbf{A} \text{ and } X \subseteq B\}.$$

If $\mathbf{A}$ and $\mathbf{B}$ are similar a function $h : A \longrightarrow B$ is a **homomorphism** if for every basic (n-ary) operation f and for all $a_1, \ldots, a_n \in A$

$$h(f_i(a_1, \ldots, a_n)) = g_i(h(a_1), \ldots, h(a_n)).$$

An injective homomorphism is a **monomorphism** or an **embedding**; a surjective homomorphism is an **epimorphism** and a bijective homomorphism is an **isomorphism**. If there is an isomorphism from $\mathbf{A}$ to $\mathbf{B}$ then the two algebras are **isomorphic** and we write $\mathbf{A} \cong \mathbf{B}$. A homomorphism from $\mathbf{A}$ to $\mathbf{A}$ is an **endomorphism** and a bijective endomorphism is an **automorphism**.

The definition of direct product of a family of algebras is more complex. Given a family of sets $(A_i)_{i\in I}$ a **choice function** ϕ on the family is a function $\phi : I \longmapsto \bigcup_{i\in I} A_i$ with the property that $\phi(i) \in A_i$ for all $i \in I$. The **direct product** of the family of similar algebras $(A_i)_{i\in I}$ is denoted by $\prod_{i\in I} \mathbf{A}_i$. It is an algebra whose universe is $\{\phi : \phi \text{ is a choice function on } (A_i)_{i\in I}\}$ and the operations are defined as follows; for every n-ary operation f in the type and for any $a^1, \ldots a^n \in \prod_{i\in I} A_i$ the realization on $\prod_{i\in I} \mathbf{A}_i$ is given by

$$(f(a^1, \ldots, a^n))_i = f(a_i^1, \ldots, a_i^n).$$

If $\mathbf{A} = \mathbf{A}_i$ for all I we will write $\mathbf{A}^I$ and we will say that A^I is a **direct power** of $\mathbf{A}$. In this case of course $A^I = \{f : f \text{ is a function from } T \text{ to } A\}$.

We observe also that if I is finite, say $I = \{1, \ldots, n\}$, then $\prod_{i=1}^n A_i$ is just the cartesian product and the operations on $\prod_{i=1}^n \mathbf{A}_i$ are defined componentwise.

If K is a class of similar algebras then

$\mathbf{H}(\mathsf{K})$ is the class of all homomorphic images of algebras in K;

$\mathbf{S}(\mathsf{K})$ is the class of all subalgebras of algebras in K;

$\mathbf{P}(\mathsf{K})$ is the class of all direct products of families of algebras in K;

$\mathbf{P}_{fin}(\mathsf{K})$ is the class of all direct products of finite families of algebras in K.

K is**closed** under homomorphic images, subalgebras or (finite) direct products if, respectively, $\mathbf{H}\mathsf{K} \subseteq \mathsf{K}$, $\mathbf{S}\mathsf{K} \subseteq \mathsf{K}$ or $\mathbf{P}\mathsf{K} \subseteq \mathsf{K}$ ($\mathbf{P}_{fin}(\mathsf{K}) \subseteq \mathsf{K}$. A class of algebras V is a **variety** if it closed under $\mathbf{H}$, $\mathbf{S}$ and $\mathbf{P}$; since the intersection of any family of varieties is still a variety it makes sense to define, for a class K, $\mathbf{V}(\mathsf{K})$ as the smallest variety containing K. We say that $\mathbf{V}(\mathsf{K})$ is the **variety generated by** K.

1.2 Posets and lattices

Given a set P a **preorder** on P, denoted by $\leq$ is a binary relation on P that is reflexive and transitive; if $\leq$ is also antisymmetric, then it is called a **partial order** and $\langle P, \leq \rangle$ is a **partially ordered set** or, briefly, a **poset**.

Lemma 1.2.1. *Given any preorder $\leq$ on a set P, then the relation*

$$a \equiv b \qquad \textit{if and only if} \qquad a \leq b \textit{ and } b \leq a$$

is an equivalence relation $\equiv$ on P. The quotient set $P/\equiv$ (i.e. the set of all $\equiv$-classes of P) is a poset where the order relation is

$$a/\equiv \leq b/\equiv \qquad \textit{if and only if} \qquad a \leq b.$$

A **semilattice** is an algebra $\langle S, \wedge$ where $\wedge$ is a binary operation called **meet** that is idempotent, commutative and associative, i.e. for all $a, b, c \in S$

$$\begin{aligned} &a \wedge a = a \\ &a \wedge b = b \wedge a \\ &a \wedge (b \wedge c) = (a \wedge b) \wedge c. \end{aligned}$$

A **lattice** is an algebra $\mathbf{L} = \langle A, \vee, \wedge \rangle$ where $\vee$ and $\wedge$ are binary (and $\vee$ is called **join**), $\langle L, \vee \rangle$ and $\langle L, \wedge \rangle$ are semilattices and moreover for all $a, b \in L$

$$\begin{aligned} &a \vee (a \wedge b) = a \\ &a \wedge (a \vee b) = a. \end{aligned}$$

There is a connection between posets and (semi)lattices.

Theorem 1.2.2. *Let P be a poset. Then:*

1. *if $\inf_{\leq} X$ exists for any finite subset $X \subseteq A$, then, upon defining $a \wedge b = \inf_{\leq}\{a, b\}$, $\langle P, \wedge \rangle$ is a semilattice.*

2. *if $\inf_{\leq} X$ and $\sup_{\leq} X$ exist for any finite subset $X \subseteq A$, then, upon defining $a \wedge b = \inf_{\leq}\{a, b\}$ and $a \vee b = \sup_{\leq}\{a, b\}$, $\langle P, \vee, \wedge \rangle$ is a lattice.*

Let $\mathbf{S} = \langle S, \wedge \rangle$ be a semilattice. Then upon defining $a \leq b$ if and only if $a \wedge b = a$, L is a poset in which every finite subset has an infimum.
Let $\mathbf{L} = \langle L, \vee, \wedge \rangle$ be a lattice. Then:

1. *for any $a, b \in L$, $a \wedge b = a$ if and only if $a \vee b = b$;*

2. *upon defining $a \leq b$ if and only if $a \wedge b = a$ if and only if $a \vee b = b$, L is a poset in which every finite subset has supremum and infimum.*

Two elements a, b of a lattice $\mathbf{L}$ are **comparable** if $a \leq b$ o $b \leq a$; otherwise they are **incomparable**. If any pair of elements in a lattice is comparable, than $\mathbf{L}$ is totally ordered; any totally oredered subset of $\mathbf{L}$ is a **chain**. Conversely an **antichain** is a subset of L consisting of elements that are pairwise incomparable.

Given $a \leq b$ in $\mathbf{L}$ we will say that a **is covered by** b (equivalently b **is a cover of** a if there is no element between a and b: ormally for all c if $a \leq c < b$, then $a = c$. In this case we will write $a \prec b$.

A lattice is **upper (lower) bounded** if it a largest element (smallest element) in the ordering; the upper and lower bound are often denoted by 1 and 0. A lattice is **bounded** if it is upper and lower bounded; it is clear form the definition that every finite lattice is bounded. If $\mathbf{L}$ is bounded an **atom** of $\mathbf{L}$ is an element $a \in L$ is a cover of 0. Similarly a **coatom** is a $b \in L$ that is covered by 1.

The best way of understanding the order structure of a lattice is to draw its **Hasse diagram**; an informal way of constructing the Hasse diagram of a lattice is to put the elements on a plane in such a way that if $a < b$ then b is above a. Then we draw a line from a to b, every time that $a \prec b$.

There are two words of caution however; first the Hasse diagram determines the order structure of a lattice if and only if the underlying ordering is the transitive closure of the relation "$a \prec b$ or $a = b$". Second it is not enough to draw a set of points in the plane joined by lines to get a Hasse diagram of a lattice. For instance the configuration in Figure 1.1 looks like a Hasse diagram of a lattice but really isn't.

A lattice $\mathbf{L}$ is **complete** if every subset of $\mathbf{L}$ has an infimum and a supremum in the ordering. It is clear that a complete lattice is always bounded (why?). An element a of a complete lattice $\mathbf{L}$ is **compact** if for every $X \subseteq L$, if $a = \bigvee X$, then $a \in X$. A lattice is **algebraic** if it is complete and every element is a join of compact elements.

It is clear that every element of a finite lattice is compact, so every finite lattice is algebraic. A more relevant example is below together with a characterization of of algebraic lattices.

Proposition 1.2.3. *If A is a set and $\mathsf{P}(A)$ is its power set, then $\langle \mathsf{P}(A), \cup, \cap \rangle$*

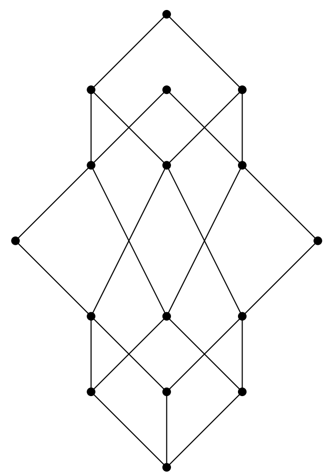


Figure 1.1: A “false” lattice

is an algebraic lattice in which the compact elements are precisely the finite subsets.

Proposition 1.2.4. *For a lattice* $\mathbf{L}$ *the following are equivalent:*

1. $\mathbf{L}$ *is complete;*

2. *every subset of* L *has a supremum;*

3. *every chain (i.e. totally ordered subset) of* $\mathbf{L}$ *has an infimum and a supremum.*

1.3 Closure operators and lattices

If A is a set, an operator $\overline{} : \mathsf{P}(A) \longmapsto \mathsf{P}(A)$ is a **closure operator** if for any $X, Y \subseteq A$

$X \subseteq \overline{X}$;

$\overline{\overline{X}} = \overline{X}$;

se $X \subseteq Y$ allora $\overline{X} \subseteq \overline{Y}$.

A subset of $X \subseteq A$ is **closed** if $X = \overline{X}$. It is easy to show that the intersection of any family of closed sets is closed; this gives substance to the following:

Theorem 1.3.1. *Let $\overline{}$ be a closure operator on A and let C_A the set of closed subsets of A. Then $\mathbf{C}_A = \langle C_A, \vee, \wedge \rangle$ is a complete lattice where, for $U, V \in C_A$*

$$U \vee V = \overline{(U \cup V)} \qquad U \wedge V = U \cap V.$$

Conversely if $\mathbf{L}$ is a complete lattice then $\mathbf{L}$ is isomorphic with the lattice of closed sets of a suitable closure operator on $\mathbf{L}$.

It is easy to show that for any closure operator a closed set is equal to the union of the closures of its finite subsets. An extension of this property gives a very important definition.

A closure operator $\overline{}$ on A is **algebraic** if for all $X \subseteq A$

$$\overline{X} = \bigcup \{ \overline{Z} : Z \subseteq X \ Z \text{ finite} \}.$$

The following characterizes algebraic closure operators.

Lemma 1.3.2. *If $\overline{}$ is a closure operator on A the following are equivalent:*

1. *$\overline{}$ is algebraic;*
2. *if $a \in \overline{X}$ then $a \in \overline{Y}$ for some finite subset $Y \subseteq X$;*
3. *every union of an upward directed set of closed sets is closed.*

This allows au to prove:

Theorem 1.3.3. *Let $\overline{}$ be an algebraic closure operator on A; then $\mathbf{C}_A = \langle C_A, \vee, \wedge \rangle$ is an algebriac lattice in which the compact elements are exactly the closures of finite subsets of A.*

Conversely any algebraic lattice $\mathbf{L}$ is isomorphic with the lattice of closed sets of a suitable algebraic closure operator on A.

Algebraic closure operators (and thus algebraic lattices) are ubiquitous in general algebra. For instance it is not hard to show that for any algebra $\mathbf{A}$ and X, the operator $X \longmapsto \mathrm{Sub}_{\mathbf{A}}(X)$ is an algebraic closure operator and thus subalgebra have the natural structures of an algebraic lattice[1]. By the same token the lattice of normal subgroups of a group and the lattices of ideals of a commutative ring are algebraic.

[1] actually this is slightly incorrect but can be fixed; can you tell why and how?

1.4 Modular and distributive lattices

A lattice **L** is **modular** if for all $a, b, c \in L$

$$c \leq a \qquad \text{implies} \qquad a \wedge (b \vee c) = (a \wedge b) \vee c;$$

a lattice **L** is **distributive** if for all $a, b, c \in L$

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c).$$

While the genesis of the term "distributive" is clear, the term "modular" is not so obvious. It turns out (and it is an easy exercise) that the lattice of normal subgroups of a group is modular. Thus the lattice of any $\mathbb{Z}$-module is modular and this is where the word came from.

It is obvious that every distributive lattice is modular; in Figure 1.2 are shown the smallest non modular lattice and the smallest modular non distributive lattice.

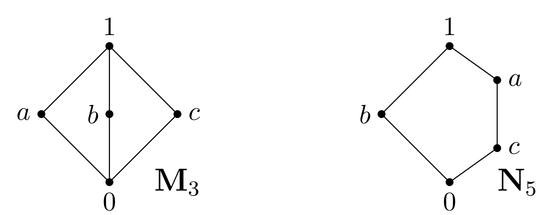


Figure 1.2: $\mathbf{M}_3$ and $\mathbf{N}_5$

It is obvious that distributive lattices form a variety; it is less obvious, but nevertheless true, that modular lattices for a variety as well. $\mathbf{M}_3$ and $\mathbf{N}_5$ turn out to be instrumental in describing those varieties, due to the two following classical results by Dedekind. If **L** e **M** are lattices we say that**L** **omits M** if **L** does not have any sublattice isomorphic **M**.

Theorem 1.4.1. *(Dedekind) For a lattice* **L** *the following are equivalent:*

1. **L** *is modular;*
2. *for all $a, b, c \in L$ if $c \leq a$, $a \wedge b = c \wedge b$ and $a \vee b = c \vee b$, then $a = c$;*
3. **L** *omits* $\mathbf{N}_5$.

Theorem 1.4.2. *(Dedekind) For a lattice* $\mathbf{L}$ *the following are equivalent:*

1. $\mathbf{L}$ *is distributive;*

2. *for all* $a, b, c \in L$ *if* $a \wedge b = c \wedge b$ *and* $a \vee b = c \vee b$*, then* $a = c$*;*

3. $\mathbf{L}$ *omits* $\mathbf{N}_5$ *and* $\mathbf{M}_3$*.*

1.5 Congruences

Let $\varphi : \mathbf{A} \longrightarrow \mathbf{B}$ be a homomorphism; we define the **kernel** of f as the relation on A

$$\ker(\varphi) = \{(a, b) : \varphi(a) = \varphi(b)\}.$$

A **congruence** of $\mathbf{A}$ is an equivalence relation θ on A which is compatible with the basic operations; i.e. if $F(x_1, \dots, x_n)$ is a basic operation of $\mathbf{A}$ and $(a_i, b_i) \in \theta$ for $i = 1. \dots, n$, then $(F(a_1, \dots, a_n), F(b_1, \dots, b_n)) \in \theta$. This yields a structure of an algebra similar to $\mathbf{A}$ on the quotient set A/θ and the map $a \longmapsto a/\theta$ is clearly an epimorphism, dubbed the **natural epimorphism**. The following is a simple exercise.

Proposition 1.5.1. *For any algebra* $\mathbf{A}$ *and* $\theta \subseteq A \times A$ *the following are equivalent:*

1. θ *is a congruence of* $\mathbf{A}$*;*

2. θ *is a reflexive, symmetric and transitive subalgebra of* $\mathbf{A} \times \mathbf{A}$*;*

3. θ *is the kernel of an homomorphism form* $\mathbf{A}$*.*

Since the intersection of any family of congruence is clearly a congruence, given $X \subseteq A \times A$ one can consider the congruence **generated by** X **in** $\mathbf{A}$ as the intersection of all congruence of $\mathbf{A}$ containing X. This congruence is denoted by $\vartheta_{\mathbf{A}}(X)$. Since the operator $X \longmapsto \vartheta_{\mathbf{A}}(X)$ is clearly a closure operator (and it is easy to show that it is also algebraic) the congruence of $\mathbf{A}$ form an algebraic lattice denoted by $\mathrm{Con}(\mathbf{A})$.

Congruences are important for the three homomorphism theorems below.

Theorem 1.5.2. *(First Homomorphism Theorem) Let* $\mathbf{A}, \mathbf{B}$ *similar algebras* $h : \mathbf{A} \longrightarrow \mathbf{B}$ *a homomorphism. If* $\theta = \ker(h)$ *and* g *is the natural* *epimorphism,there exists a (unique) monomorphism $f : A/\theta \longrightarrow B$ satisfying $fg = h$. Moreover $\widehat{f} : \mathbf{A}/\theta \longrightarrow h(\mathbf{A})$, defined as $\widehat{f}(a/\theta) = h(a)$ is an isomorphism. Hence if h is an epimorphism , then $f = \widehat{f}$ is an isomorphism from $\mathbf{A}/\theta$ to $\mathbf{B}$.*

$$\begin{tikzcd}
\mathbf{A} \arrow[r, two heads, "h"] \arrow[dr, "g"'] & h(\mathbf{A}) \arrow[r, tail, "i"] & \mathbf{B} \\
& \mathbf{A}/\theta \arrow[u, tail, two heads, "\widehat{f}"'] \arrow[ur, tail, "f"'] &
\end{tikzcd}$$

Theorem 1.5.3. *(Second Homomorphism Theorem) Let $f : \mathbf{A} \longrightarrow \mathbf{B}$ and $g : \mathbf{A} \longrightarrow \mathbf{C}$ two homomorphism such that f is surjective and $\ker(f) \subseteq \ker(g)$. Then there exists a (unique) homomorphism $h : \mathbf{B} \longrightarrow \mathbf{C}$ satisfyin $g = h$. Moreover h is a monomorphism if and only if $\ker(f) = \ker(g)$.*

$$\begin{tikzcd}
\mathbf{A} \arrow[r, two heads, "f"] \arrow[dr, "g"'] & \mathbf{B} \arrow[d, "h"] \\
& \mathbf{C}
\end{tikzcd}$$

Let $\mathbf{A}$ be and algebra nd $\alpha, \beta \in \mathrm{Con}(\mathbf{A})$ with $\alpha \leq \beta$; then it is easy to show that

$$\beta/\alpha = \{(a/\beta, b/\beta) : (a, b) \in \alpha\}$$

is a congruence of $\mathbf{A}/\alpha$.

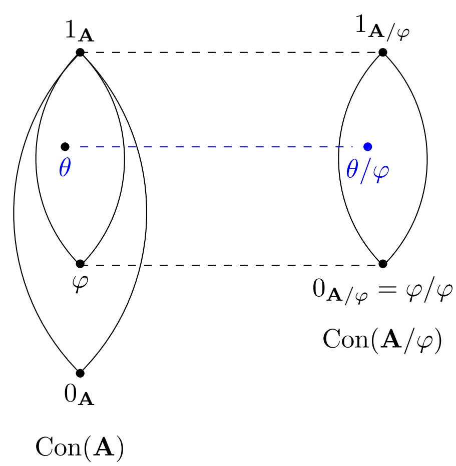
Corollary 1.5.4. *Let $\mathbf{A}$ be and algebra nd $\alpha, \beta \in \mathrm{Con}(\mathbf{A})$ with $\alpha \leq \beta$. Then*

$$(a/\alpha)/(\beta/\alpha) \longmapsto a/\beta$$

is an isomorphism from $(\mathbf{A}/\alpha)/(\beta/\alpha)$ to $\mathbf{A}/\beta$.

$$
\begin{tikzcd}
\mathbf{A} \arrow[r, two heads, "\pi_\alpha"] \arrow[dr, "\pi_\beta"'] & \mathbf{A}/\alpha \arrow[d, two heads, "f"] \arrow[dr, "\pi_{\beta/\alpha}"] & \\
& \mathbf{A}/\beta \arrow[r, tail, two heads, "g"] & (\mathbf{A}/\alpha)/(\beta/\alpha)
\end{tikzcd}
$$

Theorem 1.5.5. (Third Homomorphism Theorem) *Let* $\mathbf{A}$ *be an algebra and let* $\varphi \in \mathrm{Con}(\mathbf{A})$*. The function on* $I[\varphi, 1_{\mathbf{A}}]$ *defined by* $\theta \longmapsto \theta/\varphi$ *is an isomorphism from* $[\varphi, 1_{\mathbf{A}}]$ *to* $\mathrm{Con}(\mathbf{A}/\varphi)$*.*



1.6 Free algebras

An **identity** is a formal expression involving operation symbols. Formally, given a type τ and set X of variables we can construct the **terms** of type τ. A term is a string of symbols constructed in the following way:

- every variable is a term;

- if F is a n-ary operation symbol in τ and $g_1, \ldots, g_n$ are terms, then $f(g_1, \ldots, g_n$ is a term.

There is a fine point here that has to be stressed, since we will be using it implicitly many times. The terms of type τ have the **unique readability property**, in other words they are the same if and only if they are formally the same. That is expressed in the following lemma whose proof is essentially combinatory in nature.

Lemma 1.6.1. *(Unique readability) Let $f(g_1, \ldots, g_n)$ and $f'(g'_1, \ldots, g'_m)$ two terms of the same type τ. Then $(g_1, \ldots, g_n) = f'(g'_1, \ldots, g'_m)$ if and only if $n = m$, $f = f'$ and $g_i = g'_i$ for all $i \leq n$.*

If we denote by $T_\tau(X)$ the set of all terms of type τ then it possible to give $T_\tau(X)$ the structure of an algebra $\mathbf{T}_\tau(X)$ of type τ; if $f(x_1, \ldots, x_)$ is an operations in τ and $g_1, \ldots, g_n \in T_\tau(X)$ then

$$f(g_1, \ldots, g_n) = fg_1, \ldots, g_n.$$

We have that:

Proposition 1.6.2. *Let $\mathbf{A}$ be an algebra of type τ and let $\alpha : X \longrightarrow A$ be any function; then there exists a homomorphism $\beta : \mathbf{T}_\tau(X) \longrightarrow \mathbf{A}$ such that $\beta(x) = \alpha(x)$ for all $x \in X$.*

The property in Proposition 1.6.2 suggests a definition. Let V be ant variety; we say that an algebra $\mathbf{U}$ is **free in V for** X if

- $\mathbf{U} \in \mathsf{V}$;
- $\mathbf{U}$ is generated by X;
- for all $\mathbf{A} \in \mathsf{V}$ and for any function $\alpha : X \longrightarrow A$ there exists a homomorphism $\beta : \mathbf{U} \longrightarrow \mathbf{A}$ such that $\beta(x) = \alpha(x)$ for all $x \in X$.

It is not hard to show that a free algebra is totally determined by the cardinality of the set of generators; for that reason we may assume that X is always a cardinal. We will use this fact without any further mention. On the other hand it is entirely nontrivial to show that free algebras exist for any variety V, provided that either $X \neq \emptyset$ or else the type of V contains at least a constant, and that this algebra, denoted by $\mathbf{F}_{\mathsf{V}}(X)$ is unique up to isomorphism. Namely if

$$\Theta(\mathsf{V}) = \bigcup\{\theta \in \mathrm{Con}(\mathbf{T}_\tau(X)) : \mathbf{T}_\tau(X)/\theta \in \mathsf{V}\}$$

on can show that $\mathbf{F}_{\mathsf{V}}(X) \cong \mathbf{T}_\tau(X)/\Theta(\mathsf{V})$.

Let $\mathbf{A}$ be an algebra of type τ and $p \in T_\tau(n)$. We define an n-ary operation $p^{\mathbf{A}}$ on $\mathbf{A}$ by induction on the complexity of p. If $p = x_i$ then $p^{\mathbf{A}}(a_1, \dots, a_n) = a_i$, i.e. the i-th projection. If $p = F(p_1, \dots, p_k)$ then

$$p^{\mathbf{A}}(a_1, \dots, a_n) = F(p_1^{\mathbf{A}}(a_1, \dots, a_n), \dots, p_k^{\mathbf{A}}(a_1, \dots, a_n)).$$

Proposition 1.6.3. *Let $\mathbf{A}$ be an algebra of type τ and let $n > 0$ τ does not contain any constant. Then $p \longmapsto p^{\mathbf{A}}$ is an epimorphism of $\mathbf{T}_\sigma(n)$ onto $\mathrm{Clo}_n(\mathbf{A})$. Therefore an operation f on A is a term of $\mathbf{A}$ is and only if $f = p^{\mathbf{A}}$ for some term p of type τ.*

1.7 The HSP theorem

An **identity** of type τ (in X) is a formal expression $p(x_1, \dots, x_n) \approx q(y_1, \dots, y_m)$ where $p, q \in T_\tau(X)$. An algebra **A satisfies the identity** $p \approx q$ if there are $p^{\mathbf{A}}, q^{\mathbf{A}} \in \mathrm{Clo}(\mathbf{A})$, respectively n-ary and m-ary such that for all $a_1, \dots, a_n, b_1, \dots, b_m \in A$

$$p^{\mathbf{A}}(a_1, \dots, a_n) = q^{\mathbf{A}}(b_1, \dots, b_m).$$

In this case we will write $\mathbf{A} \vDash p \approx q$ and if K is a class of algebras $\mathsf{K} \vDash p \approx q$ if $\mathbf{A} \vDash p \approx q$ for all $\mathbf{A} \in \mathsf{K}$. If Σ is a set of equation, then $\mathbf{A} \vDash \Sigma$ if $\mathbf{A} \vDash p \approx q$ for all $p \approx q \in \Sigma$ and similarly for $\mathsf{K} \vDash \Sigma$.

The importance of free algebras is clear from the following;

Theorem 1.7.1. *Let V be a variety, p, q n-ary terms of the same type as V, X a set and $x_1, \dots, x_n \in X$. The following are equivalent:*

1. $\mathsf{K} \vDash p \approx q$.
2. $(p, q) \in \Theta_{\mathbf{T}_\sigma(\omega)}(\mathsf{K})$.
3. *Se $\mathbf{F}_{\mathsf{K}}(X)$ exists, then $\mathbf{F}_{\mathsf{K}}(X) \vDash p \approx q$.*
4. *Se $\mathbf{F}_{\mathsf{K}}(X)$ exists then*

$$p^{\mathbf{F}_{\mathsf{K}}(X)}(x_1, \dots, x_n) = q^{\mathbf{F}_{\mathsf{K}}(X)}(x_1, \dots, x_n).$$

Given a set Σ of equations we define

$$\mathrm{Mod}(\Sigma) = \{\mathbf{A} : \mathbf{A} \models \Sigma\}$$

and if K is a class of algebras we define

$$opTh(\mathsf{K}) = \{p \approx q : \mathsf{K} \models p \approx q\}.$$

Theorem 1.7.2. *(The* **HSP** *theorem) For any class* K *of algebras*

$$\mathrm{Mod}(\mathrm{Th}(\mathsf{K})) = \mathbf{HSP}(\mathsf{K}).$$

The **HSP** theorem implies (among other things) that $\mathbf{B} \in \mathbf{HSP}(\mathbf{A})$ if and only if every equation satisfied by $\mathbf{A}$ is also satisfied by $\mathbf{B}$. Even for this the general proof is too complex for these notes, but we can prove its finite version:

Theorem 1.7.3. *If* $\mathbf{A}$ *and* $\mathbf{B}$ *are finite algebras of the same type* τ *the following are equivalent:*

1. *every identity satisfied by* $\mathbf{A}$ *is also satisfied by* $\mathbf{B}$*;*
2. $\mathbf{B} \in \mathbf{HSP}_{fin}(\mathbf{A})$.

Proof. That (2) implies (1), i.e. that identities are preserved by $\mathbf{HSP}_{fin}$, is easy to prove.

For the converse suppose that $|B| = k$, and let $b_1, \dots, b_k$ be an enumeration of the elements of B. Let $C = A^k$ and let $m = |C|$; then we can enumerate the elements of C as $c^1, \dots, c^m$ and we define $c_i = (c_i^1, \dots, c_i^m)$ for $i = 1, \dots, k$.

Let $\mathbf{S} = \mathrm{Sub}_{\mathbf{A}^m}(c_1, \dots, c_k)$; then by Lemma **??** the elements of $\mathbf{S}$ are of the form

$$t(c_1, \dots, c_k)$$

for some term t in the language of $\mathbf{A}$.

Define $\alpha : \mathbf{S} \longrightarrow \mathbf{B}$ by setting

$$\alpha(t(c_1, \dots, c_k) = t(b_1, \dots, b_k)$$

and observe that α is well-defined because of (1) and it is easily seen to be surjective.

To prove that it is a homomorphism let f be an n-ary operation and let $s_1, \ldots, s_n \in S$. Then there are terms $t_1, \ldots, t_n$ such that $s_i = t_i(c_1, \ldots, c_k)$. Now

$$\begin{aligned}
\alpha(f(s_1, \ldots, s_n)) &= \alpha(f(t_1(c_1, \ldots, c_k), \ldots, t_n(c_1, \ldots, c_k))) \\
&= \alpha(f(t_1, \ldots, t_n)(s_1, \ldots, s_k)) \\
&= f(t_1, \ldots, t_n)(b_1, \ldots, b_k) \\
&= f(t_1(b_1, \ldots, b_k), \ldots, t_n(b_1, \ldots, b_k)) \\
&= f(\alpha(t_1(c_1, \ldots, c_k)), \ldots, \alpha(t_n(c_1, \ldots, c_k))) \\
&= f(\alpha(s_1), \ldots, \alpha(s_k))
\end{aligned}$$

hence α is a homomorphism.

So $\mathbf{B} \in \mathbf{HSP}_{fin}(\mathbf{A})$ and the proof is concluded. $\square$

Chapter 2

Ideals

2.1 What is an ideal?

Given an algebra $\mathbf{A}$ an ideal is an "interesting subset" of the universe A, that may or may not be a subalgebra of $\mathbf{A}$; an example of the first kind is a normal subgroup of the group and of the second kind is an ideal of a commutative ring[1]. Now defining what "interesting" means is largely a matter of taste; however there is a large consensus among the practitioners of the field that:

- an ideal must have a simple algebraic definition;
- ideals must be closed under arbitrary intersections, so that a closure operator can be defined in which the ideals are exactly the closed sets; this gives raise to an algebraic lattice whose elements are exactly the ideals;
- ideals must convey meaningful information on the structure of the algebra.

The three points above are all satisfied by classical ideals on lattices and of course by ideals on a set X, where we interpret a set as an algebra in which the set of fundamental operations is empty. We have however to be careful here; an ideal on a set X is an ideal (in the lattice sense) on the Boolean algebra of subsets of X. There also a significant difference between ideals on lattices and ideals on Boolean algebras; in Boolean algebras an ideal is always the 0-class of a suitable congruence of the algebra (really, of exactly

[1] we follow the modern *dictum* that every ring has a multiplicative unit...

one congruence), while this is not true in general for lattices. As a matter of fact, identifying the class of (lower bounded) lattices in which every ideal is the 0-class of a congruence is a difficult problem which is still unsolved, up to our knowledge. Of course the same property is shared by normal subgroups of a group and (two-sided) ideals of a ring (since they are both congruence kernels).

The problem of connecting ideals of general algebras to congruence classes has been foreshadowed in [12] but really tackled by A. Ursini in his seminal paper [28]. Later, from the late 1980's to the late 1990's, A. Ursini and the author published a long series of papers on the subject ([30], [3], [4], [5], [2]); the theory developed in those papers will constitute the basis of these notes.

2.2 Ideals in universal algebra

We postulated that an ideal must have a simple algebraic definition; as imprecise as this concept might be, in our context there is a natural path to follow. Given a type (a.k.a. a signature) σ we can consider the **σ-terms** (i.e. the elements of $\mathbf{T}_\sigma(\omega)$, the absolutely free countably generated algebra of type σ); a term is denoted by $p(x_1, \dots, x_n)$ to emphasize the variable involved and we will use the vector notation $\vec{x}$ for $x_1, \dots, x_n$. Let Γ be a set of σ-terms; we will divide the (finite) set of variables $z_1, \dots, z_{n+m}$ of each term in two subsets $\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_m\}$ so that every term in Γ can be expressed as $p(\vec{x}, \vec{y})$ and we allow $n = 0$, while m must always be at least 1. Moreover we ask that Γ be closed under *composition on* $\vec{y}$; this means that if $p(\vec{x}, \vec{y} \in \Gamma)$, $\vec{y} = (y_1, \dots, y_m)$ and $p_1(\vec{x}^1, \vec{y}^1), \dots, p_m(\vec{x}^m, \vec{y}^m) \in \Gamma$, then

$$p(\vec{x}, p_1(\vec{x}^1, \vec{y}^1), \dots, p_m(\vec{x}^m, \vec{y}^m)) \in \Gamma.$$

If $\mathbf{A}$ has type σ a Γ-ideal of $\mathbf{A}$ is an $I \subseteq A$ such that for any $a_1, \dots, a_n \in A$, $b_1, \dots, b_m \in I$ and $p(x, y) \in \Gamma$, $p(\vec{a}, \vec{b}) \in I$.

The following is a simple exercise.

Lemma 2.2.1. *Let σ be any type, Γ a set of σ-terms closed under* composition on $\vec{y}$ *and* $\mathbf{A}$ *an algebra of type σ. Then*

1. *the Γ-ideals of* $\mathbf{A}$ *are closed under arbitrary intersections;*

2. *the Γ-ideal generated by $X \subseteq A$, i.e. the intersection of all the Γ-ideals containing X, is*

$$(X)^\Gamma_\mathbf{A} = \{p(\vec{a}, \vec{b}) : \vec{a} \in A, \vec{b} \in X, p(\vec{x}, \vec{y}) \in \Gamma\};$$

3. the Γ*-ideals of* $\mathbf{A}$ *form an algebraic lattice* $\mathrm{Id}^\Gamma(\mathbf{A})$*.*

As usual a Γ-ideal of $\mathbf{A}$ is **principal** if it generated by a single element. It is **compact** if it is compact in the lattice $\mathrm{Id}^\Gamma(\mathbf{A})$; this means that it is generated by a finite set or equivalently is the join of finitely many Γ-ideals. It is evident that the compact Γ-ideals of $\mathbf{A}$ form a join semilattice, denoted by $\mathrm{CI}^\Gamma(\mathbf{A})$.

2.3 0-ideals and ideal terms

At this level of generality we cannot say much more; if the type however contains a constant we can get a more focused definition. Let V be a variety whose type contains a constant which will denote by 0; a $\mathsf{V}, 0$**-ideal term in** $y_1, \ldots, y_m$ is a term $p(\vec{x}, \vec{y})$ such that

$$\mathsf{V} \vDash p(\vec{x}, 0, \ldots, 0) \approx 0.$$

Let $ID_{\mathsf{V},0}$ be the set of all $\mathsf{V}, 0$-ideal terms in V; a $\mathsf{V}, 0$**-ideal** I of $\mathbf{A} \in \mathsf{V}$ is a $ID_{\mathsf{V},0}$-ideal of $\mathbf{A}$. If $\mathsf{V} = \mathbf{V}(\mathbf{A})$ we will simply say that I is a 0**-ideal** of $\mathbf{A}$. As before the set $\mathrm{Id}_{\mathsf{V},0}(\mathbf{A})$ of $\mathsf{V}, 0$-ideals of $\mathbf{A}$ and the set $\mathrm{Id}_0(\mathbf{A})$ of 0-ideals of $\mathbf{A}$ are algebraic lattices and $\mathrm{Id}_0(\mathbf{A}) \subseteq \mathrm{Id}_{\mathsf{V},0}(\mathbf{A})$ (and the inclusion may be strict). It is also evident that for any $\theta \in \mathrm{Con}(\mathbf{A})$, $0/\theta$ is a $\mathsf{V}, 0$-ideal of $\mathbf{A}$: if $p(\vec{x}, \vec{y}) \in ID_{\mathsf{V},0}$, $\vec{a} \in A$ and $\vec{b} \in 0/\theta$ then

$$p(\vec{a}, \vec{b}) \ \theta \ p(\vec{a}, \vec{0}) = 0.$$

In general the ideals of an algebra depend on the variety to which it belongs; we will denote by $\mathrm{Id}(\mathbf{A})$ the set of all $\mathbf{V}(\mathbf{A}), 0$-ideals of $\mathbf{A}$. Similarly $(X)_{\mathbf{A}}$ for $X \subseteq A$ will denote the $\mathbf{V}(\mathbf{A}), 0$-ideal of $\mathbf{A}$ generated by X. The following three propositions are simple exercises.

Proposition 2.3.1. *For any algebra* $\mathbf{A}$ *with* 0*,* $\mathrm{Id}(\mathbf{A})$ *is isomorphic with the ideal lattice of* $\mathrm{CI}(\mathbf{A})$ *(semilattice ideals in the usual sense).*

Proposition 2.3.2. *Let* $\mathbf{R}$ *be a subalgebra of* $\mathbf{A} \times \mathbf{A}$ *such that* $\pi_2(\mathbf{R}) = A$*, where* π_2 *denotes the second projection. If* $K \in \mathrm{Id}(\mathbf{R})$ *and* $I \in \mathrm{Id}(\mathbf{A})$*, then*

$$(I)_K = \{b \in B : \textit{for some } a \in I, (a, b) \in K\}$$

is an ideal of $\mathbf{A}$*.*

In particular $\mathbf{R}$ may be a subdirect product or a reflexive subalgebra of $\mathbf{A} \times \mathbf{A}$.

Proposition 2.3.3. *Let $\theta \in \mathrm{Con}(\mathbf{A})$. There is a one-to-one correspondence (which is in fact a complete lattice isomorphism) between the ideals I of $\mathbf{A}$ such that $0/\theta \subseteq I$ and the ideals of $\mathbf{A}/\theta$. The correspondence is*

$$I \longmapsto I/\theta = \{a/\theta : a \in I\}$$

We say that V has **normal** $\mathsf{V}, 0$**-ideals** if for all $\mathbf{A} \in \mathsf{V}$ for all $I \in \mathrm{Id}_{\mathsf{V},0}(\mathbf{A})$ there is a $\theta \in \mathrm{Con}(\mathbf{A})$ with $I = 0/\theta$. If V has normal $\mathsf{V}, 0$-ideals then of course $\mathrm{Id}_0(\mathbf{A}) = \mathrm{Id}_{\mathsf{V},0}(\mathbf{A}) = \{0/\theta : \theta \in \mathrm{Con}(\mathbf{A})\}$ so we can simply talk about 0-ideals of $\mathbf{A}$ without specifying the variety. Observe that the variety of pointed (by 0) sets has normal 0-ideals, so we can hardly expect any nice structural theorem for varieties with 0-normal ideals.

However something can be said also in this case. Let $\mathbf{A}$ be an algebra in a variety with a constant 0 and let $N(\mathbf{A}) = \{0/\theta : \theta \in \mathrm{Con}(\mathbf{A})\}$. Then

Theorem 2.3.4. *[2] For any algebra $\mathbf{A}$ the following are equivalent:*

1. *$\mathbf{A}$ has normal ideals;*
2. *$(X)_{\mathbf{A}} = 0/\vartheta_{\mathbf{A}}(X)$ for any $X \subseteq A$;*
3. *$I/\vartheta_{\mathbf{A}}(J) = I \vee J$ for any $I, J \in \mathrm{Id}(\mathbf{A})$;*
4. *$I/\vartheta_{\mathbf{A}}(J) = J/\vartheta_{\mathbf{A}}(I)$ for any $I, J \in \mathrm{Id}(\mathbf{A})$;*
5. *the mapping from $\mathrm{Id}(\mathbf{A})$ to $\mathrm{Con}(\mathbf{A})$ sending $I \longmapsto \vartheta_{\mathbf{A}}(I)$ is one-to-one;*
6. *the mapping from $\mathrm{Con}(\mathbf{A})$ to $\mathrm{Id}(\mathbf{A})$ sending $\theta \longmapsto 0/\theta$ is onto.*

Proof. (1),(2) and (6) are clearly equivalent. (2) implies (5) since, if $I, J \in \mathrm{Id}(\mathbf{A})$ and $\vartheta_{\mathbf{A}}(I) = \vartheta_{\mathbf{A}}(J)$, then $0/\vartheta_{\mathbf{A}}(I) = 0/\vartheta_{\mathbf{A}}(J)$, hence, via (2), $I = J$.

Assume now (5); we claim that, for any $I \in \mathrm{Id}(\mathbf{A})$, we have $\vartheta_{\mathbf{A}}(I) = \vartheta_{\mathbf{A}}(0/\vartheta_{\mathbf{A}}(I))$. One inclusion is obvious, since $I \subseteq 0/\vartheta_{\mathbf{A}}(I)$. Let then $(u, v) \in \vartheta_{\mathbf{A}}(0/\vartheta_{\mathbf{A}}(I))$; by the congruence generation theorem, there exists a positive integer n, $a_1, \dots, a_n, b_1, \dots, b_n \in 0/\theta(I)$ and binary polynomials $f_1, \dots, f_n$

such that

$$\begin{aligned} u &= f(a_1, b_1) \\ f_1(b_1, a_1) &= f_2(a_2, b_2) \\ &\vdots \\ f_n(b_n, a_n) &= v. \end{aligned}$$

Since $a_i, b_i \in 0/\vartheta_{\mathbf{A}}(I)$ for all i, we have

$$\begin{aligned} u &= f_1(a_1, b_1)\ \vartheta_{\mathbf{A}}(I)\ \varphi_1(0,0)\ \vartheta_{\mathbf{A}}(I)\ f_1(b_1, a_1) \\ &= f_2(a_2, b_2)\ \vartheta_{\mathbf{A}}(I)\ f_2(0,0)\ \vartheta_{\mathbf{A}}(I) \ldots\ \vartheta_{\mathbf{A}}(I)\ f_n(a_n, b_n) = v. \end{aligned}$$

So $(u, v) \in \vartheta_{\mathbf{A}}(I)$ and the claim is proved. But via (5) this implies $I = 0/\theta(I)$ and hence (2).

Assume now (2). The left-to-right inclusion in (3) is easy, since $u \in I/\vartheta(J)$ implies that $(i, u) \in \vartheta_{\mathbf{A}}(J)$ for some $i \in I$. But $0 \in I$, so $(0, i) \in \vartheta_{\mathbf{A}}(I)$ and hence $(0, u) \in \vartheta_{\mathbf{A}}(I) \vee \vartheta_{\mathbf{A}}(J) \subseteq \vartheta_{\mathbf{A}}(I \vee J)$. Hence, by (2) $u \in I \vee J$. On the other hand, if $u \in I \vee J$, then there is an ideal term $t(\vec{x}, \vec{y})$, $\vec{a} \in A$, $\vec{i} \in I$ and $\vec{j} \in J$, such that $u = t(\vec{a}, \vec{i}, \vec{j})$. But then

$$u = t(\vec{a}, \vec{i}, \vec{j})\ \vartheta_{\mathbf{A}}(J)\ t(\vec{a}, \vec{i}, 0, \ldots, 0) \in I.$$

Therefore $u \in I/\vartheta_{\mathbf{A}}(J)$ and the other inclusion is proved.

That (3) implies (4) is obvious. On the other hand if in the equality in (4) we set $J = \langle 0 \rangle_I = \{0\}$ we get at once $I = I/\vartheta_{\mathbf{A}}(\{0\}) = 0/\vartheta_{\mathbf{A}}(I)$. Hence (4) implies (2) and the proof is finished. $\square$

In analogy to subdirect irreducibility let us define an algebra $\mathbf{A}$ to be **ideal irreducible** if for any family $(I_\lambda)_{\lambda \in \Lambda}$ of ideals of $\mathbf{A}$, if $\bigcap_{\lambda \in \Lambda} I_\lambda = (0)$, then, for some λ, $I_\lambda = (0)$; the concept of **finitely ideal irreducible** is defined in an obvious way. An algebra $\mathbf{A}$ is ideal irreducible if and only if there is a minimal nonzero ideal, which then must be principal, generated by a **monolithic element** a (namely $a \neq 0$ and $a \in I$ for any nonzero $I \in \mathrm{Id}(\mathbf{A})$).

Proposition 2.3.5. *Assume* $\mathrm{N}(\mathbf{A}) = \mathrm{Id}(\mathbf{A})$. *If* $a \neq 0$, $a \in A$, *then there is a* $\theta \in \mathrm{Con}(\mathbf{A})$ *such that* a/θ *is monolithic in* $\mathbf{A}/\theta$.

Proof. In fact, let H be maximal in $\{I \in \mathrm{Id}(\mathbf{A}) : a \notin I\}$, via Zorn Lemma. Let $H = 0/\theta$; if $J \in \mathrm{I}(\mathbf{A}/\theta)$ then by Proposition 2.3.3 $J = I/\theta$ with $H \subseteq I$. Take $b/\theta \in J$ with $b/\theta \neq 0/\theta$; then $b \in I - H$. Hence, by the maximality of H, $a \in I$, Hence $a/\theta \in J$. □

Proposition 2.3.6. *Let $\theta \in \mathrm{Con}(\mathbf{A})$. If $a \notin 0/\theta$ and $\mathrm{N}(\mathbf{A}/\theta) = \mathrm{I}(\mathbf{A}/\theta)$, then there is a $\varphi \in \mathrm{Con}(\mathbf{A})$, $\varphi \supseteq \theta$ such that a/φ is monolithic in $\mathbf{A}/\varphi$.*

Proof. In fact, apply 2.3.5 to $\mathbf{A}/\theta$ and recall that congruences of $\mathbf{A}/\theta$ corresponds to congruences of $\mathbf{A}$ containing θ. We then get a congruence $\varphi \supseteq \theta$ such that $(a/\theta)/(\varphi/\theta)$ is monolithic in $(\mathbf{A}/\theta)/(\varphi/\theta)$, namely a/φ is monolithic in $\mathbf{A}/\varphi$. □

2.4 Subtractive varieties

Almost all varieties with a good theory of ideals have a binary term whose behavior reminds the difference between ordinary numbers. This is not a coincidence as we will see. From now on all algebras and varieties will have a constat 0 in their type.

A variety V is **subtractive** if there exists a binary term $s(x, y)$ such that V satisfies the equations

$$s(x, x) \approx 0 \qquad s(x, 0) \approx x.$$

An algebra $\mathbf{A} \in \mathsf{V}$ is said to be 0**-permutable**, or to have 0**-permutable congruences** if for all $a \in A$ and $\theta, \varphi \in \mathrm{Con}(\mathbf{A})$, if $(a, 0) \in \theta \circ \varphi$, then $(a, 0) \in \varphi \circ \theta$.

An algebra $\mathbf{A}$ is called **ideal-coherent** if, for any $I \in \mathrm{Id}(\mathbf{A})$ and $\theta \in \mathrm{Con}(\mathbf{A})$, $0/\theta \subseteq I$ yields that I is a union of θ-blocks.

Theorem 2.4.1. *[2] For a variety V the following are equivalent:*

1. *for all $\mathbf{A} \in \mathsf{V}$ and $\theta, \varphi \in \mathrm{Con}(\mathbf{A})$ we have $0/(\theta \vee \varphi) = 0/(\theta \circ \varphi)$;*
2. *every algebra in V has 0-permutable congruences;*
3. *V is subtractive ;*
4. *there is a ternary term $w(x, y, z)$ of V such that*

$$w(x, y, y) \approx x \qquad\qquad w(x, x, 0) \approx 0$$

hold in V;

5. *there exists a positive integer* m*, binary terms* $d_1(x,y),\dots,d_m(x,y)$ *and an* $m+3$*-ary term* $q(x_1,\dots,x_{m+3})$ *of* V *such that*

$$\begin{aligned}&d_i(x,x)\approx 0 \quad \textit{for} \quad i=1,\dots,m\\ &q(x,y,0,0,\dots,0)\approx 0\\ &q(x,y,y,d_1(x,y),\dots,d_m(x,y))\approx x\end{aligned}$$

hold in V*;*

6. V *is* **ideal-coherent***;*

7. *for all* $\mathbf{A}\in\mathsf{V}$*, the mapping* $\mathrm{Con}(\mathbf{A})\longrightarrow\mathrm{Id}(\mathbf{A})$ *defined by* $\theta\longmapsto 0/\theta$ *is a complete and onto lattice homomorphism.*

Proof. (1) trivially implies (2). Assume (2) and consider $\mathbf{F}=\mathbf{F}_{\mathsf{V}}(x,y)$; if $\theta=\vartheta_{\mathbf{F}}(x,y)$ and $\varphi=\vartheta_{\mathbf{F}}(y,0)$, then $(x,0)\in\theta\circ\varphi$. Then $(x,0)\in\varphi\circ\theta$ and so the usual Mal'cev argument yields a term $s(x,y)$ satisfying the equations. If (3) holds we set $w(x,y,z):=s(x,s(y,z))$ and we check that the equations in (4) hold. Finally if (4) holds and $(a,0)\in\theta\circ\varphi$ then there is a b such that $a\ \theta\ b\ \varphi\ 0$; hence

$$a=w(a,0,0)\ \varphi\ w(a,b,0)\ \theta\ w(b,b,0)=0.$$

So $(a,0)\in\theta\circ\varphi$. This implies that $0/\theta\circ\varphi=0/\varphi\circ\theta$ and also very easily (1). So (1)-(4) are equivalent.

Moreover (3) implies (5) if one puts $m=1$, $d_1(x,y)=s(x,y)$ and $q(x,y,z,w)=s(x,s(s(x,z),w))$.

Assume then (5). Let $I\in\mathrm{Id}(\mathbf{A})$, $\theta\in\mathrm{Con}(\mathbf{A})$ and $0/\theta\subseteq I$. Let $v\in I$ with $(u,v)\in\theta$; then for all $i\le m$ we have $d_i(u,v)\ \theta\ d_i(v,v)=0$, hence $d_i(u,v)\in I$. Note that $q(x,y,\vec{z})$ is an ideal term in $\vec{z}$, so we must have

$$u=q(u,v,v,d_1(u,v),\dots,d_m(u,v))\in I.$$

Hence (5) implies (6). For the converse, assume that V is ideal-coherent and look at $\mathbf{F}_{\mathsf{V}}(x,y)$. Let θ_f be the congruence associated with the endomorphism of $\mathbf{F}_{\mathsf{V}}(x,y)$ defined by $f(x)=f(y)=x$ and $f(0)=0$. Let

$$I=(\{y\}\cup\{d(x,y)\in\mathbf{F}_{\mathsf{V}}x,y:d(x,y)\in 0/\theta_f\})_{\mathbf{A}}.$$

Then clearly $0/\theta_f \subseteq I$, $y \in I$ and $(x, y) \in \theta_f$. Thus ideal-coherency yields $x \in I$. Then there is an ideal term $t(\vec{x}, y, \vec{z})$ in $y \cup \vec{z}$ and $d_1(x, y), \ldots, d_m(x, y) \in 0/\theta_f$ with

$$t(\vec{u}, y, d_1(x, y), \ldots, d_m(x, y)) = x.$$

Since any $u_j = u_j(x, y)$ we do get $m + 3$-ary term by setting

$$q(x, y, y, z_1, \ldots, z_m) = t(\vec{u}, y, z_1, \ldots, z_m).$$

But then $q(x, y, 0, 0, \ldots, 0) = 0$, since q is an ideal term in $y \cup \vec{z}$. As showed above $q(x, y, y, d_1(x, y), \ldots, d_m(x, y)) = x$ and finally, for all i, $d_i(x, y) \in 0/\theta_f$ that yields $d_i(x, x) = f(d_i(x, y)) = f(0) = 0$. Therefore (5) and (6) are equivalent.

Assume again (5). Let $\mathbf{A} \in \mathsf{V}$, $\theta, \varphi \in \mathrm{Con}(\mathbf{A})$ and $a \in 0/(\theta \circ \varphi)$. Then there is a $b \in A$ with $(0, b) \in \theta$ and $(b, a) \in \varphi$. Hence we get $d_i(a, b)\ \varphi\ d_i(b, b)\ \varphi\ 0$ for all i. So

$$\begin{aligned} 0 &= q(a, b, 0, 0, \ldots, 0) \\ &\ \varphi\ q(a, b, 0, d_1(a, b), \ldots, d_m(a, b)) \\ &\ \theta\ q(a, b, b, d_1(a, b), \ldots, d_m(a, b)) = a, \end{aligned}$$

hence $(0, a) \in \varphi \circ \theta$ and V is 0-permutable. Therefore (5) implies (2).

Assume now (3). Let $\mathbf{A} \in \mathsf{V}$, $\theta, \varphi \in \mathrm{Con}(\mathbf{A})$ and $a \in 0/(\theta \vee \varphi)$. Then there are $a_1, \ldots, a_n \in A$ with

$$a\ \theta\ a_1\ \varphi\ \alpha_2\ \theta\ \ldots\ \theta\ a_n\ \varphi\ 0.$$

Let us set $t(x, y, z,) = s(x, s(s(x, y), z,))$ and let us induct on n. If $n = 1$ then $a\ \theta\ a_1\ \varphi\ 0$. Hence $s(a, a_1)\ \theta\ 0$, therefore

$$a = t(a, a_1, s(a, a_1)) \in 0/(\theta \vee \varphi)$$

being $t(x, y, z)$ an ideal term in y, z. Let now assume the statement true for n and let

$$a\ \theta\ a_1\ \varphi\ \alpha_2\ \theta\ \ldots\ \theta\ a_n\ \varphi\ a_{n+1}\ \theta\ 0.$$

Then $s(a, a_{n+1})\ \varphi\ s(a, a_n)\ \theta\ \ldots\ \theta\ s(a, a) = 0$, so, by induction hypothesis, $s(a, a_{n+1}) \in 0/\theta \vee 0/\varphi$. But since $a_{n+1} \in 0/\theta$ we get again

$$a = t(a, a_{n+1}, s(a, a_{n+1})) \in 0/\theta \vee 0/\varphi.$$

The case $a_{n+1}\ \varphi\ 0$ is totally similar hence we conclude that $0/(\theta \vee \varphi) \subseteq 0/\theta \vee 0/\varphi$. For the converse let $a \in 0/\theta \vee 0/\varphi$; then there are an ideal term $p(\vec{x}, \vec{y}, \vec{z})$ in $\vec{y} \cup \vec{z}$, $\vec{a} \in A$, $\vec{u} \in 0/\theta$ and $\vec{v} \in 0/\varphi$ with $a = p(\vec{a}, \vec{u}, \vec{v})$. If we now set $b = p(\vec{a}, \vec{u}, 0, \dots, 0)$ we get $(b, 0) \in \theta$ and $(a, b) \in \varphi$; therefore $a \in 0/(\theta \vee \varphi)$. Hence we conclude that (3) implies (7). That (7) implies (2) follows from the fact that $a \in 0/(\theta \vee \varphi)$ implies $a \in 0/\theta \vee 0/\varphi$, hence $(a, 0) \in \theta \circ \varphi$. □

If $s(x, y)$ witnesses subtractivity for V we define a ternary term $u(x, y, z) := s(x, s(s(x, y), z))$ and we observe note that the following identities hold:

1. $u(x, y, s(x, y)) \approx x$
2. $u(x, 0, 0) \approx 0$
3. $u(x, x, 0) \approx x$
4. $u(x, 0, y) \approx u(x, y, 0)$.

This term is very useful; the first application is yet other equivalent conditions for subtractivity that one may add to the ones in Theorem 2.4.1.

Proposition 2.4.2. *For a variety* V *with* 0 *the following are equivalent:*

1. V *is subtractive;*
2. *there is a binary term* $t(x, y)$ *of* V *such that* $t(x, x) \approx 0$ *and for any* $\mathbf{A} \in \mathsf{V}$ *and* $a, b \in A$
$$a \in (b)_{\mathbf{A}} \vee (t(a, b))_{\mathbf{A}};$$
3. *there is a binary term* $t(x, y)$ *of* V *such that* $t(x, x) \approx 0$ *and for any* $\mathbf{A} \in \mathsf{V}$ *and* $a \in A$
$$a \in (t(a, 0))_{\mathbf{A}};$$

Proof. If V is subtractive, $\mathbf{A} \in \mathsf{V}$ and $a, b \in A$, then $a = u(a, b, s(a, b)) \in (b)_{\mathbf{A}} \vee (t(a, b))_{\mathbf{A}}$. Therefore (2) holds and trivially implies (3).

Assume now (3) and let $\mathbf{A} \in \mathsf{V}$, $\theta, \varphi \in \mathrm{Con}(\mathbf{A})$ and $a \in A$ with $a \in 0/\theta \circ \varphi$. Then there is a $b \in A$ with $a\ \theta\ b\ \varphi\ 0$. Now

$$t(a, 0)\ \varphi\ t(a, b)\ \theta\ t(a, a) = 0$$

so $t(a, 0) \in 0/\varphi \circ \theta$ which is an ideal. As $a \in (t(a, 0))_{\mathbf{A}}$ we get $a \in 0/\varphi \circ \theta$, thus V is 0-permutable and hence, by Theorem 2.4.1, subtractive. □

Another consequence is that subtractive varieties have normal ideals and the quickest way to show it is to use the so called *Mal'cev criterion* whose proof is left as an exercise,

Lemma 2.4.3. *(Mal'cev) Let* $\mathbf{A}$ *be an algebra and let* $I \subseteq A$*,* I *non empty; then the following are equivalent*

1. *there is a* $\theta \in \mathrm{Con}(\mathbf{A})$ *with* $I = a/\theta$ *for some* $a \in A$*;*
2. *for every unary polynomial* $g(x)$ *of* $\mathbf{A}$ *if* $a, b, g(a) \in I$ *then* $g(b) \in I$*.*

Remark 2.4.4. We observe that if V is subtractive, $\mathbf{A} \in \mathsf{V}$ and $I \in \mathrm{Id}(\mathbf{A})$ and $a, b \in A$ then

$$b, s(a,b) \in I \qquad \text{implies} \qquad a \in I.$$

If we write suggestively $s(a,b)$ as $b \to a$, then we get

$$b, b \to a \in I \qquad \text{implies} \qquad a \in I$$

which reminds the logical rule of *modus ponens*. This explains why people in algebraic logic got interested in the general theory of ideals,

Now we can show:

Proposition 2.4.5. *Every subtractive variety* V *has normal ideals.*

Proof. Let $\mathbf{A} \in \mathsf{V}$ and $I \in \mathrm{Id}(\mathbf{A})$. Let $g(x)$ be a unary polynomial of $\mathbf{A}$; then there is an $n+1$-term $t(\vec{y}, x)$ and $\vec{a} \in A$ with $t(\vec{a}, x) = g(x)$. Let $a, b, g(a) \in I$ and observe that $s(t(\vec{y}, x_1), t(\vec{y}, x_2))$ is an ideal term in x_1, x_2. Therefore $s(g(b), g(a)) \in I$; as $g(a) \in I$ we get that $g(b) = u(g(b), g(a), s(g(b), g(a))) \in I$. By Lemma 2.4.3 I is a congruence class, hence it is $0/\theta$ for some $\theta \in \mathrm{Con}(\mathbf{A})$. $\square$

In a subtractive variety V we can describe the join of two ideals in the ideal lattice very effectively.

Lemma 2.4.6. *If* V *is subtractive,* $\mathbf{A} \in \mathsf{V}$ *and* $I, J \in \mathrm{Id}(\mathbf{A})$ *then*

$$I \vee J = \{u(a,b,c) : a \in A, b \in I, c \in J\}.$$

Proof. Let $K = \{u(a, b, c) : a \in A, b \in I, J \in J\}$; a generic element $b \in (K)_{\mathbf{A}}$ is of the form

$$p(\vec{a}, u(d_1, e_1, f_1), \dots, u(d_m, e_m, f_m))$$

where $p(\vec{x}, \vec{y})$ is an ideal term in $\vec{y}$, $\vec{a}, d_1, \dots, d_m \in A$, $e_1, \dots, e_m \in I$, $f_1, \dots, f_m \in J$. Let

$$c = p(\vec{a}, u(d_1, e_1, 0), \dots, u(d_m, e_m, 0));$$

hence $c \in I$. Since

$$s(p(\vec{x}, u(z_1, y_1, w_1), \dots, u(z_k, y_k, w_k))), p(\vec{x}, u(z_1, y_1, 0), \dots, u(z_k, y_k, 0))$$

is an ideal term in $\vec{w}$ we have that $s(b, c) \in J$. It follows that $b = u(b, c, s(b, c)) \in K$. Hence $(K)_{\mathbf{A}} \subseteq K$ and thus equality holds.

So K is an ideal containing I, J and it is clearly the smallest. This proves the thesis. □

Proposition 2.4.7. *If* V *is subtractive, then for all* $\mathbf{A} \in \mathsf{V}$, $\mathrm{Id}(\mathbf{A})$ *is a modular lattice.*

Proof. Let $I, J, H \in \mathrm{Id}(\mathbf{A})$ and suppose that $I \subseteq J$, $I \vee H = J \vee H$ and $I \cap H = J \cap H$. If $a \in J$, then $a \in I \vee H$, so by (the proof of) Lemma 2.4.6 for some $c \in I$ we have that $s(a, c) \in H$. As $I \subseteq J$, $c \in J$ and thus $s(a, c) \in J \cap H = I \cap H$. In particular $s(a, c) \in I$ and thus $a = u(a, c, s(a, c)) \in I$. So $J \subseteq I$ and hence $I = J$; this proves modularity of $\mathrm{Id}(\mathbf{A})$. □

Chapter 3

Congruence regularity

3.1 Ideal determined varieties

A variety V is congruence 0-regular if for all $\mathbf{A} \in \mathsf{V}$ and $\theta, \varphi \in \mathrm{Con}(\mathbf{A})$,$0/\theta = 0/\varphi$ implies $\theta = \varphi$. 0-regularity was introduced and characterized in [12]; when the variety is also subtractive, then it is said to be **ideal determined**.

Theorem 3.1.1. *[18] For a variety V the following are equivalent:*

1. *V is ideal determined;*

2. *any algebra in V has 0-regular and 0-permutable congruences;*

3. *there exists a natural number m, binary terms $d_1(x,y), \dots, d_m(x,y)$ and a $m+3$-term q such that*

$$\begin{aligned}
&d_i(x,y) \approx 0 \quad \textit{for} \quad i = 1, \dots, m \textit{ implies } x \approx y \\
&d_i(x,x) \approx 0 \quad \textit{for} \quad i = 1, \dots, m \\
&q(x,y,0,0,\dots,0) \approx 0 \\
&q(x,y,y,d_1(x,y),\dots,d_m(x,y)) \approx x
\end{aligned}$$

   hold in V;

4. *the mapping from $\mathrm{Con}(\mathbf{A}) \longrightarrow \mathrm{Id}(\mathbf{A})$ defined by $\theta \longmapsto 0/\theta$ is a lattice isomorphism.*

The proof can easily patterned after the one of Theorem 2.4.1 and it is left to the reader. Examples of ideal determined varieties: groups, rings,

$\mathbf{R}$-modules, $\mathbf{R}$-algebras, residuated lattices (and any of their fragments containing $\to$ and 1) and many others.

In an ideal determined variety the congruence permute at 0 and they are completely determined by the ideals. This does not mean however that the congruence must permute *away* from zero.

Remark 3.1.2. (Implication algebras) An **implication algebra** (a.k.a. Hilbert algebra) is a $\to, 1$-subreduct of a Heyting algebra. It is well-known [11] that implication algebras form a variety axiomatized by

$$\begin{aligned}
&x \to x \approx 1 \\
&(x \to y) \to x \approx x \\
&(x \to y) \to y \approx (y \to x) \to x \\
&x \to (y \to z) \approx y \to (x \to z).
\end{aligned}$$

Now $1 \to x \approx (x \to x) \to x \approx x$ by the first two equations, so $y \to x$ is a subtraction term relative to 1. Next if $x \to y \approx y \to x \approx 1$ then

$$\begin{aligned}
x \approx 1 \to x \approx (y \to x) \to x \\
(x \to y) \to y \approx 1 \to y \approx y.
\end{aligned}$$

which of course implies 1-regularity of congruences. So the variety of implication algebras is ideal determined; it is not congruence permutable though as shown in [25]. In the same paper it is shown that it is congruence 3-permutable; this means that for any implication algebra $\mathbf{A}$ and $\theta\varphi \in \mathrm{Con}(\mathbf{A})$, $\theta \circ \varphi \circ \theta = \varphi \circ \theta \circ \varphi$.

However this is not true in general; in [27] the author proved that the variety of *lower BCK-semilattices* is ideal determined and 4-permutable but not 3-permutable. As a final fact in [15] it is shown that for every n there is an ideal determined variety that is congruence n-permutable but not congruence $n+1$-permutable.

In many cases there is no need to check for closure under all the ideal terms to ascertain if a subset of an algebra is an ideal. This concept can be formalized as follows: if V is a variety a **base for the V-ideal terms** is any set T of ideal terms such that Tcontains 0, T is closed under compositions and the following holds: for any $\mathbf{A} \in \mathsf{V}$ and any $I \subseteq A$, $I \in \mathrm{Id}(\mathbf{A})$ if an only if for any $t(\vec{x}, \vec{y}) \in T$, $\vec{a} \in A$ and $\vec{b} \in I$, $t(\vec{a}, \vec{b}) \in I$. The interesting case is the one in which the base is finite and the reader can check in a minute that groups and rings have a finite base for their ideal terms. This is not a coincidence and to see why we first need a technical lemma.

Lemma 3.1.3. *[12] For a variety V the following are equivalent:*

1. *there is an m and binary terms $d_1, \ldots, d_m$ such that the equivalence*

$$d_1(x,y) \approx \ldots \approx d_m(x,y) \approx 0 \qquad \textit{if and only if} \qquad x \approx y$$

   hold in V;

2. *there is an m, binary terms $d_1, \ldots, d_m$ an quaternary terms $g_1, \ldots, g_m$ such that the equations*

$$\begin{aligned} &g_1(x,y,d_1(x,y),0) \approx x \\ &g_i(x,y,0,d_i(x,y)) \approx g_{i+1}(x,y,d_{i+1}(x,y),0) \quad i = 1, \ldots, m-1 \\ &g_m(x,y,0,d_m(x,y)) \approx y \end{aligned}$$

   hold in V;

3. *V is congruence 0-regular.*

Theorem 3.1.4. *[29] Let V be an ideal determined variety of finite type. Then V has a finite base for ideal terms.*

Proof. Let $d_1, \ldots, d_m$ be the terms whose existence is guaranteed by Lemma 3.1.3. We first observe that if $\mathbf{A} \in \mathsf{V}$, $\theta \in \mathrm{Con}(\mathbf{A})$ and $I = 0/\theta \in \mathrm{Id}(\mathbf{A})$ then for all $a, b \in A$

$$(a,b) \in \theta \qquad \text{if and only if} \qquad d_i(a,b) \in I \quad i = 1, \ldots, m.$$

Next if f is an n-ary basic operation of V we consider the free algebra in V generated by $x_1, \ldots, x_n, y_1, \ldots, y_n$ and the ideal I generated by $\{d_i(x_k, y_k) : i = 1, \ldots, m,\ k = 1. \ldots, n\}$; clearly $d_i(f(\vec{x}), f(\vec{y})) \in I$ for $i = 1, \ldots, m$ and thus there exist ideal terms $r_{i,f}$, $i = 1, \ldots, n$ such that

$$\begin{aligned} &r_{i,f}(x,y,0,\ldots,0) \approx 0 \\ &r_{i,f}(x,y,d_1(x_1,y_1),\ldots,d_m(x_1,y_1),\ldots,d_1(x_n,y_n),\ldots,d_m(x_n,y_n)) \approx d_i(f(\vec{x}),f(\vec{y})) \end{aligned}$$

hold in V.

Next since congruences are symmetric and transitive relations this means that for $i = 1, \ldots, n$ $d_i(x, y), d_i(z, y) \in I$ implies $d_i(x, z) \in I$. Hence there are terms q_i, $i = 1, \ldots, n$ such that

$$q_i(x, y, z, 0, \ldots, 0) \approx 0$$
$$q_i(x, y, z, d_1(x, y), \ldots, d_m(x, y), d_1(z, y), \ldots, d_m(z, y)) \approx d_i(x, z)$$

hold in V.

Finally let q be the term whose existence is requested in point (4) of Theorem 3.1.1. We claim that

$$T = \{0, d_i, r_{i,f}, q_i, q\}$$

is a base for ideal terms for V. We have to check that T is closed under composition and that is is in fact a base. Both proofs are routine and are left as an exercise. $\square$

3.2 Protomodular varieties

In many ideal determined varieties there is a *strong additive structure* in the following sense: if $s(x, y)$ is the subtraction term, then there is another binary term $t(x, y)$ such that $t(y, s(x, y)) \approx y$ holds in the variety. This happens for instance in groups, rings and Boolean algebras.

This property, when properly generalized, corresponds to an interesting categorical property called **protomodularity**. Let us stress that protomodularity is a concept defined in category theory; besides the rather unfortunate choice of the name (more on that later) when one tries to translate it into the universal algebraic language some adjustments must be made.

Let V be a variety of algebras; if $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathsf{V}$ and $f : \mathbf{A} \longrightarrow \mathbf{C}$, $g : \mathbf{B} \longrightarrow \mathbf{C}$ are homomorphisms, the **pullback of A and B along C**, denoted by $\mathbf{A} \times_{\mathbf{C}} \mathbf{B}$ is the subalgebra of $\mathbf{A} \times \mathbf{B}$ consisting of all the pairs (a, b) such that $f(a) = g(b)$. It is readily checked that $\mathbf{A} \times_{\mathbf{C}} \mathbf{B}$ is a subalgebra of $\mathbf{A} \times \mathbf{B}$. Moreover if $p_{\mathbf{A}}$, $p_{\mathbf{B}}$ are the projections of $\mathbf{A} \times_{\mathbf{C}} \mathbf{B}$ into $\mathbf{A}$, $\mathbf{B}$ then the square in Figure 3.1 has the **universal mapping property** in the following sense.

Lemma 3.2.1. *Let $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathsf{V}$, consider the pullback of $\mathbf{A}$ and $\mathbf{B}$ along $\mathbf{C}$, let $\mathbf{D} \in \mathsf{V}$ such that $f' : \mathbf{D} \longrightarrow \mathbf{A}$, $g' :\longrightarrow \mathbf{B}$ be homomorphism. If $ff' = gg'$, then the function $h : \mathbf{D} \longrightarrow \mathbf{A} \times_{\mathbf{C}} \mathbf{B}$ defined by $h(d) = (f'(d), g'(d))$ is the unique homomorphism such that the following diagram commutes:*

$$\begin{tikzcd}
\mathbf{D} \arrow[dr, "h"] \arrow[drr, bend left, "g'"] \arrow[ddr, bend right, "f'"'] & & \\
& \mathbf{A} \times_{\mathbf{C}} \mathbf{B} \arrow[r, "p_{\mathbf{B}}"] \arrow[d, "p_{\mathbf{A}}"] & \mathbf{B} \arrow[d, "g"] \\
& \mathbf{A} \arrow[r, "f"] & \mathbf{C}
\end{tikzcd}$$

Figure 3.1: Pullback

Let now $\mathbf{B}$ in V and let $r(\mathbf{B}) = \{\mathbf{A} \in \mathsf{K} : \mathbf{B} \text{ is a retract of } \mathbf{A}\}$.

Theorem 3.2.2. *Let $\mathbf{E}, \mathbf{B} \in \mathsf{V}$ and let $f : \mathbf{E} \longrightarrow \mathbf{B}$ be a homomorphism; if $\mathbf{A}, \mathbf{A}' \in r(\mathbf{B})$ and $g : \mathbf{A}' \longrightarrow \mathbf{A}$ is a homomorphism, then there is a unique homomorphism $f^*(g) : \mathbf{E} \times_{\mathbf{B}} \mathbf{A}' \longrightarrow \mathbf{E} \times_{\mathbf{B}} \mathbf{A}$ that makes the diagram in Figure 3.2 commute.*

$$\begin{tikzcd}
\mathbf{E} \times_{\mathbf{B}} \mathbf{A}' \arrow[rrr, "p_{\mathbf{A}'}"] \arrow[dr, "f^*(g)"] \arrow[dd, shift left] & & & \mathbf{A}' \arrow[dr, "g"] \arrow[dd, shift left] & \\
& \mathbf{E} \times_{\mathbf{B}} \mathbf{A} \arrow[rrr] \arrow[dl, shift left] & & & \mathbf{A} \arrow[dl, shift left] \\
\mathbf{E} \arrow[uu, shift left] \arrow[rrr, "f"] \arrow[ur, shift left] & & & \mathbf{B} \arrow[uu, shift left] \arrow[ur, shift left] &
\end{tikzcd}$$

Figure 3.2:

Proof. It is enough to apply Lemma 3.2.1 to the pullback of $\mathbf{E}$ and $\mathbf{A}$ along $\mathbf{B}$. $\square$

Let $\mathcal{R}(\mathbf{B})$ be the category whose objects are in $r(\mathbf{B})$ and whose morphisms are just homomorphisms between algebras in $r(\mathbf{B})$; then f^* can be seen as a functor from $\mathcal{R}(\mathbf{B})$ to $\mathcal{R}(\mathbf{E})$, where $f^*(\mathbf{A}) = \mathbf{E} \times_{\mathbf{B}} \mathbf{A}$.

A variety $\mathbf{V}$ of algebras is **protomodular** if for all $\mathbf{E}, \mathbf{B} \in \mathsf{V}$ and for all $f : \mathbf{E} \longrightarrow \mathbf{B}$ the functor f^* reflects isomorphisms. In other words if for any $\mathbf{A}, \mathbf{A}' \in r(\mathbf{A})$ and $g : \mathbf{A}' \longrightarrow \mathbf{A}$

$$f^*(g) \text{ is an isomorphism} \qquad \text{implies} \qquad g \text{ is an isomorphism.}$$

Lemma 3.2.3. *A variety* V *is protomodular if and only if for all* $\mathbf{E}, \mathbf{B} \in \mathsf{V}$ *and for all monomorphisms* $f : \mathbf{E} \longrightarrow \mathbf{B}$*, if* $\mathbf{A}, \mathbf{A}' \in r(\mathbf{A})$ *and* $g : \mathbf{A}' \longrightarrow \mathbf{A}$ *is a monomorphism, then*

$$f^*(g) \text{ is an isomorphism} \qquad \text{implies} \qquad g \text{ is an isomorphism.}$$

As a monomorphism is just a subalgebra injection up to isomorphisms we can reformulate the theorem in the following way.

Theorem 3.2.4. *For a prevariety* V *the following are equivalent:*

1. V *is protomodular;*
2. *if* $\mathbf{E} \leq \mathbf{B} \leq \mathbf{A}' \leq \mathbf{A} \in \mathsf{V}$ *with* $\mathbf{B}$ *a retract of* $\mathbf{A}$*, witness* α*, if* $\alpha^{-1}(\mathbf{E}) \leq \mathbf{A}'$*, then* $\mathbf{A}' = \mathbf{A}$*;*
3. *if* $\mathbf{E} \leq \mathbf{B} \leq \mathbf{A} \in \mathsf{V}$ *with* $\mathbf{B}$ *a retract of* $\mathbf{A}$*, witness* α*, then* $\mathbf{A} = \mathrm{Sub}_{\mathbf{A}}(\alpha^{-1}(E) \cup B)$*.*

Point (3) above can be taken as the simplest algebraic definition of a protomodular variety; really nothing has been done with it since the formulation which is more common for categories is the one in which the category has an initial object. When we go to algebraic categories, then an initial object is not necessarily present. However since any variety V can be seen as a concrete category with free objects, the initial object, if it exists, is exactly the free algebra over the empty set. Now for any variety $\mathbf{F}_{\mathsf{V}}(\emptyset)$ exists if and only if the language of V contains at least a constant and in this case it is the algebra generated by the constant elements. We will see in the next section that in this case protomodularity has a nice algebraic description.

Remark 3.2.5. There is a problem which, to the best of my knowledge, is still open: are there protomodular varieties with no initial objects? Which means are there protomodular (pre)varieties in which the free algebra over the empty set does not exist?

Theorem 3.2.6. *For a variety* V *with a constant* 0 *the following are equivalent:*

1. V *is protomodular;*
2. *for all* $\mathbf{A}, \mathbf{B} \in \mathsf{V}$*, where* $\mathbf{B}$ *is a retract of* $\mathbf{A}$ *via* α *and* $\mathbf{E}$ *is the subalgebra of* $\mathbf{B}$ *generated by* 0*, then* $\mathbf{A} = \mathrm{Sub}_{\mathbf{A}}(\alpha^{-1}(E) \cup B)$*;*

3. *there is an $n \in \mathbb{N}$, $e_1, \dots, e_n \in E$, an $n+1$-ary term t and binary terms $d_1, \dots, d_n$ such that*

$$d_i(x,x) \approx e_i \qquad i = 1, \dots, n$$
$$t(y, d_1(x,y), \dots, d_n(x,y)) \approx x$$

holds in V.

Proof. (2) is just an instance of Theorem 3.2.4(3), so (1) implies (2).

Assume then (2) and let $\mathbf{A} = \mathbf{F}_{\mathsf{V}}(x,y)$ and $\mathbf{B} = \mathbf{F}_{\mathsf{V}}(y)$; then $\alpha(x) = \alpha(y) = y$ and $\mathbf{A} = \mathrm{Sub}_{\mathbf{A}}(\alpha^{-1}(E) \cup B)$ where $\mathbf{E}$ is the subalgebra of $\mathbf{B}$ generated by the constants. Since $x \in A$, there is an $n+1$-ary term t and binary terms $d_1, \dots, d_n$ such that

$$x \approx t(y, d_1(x,y), \dots, d_n(x,y))$$

where $t_1, \dots, t_n \in E$. This means that $d_i(y,y) = \alpha(d_i(x,y))$ is in the subalgebra generated by the constants. It follows that there are $e_1, \dots, e_n \in E_{\mathsf{V}}$ such that $d_i(x,x) \approx e_i$, $i = 1, \dots, n$. This proves (3).

Assume now (3) and let $\mathbf{B} \leq \mathbf{A} \in \mathsf{V}$ where $\mathbf{B}$ is a retract of $\mathbf{A}$ via α. Then if $\mathbf{E}$ is the subalgebra of $\mathbf{B}$ generated by the constants and $a \in A$ we have

$$a = t(\alpha(a), d_1(\alpha(a), a), \dots, d_n(\alpha(a), a))$$

and $\alpha(d_i(\alpha(a), a)) = d_i(\alpha(a), \alpha(a)) = e_i \in E$. Therefore $\mathbf{A} = \mathrm{Sub}_{\mathbf{A}}(\alpha^{-1}(E) \cup B)$. Now if $\mathbf{E}' \leq \mathbf{B} \leq \mathbf{A} \in \mathsf{V}$, then $\mathbf{E} \leq \mathbf{E}'$ and, a fortiori, $\mathbf{A} = \mathrm{Sub}_{\mathbf{A}}(\alpha^{-1}(E') \cup B)$. Thus V is protomodular by Theorem 3.2.4. $\square$

We stress, even if there is no need, that we are not asking that the constants $e_1, \dots, e_n$ be distinct.

Corollary 3.2.7. *If* V *is protomodular then it is congruence permutable. If Theorem 3.2.4 holds for $e_1 = \dots = e_n = 0$, then it is ideal determined.*

Proof. If V is protomodular, then consider the term

$$m(x,y,z) := t(z, d_1(x,y), \dots, d_n(x,y)).$$

Then

$$\begin{aligned} m(x,y,y) &\approx t(y, d_1(x,y), \dots, d_n(x,y)) \approx x \\ m(x,x,y) &\approx t(y, d_1(x,x), \dots, d_n(x,x)) \approx t(x, e_1, \dots, e_m) \\ &\approx t(x, d_1(x,x), \dots, d_n(x,x)) \approx x \end{aligned}$$

So $m(x, y, z)$ is a Mal'cev term for V, which is then congruence permutable. If it is pointed then $E = \{0\}$ and so $d_i(x, x) \approx 0$ for $i = 1, \dots, n$. Hence the term $s(x, y) = t(0, d_1(x, y), \dots, d_n(x, y))$ is a subtraction term for V. Moreover if $d_i(x, y) \approx 0$ for $i = 1, \dots, n$ then

$$x \approx t(y, d_1(x, y), \dots, d_n(x, y)) \approx t(y, 0, \dots, 0) \approx t(y, d_1(y, y), \dots, d_n(y, y)) \approx y.$$

This shows that V is 0-regular and hence ideal determined. □

3.3 Classically ideal determined varieties

Clearly if a variety is protomodular and **pointed**, i.e. there is exactly one constant, then the hypotheses of Corollary 3.2.7 are automatically satisfied. However the variety of Boolean algebras is ideal determined, not pointed since $\mathbf{E} = \{0, 1\}$ and still satisfies the hypotheses of Corollary 3.2.7 for $e_1 = \cdots = e_n = 1$. This suggests a definition: a variety V is **classically ideal determined** if it satisfies (4) of Theorem 3.2.4 with $e_1 = \cdots = e_n = 0$. In other words a variety V is classically ideal determined is there is an $n \in \mathbb{N}$, binary terms $d_1, \dots, d_n$ and a $n+1$-ary term t such that

$$\begin{aligned} &d_i(x, x) \approx 0 \quad i = 1. \dots, n \\ &t(y, d_1(x, y), \dots, d_n(x, y)) \approx x. \end{aligned}$$

The following is obvious:

Proposition 3.3.1. *A classically ideal determined variety is* 0*-regular and congruence permutable, hence ideal determined.*

Varieties that are 0-regular and congruence permutable have been described in [2]:

Theorem 3.3.2. *[2] For a variety* V *the following are equivalent:*

1. V *is* 0*-regular and congruence permutable;*

2. *there is an* $n \in \mathbb{N}$*, an* $n+2$*-ary term* p *and binary terms* $d_1, \dots, d_n$ *such that*

$$\begin{aligned} &d_i(x, x) \approx 0 \qquad i = 1, \dots, n \\ &p(x, y, 0, \dots, 0) \approx y \\ &p(x, y, d_1(x, y), \dots, d_n(x, y)) \approx x \end{aligned}$$

Proof. (Sketch) For (2) implies (1) observe that

$$m(x, y, z) = p(x, z, d_1(x, y), \ldots, d_n(x, y))$$

is a Mal'cev term for V. Conversely, let $\mathbf{F} = \mathbf{F}_{\mathsf{V}}(x, y)$ and consider the congruence

$$\theta = \vartheta_{\mathbf{F}}(\{(0, d(x, y)) : d(x, x) = 0\}).$$

Then we can show that $0/\theta = 0/\vartheta_{\mathbf{F}}(x, y)$ and thus 0-regularity forces $\theta = \vartheta_{\mathbf{F}}(x, y)$. In particular $(y, x) \in \theta$; now by an old result by H. Werner [31] in a congruence permutable variety a congruence is just a reflexive subalgebra of the square, so there is an $n \in \mathbb{N}$, and $d_1, \ldots, d_n$ such that

$$(y, x) \in \mathrm{Sub}_{\mathbf{F}^2}(\{(y, y), (x, x), (0, d_1(x, y)), \ldots, (0, d_n(x, y))\}).$$

From here we proceed a s usual to get the thesis. $\square$

So apparently varieties that are 0-regular and congruence permutable do not coincide with classically ideal determined varieties. To substantiate this with an example we need a better characterization of classically ideal determined variety.

A subalgebra $\mathbf{S} \leq \mathbf{A} \times \mathbf{A}$ is **classical** if $(a, b) \in S$ implies $(a, a) \in S$. Of course any congruence is a classical subalgebra of $\mathbf{A} \times \mathbf{A}$ and a standard argument (really Lemma 3.3.3 below) shows that the classical subalgebras of $\mathbf{A} \times \mathbf{A}$ form an algebraic lattice $\mathrm{CS}(\mathbf{A})$. Classical subalgebras were introduced in [30] and there are two facts proved there that will be useful in the sequel.

For any subset $\mathbf{X}$ of $A \times A$ we define

$$X^{\Delta} = \{(a, a) : \text{ there is a } b \in A \text{ with } (a, b) \in X\}.$$

Lemma 3.3.3. *[30] Let $\mathbf{A}$ be any algebra and let $X \subseteq A \times A$; then the classical subalgebra generated by X, denoted by $\mathrm{CS}_{\mathbf{A}}(X)$ can be defined in the following way. Let*

$$\begin{aligned} X_0 &= X \\ X_{n+1} &= \mathrm{Sub}_{\mathbf{A}^2}(X_n \cup X_n^{\Delta}); \end{aligned}$$

then $\mathrm{CS}_{\mathbf{A}}(X) = \bigcup_{n \in \mathbb{N}} X_n$.

The following technical result, appearing in [30], is not hard to prove and it is left to the reader.

Lemma 3.3.4. *[30] Let* $\mathbf{A}$ *be an algebra, and let* $\mathbf{S}, \mathbf{T} \in \mathrm{CS}(\mathbf{A})$*. If* $\mathbf{S} = \mathrm{CS}_{\mathbf{A}}(X)$ *and* $X^{\Delta} \subseteq T^{\Delta}$*, then* $S^{\Delta} \subseteq T$*.*

A variety V is **classically** 0**-regular** if for all $\mathbf{A} \in \mathsf{V}$ and $\mathbf{S}, \mathbf{T} \in \mathrm{Cs}(\mathbf{A})$ if $0/\mathbf{S} = 0/\mathbf{T}$ and $S^{\Delta} \subseteq T$ and $T^{\Delta} \subseteq S$, then $\mathbf{S} = \mathbf{T}$. Clearly every classically 0-regular variety is 0-regular as well.

A variety V is 0**-coherent** if for all $\mathbf{A} \in \mathsf{B}$, for all $\theta \in \mathrm{Con}(\mathbf{A})$ and for all $\mathbf{B} \leq \mathbf{A}$, if $0/\theta \subseteq B$, then B is a union of θ-blocks.

Theorem 3.3.5. *[30] For a variety* V *the following are equivalent:*

1. V *is classically ideal determined;*
2. V *is classically* 0*-regular;*
3. V *is* 0*-coherent.*

Proof. Assume (1) and let t be the terms witnessing classical ideal determinacy. Let $\mathbf{A} \in \mathsf{V}$ and $\mathbf{S}, \mathbf{T} \in \mathrm{Cs}(\mathbf{A})$ with $0/\mathbf{S} = 0/\mathbf{T}$ and $S^{\Delta} = T^{\Delta}$. Suppose that $(a, b) \in S$; since $\mathbf{S}$ is classical $(a, a) \in S$ and thus $(0, d_i(b, a)) \in S$ for all i. Hence $(0, d_i(b, a)) \in T$ for $i = 1, \dots, n$ and moreover $(a, a) \in T^{\Delta} \subseteq T$. It follows that

$$
\begin{aligned}
(a, b) &= (t(a, d_1(a, a), \dots, d_n(a, a)), t(a, d_1(b, a), \dots, d_n(b, a)) \\
&= (t(a, 0, \dots, 0)), t(a, d_1(b, a), \dots, d_n(b, a)) \\
&= t((a, a), (0, d_1(b, a)), \dots, (0, d_n(b, a))) \in T.
\end{aligned}
$$

A symmetric argument proves that (2) holds.

Assume now (2); to prove (1) we will proceed in stages. Let $\mathbf{F} = \mathbf{F}_{\mathsf{V}}(x, y)$ and let f be the endomorphism of $\mathbf{F}$ defined by $f(x) = f(y) = x$. We define three classical subalgebras of $\mathbf{F} \times \mathbf{F}$:

$$
\begin{aligned}
&R = \mathrm{CS}_{\mathbf{A}}(\{(x, x)\} \cup \{(0, d(x, y)) : d(x, x) = 0\}) \\
&T = \mathrm{CS}_{\mathbf{A}}(\{(x, y)\}.
\end{aligned}
$$

We will show that

$$R = S \subseteq \ker(f),$$

from which the conclusion follows easily. First one can show that $R, T \subseteq \ker(f)$ with a standard induction on the generating sets of $\mathbf{T}$ and $\mathbf{R}$, using

Lemma 3.3.3. Next let us show that $\mathbf{T} = \mathbf{R}$; since V is 0-regular it is enough to check that $0/\mathbf{T} = 0/\mathbf{R}$, that $T^\Delta = R$ and that $R^\Delta \subseteq T$.

Let $u \in 0/\mathbf{T}$; since $\mathbf{T} \leq \ker(f)$, if $u = t(x,y)$, then $0 = f(u) = f(t(x,y)) = t(x,x)$, so $(0,u) \in R$ and $u \in 0/\mathbf{R}$. Conversely let $(0,d(x,y)) \in R$; then $d(x,x) = 0$ and as $(x,x),(x,y) \in T$ we get

$$(0,d(x,y)) = d((x,x),(x,y)) \in T$$

and thus $0/\mathbf{S} = 0/\mathbf{T}$.

To show that $T^\Delta = R$ and $R^\Delta \subseteq T$ we use Lemma 3.3.4. As $T^\Delta = \{(x,x)\}$, clearly $T^\Delta \subseteq R$. Conversely suppose that $(u,u) \in R^\Delta$; then either $u = x$ and $(x,x) \in T$ since it is a classical subalgebra, or else $u = 0$. But then $(0,0) \in R^\Delta \subseteq R$ and since $0/R = 0/T$, $(0,0) \in T$ as well. Thus we have shown that $R = T$.

In particular $(x,y) \in R$ and

$$R = \mathrm{CS}_{\mathbf{A}}(\{(x,x)\} \cup \{(0,d(x,y)) : d(x,x) = 0\}).$$

Since the classical subalgebras form an algebraic lattice, then is an $n \in \mathbb{N}$ and $d_1,\ldots,d_n \in F$ with $d_i(x,x) = 0$ such that

$$(x,y) \in \mathrm{CS}_{\mathbf{A}}(\{(x,x),(0,d_1(x,y)),\ldots,(0,d_n(x,y))\}).$$

From here a standard Mal'cev argument (using the description of generation of classical subalgebras in Lemma 3.3.3) yields the terms witnessing classical ideal determinacy of V.

Assume now (1), i.e. that V is classically ideal determined witness $t, d_1, \ldots, d_n$. Let $\mathbf{A} \in \mathsf{V}$, $\theta \in \mathrm{Con}(\mathbf{A})$ and $\mathbf{B} \leq A$; if $0/\theta \subseteq B$ for $i = 1,\ldots,n$ $a \in b/\theta$ and $b \in B$ then $d_i(a,b)\ \theta\ d_i(b,b) = 0$ and thus $d_i(a,b) \in B$ for $i = 1,\ldots,n$. But then

$$a = t(b, d_1(a,b), \ldots, d_n(a,b)) \in B$$

so $b/\theta \subseteq B$ and B is a union of θ-blocks. Hence (3) holds.

Conversely, assume (3), let $\mathbf{F} = \mathbf{F}_(x,y)$ and let f be the endomorphism of $\mathbf{F}$ defined by $f(x) = f(y) = x$. Let $\mathbf{B}$ be the subalgebra of $\mathbf{F}$ generated by $\{y\} \cup \{d(x,y) : f(d(x,y)) = 0\}$; then $0/\ker(f) \subseteq B$ and by 0-coherence $x \in B$, since $(x,y) \in \ker(f)$. From here the usual Mal'cev argument gives the terms for classical ideal determinacy. $\square$

We now can show that not every congruence permutable 0-regular variety is classically ideal determined.

Example 3.3.6. This example is a subalgebra of an algebra described in [3], Example 6.3. Let $\mathbf{A} = \{0, a, b, c\}$; on A we define the following operations:

- $d(x, y)$ is a binary operation whose table is

$$\begin{array}{c|cccc} d & 0 & a & b & c \\ \hline 0 & 0 & a & b & c \\ a & a & 0 & c & a \\ b & b & c & 0 & a \\ c & c & a & a & a \end{array}$$

- $t(x, y, z)$ is a ternary operation defined by

$$g(x, y, z) = \begin{cases} x, & \text{if } z = 0; \\ y, & \text{if } d(x, y) = z; \\ z, & \text{otherwise.} \end{cases}$$

We spare the reader the tedious verification that in $\mathbf{A}$ the following equations hold:

$$\begin{aligned} &d(x, x) \approx 0 \\ &t(x, y, 0) \approx x \\ &t(x, y, d(x, y)) \approx x. \end{aligned}$$

By Theorem 3.3.2, $\mathbf{V}(\mathbf{A})$ is a congruence permutable 0-regular variety. However it is easy to check that the partition $\{\{a, b\}, \{0, c\}\}$ induces a congruence on $\mathbf{A}$ and that $\{0, a, c\}$ is the universe of a subalgebra of $\mathbf{A}$. So $\mathbf{V}(\mathbf{A})$ is not 0-coherent and thus, by Theorem 3.3.5, it is not classically ideal determined.

3.4 Strongly subtractive varieties

Let V be a subtractive variety, witness $s(x, y)$; we say that V is **strongly subtractive** if for all $\mathbf{A} \in \mathsf{V}$ and $I \in \mathrm{Id}(\mathbf{A})$ the relation

$$(a, b) \in I^* \qquad \text{if and only if} \qquad s(b, a) \in I$$

is a congruence of $\mathbf{A}$. Note that I^* is necessarily reflexive and this allows us to prove a result similar to the one by Werner [31] in congruence permutable variety.

Proposition 3.4.1. *Let* $\mathbf{A}$ *be an algebra in a subtractive variety and* $I \in \mathrm{Id}(\mathbf{A})$*; then the following are equivalent:*

1. I^* *is a congruence;*
2. I^* *is a subalgebra of* $\mathbf{A} \times \mathbf{A}$.

Proof. Suppose that I^* is a subalgebra of $\mathbf{A} \times \mathbf{A}$. To prove it is a congruence we only need to prove that I^* is symmetric and transitive. First observe that

$$0/I^* = \{a : (0, a) \in I^*\} = \{a : s(a, 0) \in I\} = I.$$

Then suppose that $(a, b) \in I^*$; then since $(a, a) \in I^*$ we get that $(0, s(a, b)) \in I^*$ and so $s(a, b)\ inI$. By definition $(b, a) \in I^*$. Similarly if $(a, b), (b, c) \in I^*$ since symmetry holds we get $(0, s(c, a)) \in I^*$ and as before $(a, c) \in I^*$. □

Theorem 3.4.2. *If* V *is* 0*-regular and strongly subtractive then it is classically ideal determined.*

Proof. If the hypotheses hold then V is subtractive witness, say, $s(x, y)$. Let $\mathbf{F} = \mathbf{F}_{\mathsf{V}}(x, y)$ and let $\theta = \vartheta_{\mathbf{F}}(x, y)$ and let $I = 0/\theta$; since V is strongly subtractive the relation

$$(u, v) \in I^* \qquad \text{if and only if} \qquad s(v, u) \in I$$

is a congruence of $\mathbf{F}$. Now $u \in 0/I^*$ if and only if $u \in I = 0/\theta$; as V is 0-regular, $\theta = I^*$ and in particular $(x, y) \in I^*$. Therefore (x, y) belongs to the subalgebra of $\mathbf{F}^2$ generated by $\{(y, y)\} \cup \{(s(u, v), 0) : s(u, v) \in 0/\theta\}$; since the lattice of subalgebras is algebraic, there is an n and $u_1, \ldots, u_n, v_1, \ldots, v_n \in F$ such that (x, y) belongs to the subalgebra generated by

$$\{(y, y)\} \cup \{(s(u_i(x, y), v_i(x, y)), 0) : i = 1, \ldots, n\}.$$

Let now $d_i := s(u_i, v_i)$ for $i = 1, \ldots, n$; then there is an $n+1$-ary term t such that

$$(x, y) = t((y, y), (d_1(x, y), 0), \ldots, (d_n(x, y), 0))$$

and thus $x = t(y, d_1(x, y), \ldots, d_n(x, y))$. Next let φ be endomorphism of $\mathbf{F}$ sending $x, y \longmapsto x$; then $\theta \subseteq \ker(\varphi)$. So

$$d_i(x, x) = s(u_i(x, x), v_i(x, x))\ \theta\ s(u_i(x, y), v_i(x, y))\ \theta\ 0$$

and thus

$$d_i(x, x) = f(d_i(x, x)) = f(0) = 0$$

for $i = 1, \ldots, n$. Hence V is classically ideal determined. □

Many examples of strongly subtractive varieties come either from classical algebras or the algebraization of logical systems. In the first class we quote groups, rings, **R**-modules and more generally associative algebras over a ring. Note that they are all congruence permutable and 0-regular (for a suitable constant) so they are classically ideal determined. In the second class we start from **left-complemented monoids**. A left-complemented monoid is an algebra $\mathbf{A} = \langle A, \to, \cdot, 1\rangle$ such that

1. $\langle A, \cdot, 1\rangle$ is a monoid;
2. for $a, b, c \in A$

$$\begin{aligned}
&a \to a = 1\\
&(a \to b)a = (b \to a)b\\
&ab \to c = a \to (b \to c).
\end{aligned}$$

Left-complemented monoid were introduced by Bosbach [9] and they have been investigated at large mainly because of their connection with algebraic logic. We leave it to the reader to show that they are subtractive (relative to 1) with $s(x, y) := y \to x$; to prove that they are strongly subtractive however, one has to dive deeply into the arithmetic of those structures and this is beyond the scope of this note. We will only say that they are congruence permutable and 1-regular so they are classically ideal determined as well. For a general discussion about these structures we refer the reader to [6].

We observe also that there are strongly subtractive varieties that fail to be 0-regular, but for an example we will have to wait till Section 4.5. The last result of this section is a characterization of strongly subtractive varieties.

Theorem 3.4.3. *For a variety* V *the following are equivalent:*

1. V *is strongly subtractive witness* $s(x, y)$*;*
2. V *is subtractive and for all* n*-ary basic operation* f *of* V *there is an* $3n$*-ary term* r_f *such that*

$$s(f(\mathbf{x}), f(\mathbf{y})) \approx r_f(\mathbf{x}, \mathbf{y}, s(x_1, y_1), \ldots, s(x_n, y_n))$$

holds in V*.*

Proof. Assume (1) and let f be an n-ary operation. Consider the free algebra $\mathbf{F}$ in V generated by $x_1, \ldots, x_n, y_1, \ldots, y_n$ and let I be the ideal generated by $\{s(x_i, y_i), i = 1, \ldots, n\}$. Since V is subtractive, the is a congruence θ of $\mathbf{F}$ with $I = 0/\theta$. Since V is strongly subtractive, $(f(\vec{x}), f(\vec{y})) \in \theta^*$. Hence $(s(f(\vec{x}), f(\vec{y})), 0) \in \theta$ and so $s(f(\vec{x}), f(\vec{y})) \in I$. From here a standard argument yields a term r_f with the desired properties. Thus we can conclude that V satisfies (2).

Conversely assume (2) and let $\mathbf{A} \in \mathsf{V}$ and $\theta \in \mathrm{Con}(\mathbf{A})$. Let φ be the subalgebra generated by θ^*; then if $(a, 0) \in \varphi$ there are $a_1, \ldots, a_n, b_1, \ldots, b_n$ with $(a_i, b_i) \in \theta^*$ and a term u such that $u(a_1, \ldots, a_n) = a$ and $u(b_1, \ldots, b_n) = 0$. Then

$$a = s(a, 0) = s(u(a_1, \ldots, a_n), u(b_1, \ldots, b_n)) = r_{i,u}(\mathbf{a}, \mathbf{b}, s(a_1, b_1), \ldots, s(a_n, b_n));$$

since $s(a_i, b_i) \in 0/\theta$ for all i we may conclude that $a \in 0/\theta$ and hence that $(a, 0) \in \theta^*$. This implies that $0/\varphi = 0/\theta^*$ for all i. The fact that this implies that θ^* is a subalgebra is left as an exercise to the reader. $\square$

Chapter 4

The ideal commutator theory

4.1 A primer on the TC-commutator

Here we recall the basic definitions of the **TC-commutator** (a.k.a. the term condition commutator(for congruences; our reference text is [13]. If α, β are congruences of any algebra $\mathbf{A}$, then

1. $M(\alpha, \beta)$ is the set of all 2×2 matrices

$$\begin{pmatrix} t(\vec{a}^1, \vec{b}^1) & t(\vec{a}^2, \vec{b}^2) \\ t(\vec{a}^2, \vec{b}^1) & t(\vec{a}^2, \vec{b}^2) \end{pmatrix}$$

   where t is an $n+m$-ary term, $\vec{a}^1 \ \alpha \ \vec{a}^2$ (componentwise) and $\vec{b}^1 \ \beta \ \vec{b}^2$ (componentwise).

2. α centralizes β modulo γ (in symbols $C(\alpha, \beta; \gamma)$) if

$$\text{whenever } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(\alpha, \beta) \text{ and } a \ \gamma \ b \text{ then also } c \ \gamma \ d.$$

3. $[\alpha, \beta] = \bigwedge \{\gamma : C(\alpha, \beta; \gamma)\}$.

This definition is perfectly general. In case $\mathbf{A}$ belongs to a congruence modular variety, three things happen:

1. the TC-commutator is the unique binary operation on $\mathrm{Con}(\mathbf{A})$ that satisfies the set of conditions we consider meaningful for a *honest to God* commutator;

2. the TC-commutator is symmetric, i.e. $[\alpha, \beta] = [\beta, \alpha]$;

3. $[\alpha, \beta]$ may be characterized in a different way (see [13], Chap. IV) and in particular the Hagemann-Herrmann definition is no more dependent on terms. Let $\Delta_{\alpha,\beta}$ be the congruence on α (regarded as a subalgebra of $\mathbf{A} \times \mathbf{A}$), generated by all pairs $\langle\langle u, u\rangle, \langle v, v\rangle\rangle$ where $u\ \beta\ v$. Then $\langle a, b\rangle \in [\alpha, \beta]$ if and only if $\langle\langle a, b\rangle\langle b, b\rangle\rangle \in \Delta_{\alpha,\beta}$ if and only if for some c, $\langle\langle a, b\rangle\langle c, c\rangle\rangle \in \Delta_{\alpha,\beta}$.

4.2 The ideal commutator

In this section we develop a basic theory of the commutator of ideals in a subtractive variety, borrowing heavily from [3], [4] and [30]. Let V be any variety; $t(\vec{x}, \vec{y}.\vec{z})$ is a **commutator term** in $\vec{y}, \vec{z}$ if it is an ideal term in $\vec{y}$ and and ideal term in $\vec{z}$.

For $\mathbf{A} \in \mathsf{V}$ and nonempty $H, K \subseteq A$ we define the **commutator** of K and H as

$$[K, H] = \{t(\vec{a}, \vec{b}, \vec{c}) : t \text{ a commutator term in } \vec{y}, \vec{z},\ \vec{a} \in A, \vec{b} \in K, \vec{c} \in H\}$$

We should have written $[K, H]_{\mathbf{A}}$ to stress the algebra or even $[K, H]_{\mathbf{A},\mathsf{V}}$ to stress the variety too. However we will see that at least the dependency from V can be avoided. Here is a summary of the property of the ideal commutator whose proof is left to the reader.

Proposition 4.2.1. *[30] If V is any variety, $\mathbf{A} \in \mathsf{V}$ and $H, K \subseteq A$ then:*

1. $[H, K]_{\mathbf{A},\mathsf{V}} \in \mathrm{Id}_{\mathsf{V}}(\mathbf{A})$;

2. $[H, K]_{\mathbf{A},\mathsf{V}} = [K, H]_{\mathbf{A},\mathsf{V}}$;

3. $[H, K]_{\mathbf{A},\mathsf{V}} = [(H)^{\mathsf{V}}_{\mathbf{A}}, (K)^{\mathsf{V}}_{\mathbf{A}}]_{\mathbf{A},\mathsf{V}}$.

It follows that the commutator is symmetric and it is worthless to consider commutators of subsets other than ideals. In particular, when there is no danger of confusion, we will write $[a, b]_{\mathbf{A}}$ instead of $[(a)_{\mathbf{A}}, (b)_{\mathbf{A}}]_{\mathbf{A}}$. To get more information (and a definition of commutator that is not term-dependent) we need to assume that V be subtractive.

First an auxiliary definition: if $\mathbf{A} \in \mathsf{V}$ and $I \in \mathrm{Id}(\mathbf{A})$ we define

$$I^{\#} = \mathrm{Sub}_{\mathbf{A}^2}(I \cup \{(a, a) : a \in J\}.$$

Then it is easy to show (but see [2]) that $I \in \mathrm{Id}(\mathbf{A})$ if and only if $0/I^{\#} = I$.

Let now $\mathbf{A} \in \mathsf{V}$ be an algebra and $I, J \in \mathrm{Id}(\mathbf{A})$; we define

$$\begin{aligned} K_{I,J} &= \text{the ideal of } I^{\#} \text{ generated by } \{(a,a) : a \in J\} \\ [I,J]_0 &= 0/K_{I,J} = \{a : (0,a) \in K_{I,J}\} \end{aligned}$$

Proposition 4.2.2. *For any $\mathbf{A} \in \mathsf{V}$ and $I, J \in \mathrm{Id}(\mathbf{A})$, $[I,J]_0$ is an ideal and $[I,J] \subseteq [I,J]_0$. If V is s-subtractive then $[I,J] = [I,J]_0$.*

Proof. The first claim follows from Proposition 2.3.2. To avoid cumbersome notations we will consider terms with a minimal number of variables; the argument is clearly general.

Let $a = t(b,i,j) \in [I,J]$, where t is a commutator term in y, z and $b \in A$, $i \in I$, $j \in J$. Then in $I^{\#}$

$$(0,a) = t((b,b),(0,i),(j,j)) \in K_{I,J}.$$

On the other hand suppose that V is s-subtractive and $a \in [I,J]_0$, i.e. $(0,a) \in K_{I,J}$. Then for some ideal term $t(x,y)$ in y and for some $(u,v) \in I^{\#}$ and $r \in J$ we have

$$(0,a) = t((u,v),(r,r))$$

i.e. $0 = t(u,r)$ and $a = t(v,r)$. On the other hand, since $(u,v) \in I^{\#}$, there is a term $q(x,y)$, $h \in I$ and $b \in A$ with

$$(u,v) = q((0,h),(b,b)),$$

therefore $0 = t(q(0,b),r)$ and $a = t(q(h,b),r)$. Hence we get

$$\begin{aligned} a &= s(s(a,0),s(0,0)) \\ &= s(s(t(q(h,b),r),t(q(h,b),0)),s(t(q(0,b),r),t(q(0,b),0))) \end{aligned}$$

But the term

$$s(s(t(q(y,x),z),t(q(y,x),0)),s(t(q(0,x),z),t(q(0,x),0)))$$

is a commutator term in y, z. Since $h \in I$ and $r \in J$ we get $a \in [I,J]$. $\square$

Now we can show that the commutators of two ideals in an algebra in a subtractive variety depends only on the algebra and not on the variety.

Proposition 4.2.3. *If* V *is subtractive,* $\mathbf{A} \in \mathsf{V}$ *and* $I, J \in \mathrm{Id}(\mathbf{A})$*, then*

$$[I,J]_{\mathbf{A}} = \{t(\vec{a},\vec{i},\vec{j}) : t \textit{ any term, } \vec{a} \in A, \vec{i} \in I, \vec{j} \in J \textit{ and } t(\vec{a},\vec{0},\vec{0}) = t(\vec{a},\vec{i},\vec{0}) = t(\vec{a},\vec{0},\vec{j}) = 0\}$$

Proof. Let

$$\Sigma_{I,J} = \mathrm{Sub}_{I^{\#}\times I^{\#}}(\{((0,0),(a,a)) : a \in J\};$$

then one easily checks that $K_{I,J} = 0/\Sigma_{I,J}$.

Notice also that, if $X, Y \subseteq A \times A$, then

$$\mathrm{Sub}_{\mathbf{A}\times\mathbf{A}}(X \cup \mathrm{Sub}_{\mathbf{A}\times\mathbf{A}}(Y)) = \mathrm{Sub}_{\mathbf{A}\times\mathbf{A}}(X \cup Y).$$

Hence

$$\Sigma_{I,J} = \mathrm{Sub}_{\mathbf{I}^{\#}\times\mathbf{I}^{\#}}(\{\langle(0,0),(b,b)\rangle : b \in J\} \cup \{\langle(a,a),(a,a)\rangle : a \in A\} \cup \{\langle(0,c),(0,c)\rangle : c \in I\}).$$

Therefore $(c,d) \in K_{I,J}$ if and only if $\langle(0,0),(c,d)\rangle \in \Sigma_{I,J}$ if and only if there is a term $t(\vec{x},\vec{y},\vec{z})$ such that

$$\langle(0,0),(c,d)\rangle = t(\overrightarrow{\langle(0,0)(b,b)\rangle}, \overrightarrow{\langle(a,a),(a,a)\rangle}, \overrightarrow{\langle(0,i),(0,i)\rangle})$$

for some $\vec{b} \in J$, $\vec{a} \in A$ and $\vec{i} \in I$. The conclusion follows. $\square$

From Proposition 4.2.3 we can infer other similar characterizations for $[I,J]$ in a subtractive algebra. We list two of them.

Proposition 4.2.4. *If* $\mathbf{A}$ *is subtractive and* $I, J \in \mathrm{Id}(\mathbf{A})$*, then*

1. $[I,J]_{\mathbf{A}} = \{s(t(\vec{i},\vec{j}),t(\vec{i},\vec{0})) : t$ *a polynomial of* $\mathbf{A}$, $\vec{i} \in I,\ \vec{j} \in J$ *and* $s(t(\vec{0},\vec{j}),t(\vec{0},\vec{0})) = 0\}$*;*
2. $[I,J]_{\mathbf{A}} = \{s(s(t(\vec{i},\vec{j}),t(\vec{i},\vec{0})),s(t(\vec{0},\vec{j}),t(\vec{0},\vec{0}))) : t$ *a polynomial of* $\mathbf{A}, \vec{i} \in I, \vec{j} \in J\}$.

Remark 4.2.5. The dependency on $\mathbf{A}$ of the commutator cannot be avoided even in case of ideal determined varieties: in [13] there is an example of a loop $\mathbf{G}$ having a normal subloop $\mathbf{N}$ such that $[\mathbf{N},\mathbf{N}]_{\mathbf{N}} = \{1\}$ but $[\mathbf{N},\mathbf{N}]_{\mathbf{G}} \neq \{1\}$.

In groups this cannot happen, since in groups we can describe the commutator of two (normal) subgroups using the commutators. Those, in our language, are *pure* (i.e. without parameters) commutator terms. Namely if $\mathbf{G}$ is a group and $\mathbf{N}, \mathbf{M} \lhd \mathbf{G}$ then

$$[\mathbf{N}, \mathbf{M}]_{\mathbf{G}} = Sub_{\mathbf{G}}(\{n^{-1}m^{-1}nm : n \in N, m \in M\}).$$

In other words the only commutator term we have to concern about is $y^{-1}z^{-1}yz$ and this clearly implies that the commutator of $\mathbf{N}, \mathbf{M}$ is the same in any group that contains both of them.

4.3 Commutator identities

Consider an algebraic language having symbols for the join, intersection, 0,1 and the commutator; identities in that language are called *commutator identities*. We say that a class K of algebras **satisfies the commutator identity** $p \approx q$ and we will write

$$\mathsf{K} \vDash_{id} p \approx q,$$

if $p \approx q$ holds in $\mathrm{Id}(\mathbf{A})$ for all $\mathbf{A} \in \mathsf{K}$.

The proof of the following proposition is routine and it is left as an exercise.

Proposition 4.3.1. *For any algebra* $\mathbf{A}$ *the following are equivalent:*

1. $\mathbf{A} \vDash_{id} [x, y] = x \cap y \cap [A, A]$;
2. $\mathbf{A} \vDash_{id} [x, y \cap z] = [x, y] \cap z$;
3. $\mathbf{A} \vDash_{id} [x, y] = [x, A] \cap y$;
4. $\mathbf{A} \vDash_{id} [x, x] = x \cap [A, A]$;
5. $\mathbf{A} \vDash_{id} x \subseteq [A, A] \Longrightarrow x = [x, x]$;
6. *for all* $a \in A$, *if* $a \in [A, A]$ *then* $[a, a] = (a)_{\mathbf{A}}$.

Is there an equivalent algebraic condition corresponding to the satisfaction of any of the conditions in Proposition 4.3.1? Yes, but we need some definitions. An algebra $\mathbf{A}$ is **(finitely) ideal irreducible** if every (finite) family of ideals different form $\{0\}$ does not intersect to $\{0\}$. Ideal irreducibility is equivalent to the existence of a minimal nonzero ideal which has to be generated by a single element, called the **monolithical** element of $\mathbf{A}$.

An algebra $\mathbf{A}$ is **ideal abelian** if $[A, A] = \{0\}$; it is **ideal prime** if for all $I, J \in \mathrm{Id}(\mathbf{A})$, $[I, J] = \{0\}$ implies $I = \{0\}$ or $J = \{0\}$.

Theorem 4.3.2. *[3] For a subtractive variety V the following are equivalent:*

1. $\mathsf{V} \vDash_{id} [x, y \cap z] \approx [x, y] \cap z$;

2. *every ideal irreducible algebra in V is either ideal abelian or ideal prime.*

Proof. Assume (1) and let $I, J \in \mathrm{Id}(\mathbf{A})$ with $[I, J] = \{0\}$. Then

$$[I, J] = I \cap J \cap [A, A].$$

and if $\mathbf{A}$ is not abelian then $[A, A] \neq \{0\}$. Since $\mathbf{A}$ is ideal irreducible, $I \cap J = \{0\}$ and again either $I = \{0\}$ or $J = \{0\}$; i.e. $\mathbf{A}$ is ideal prime.

Assume (2); by Proposition 4.3.1 it is enough to show that if $\mathbf{A} \in \mathsf{V}$, $I \in \mathrm{Id}(\mathbf{A})$ and $I \subseteq [A, A]$ then $[I, I] = I$. Assume by contradiction that there exists $a \in I \setminus [I, I]$; using Zorn Lemma let U be maximal in

$$\{J \in \mathrm{Id}(\mathbf{A}) : [I, I] \subseteq J, a \notin J\}$$

and let $\theta \in \mathrm{Con}(\mathbf{A})$ such that $U = 0/\theta$. Let L be a nonzero ideal of $\mathbf{A}/\theta$; for some $J \supsetneq U$ we have $L = \{b/\theta : b \in J\}$ and for some $b \in J$, $(0, b) \notin \theta$, i.e. $b \in J - U$. So $a \in J$, namely $a/\theta \in L$ and $\mathbf{A}/\theta$ is ideal irreducible; by hypothesis $\mathbf{A}/\theta$ is either ideal abelian or ideal prime.

Observe that $[I, I] \subseteq U$, $I \nsubseteq U$ and

$$[U \vee I, U \vee I] \subseteq U \vee [I, I] = U,$$

therefore

$$[(U \vee I)/\theta, (U \vee I)/\theta]_{\mathbf{A}/\theta} = [U \vee I, U \vee I]/\theta \subseteq U/\theta = \{0/\theta\},$$

while $(U \vee I)/\theta \neq \{0/\theta\}$, since $a/\theta \in (U \vee I)/\theta$. Hence $\mathbf{A}/\theta$ is not ideal prime and so it must be ideal Abelian. This implies

$$\{0/\theta\} = [A/\theta, A/\theta]_{\mathbf{A}/\theta} = [A, A]/\theta$$

and since $I \subseteq [A, A]$ we would have $I/\theta = \{0/\theta\}$, which is absurd since $a/\theta \neq 0/\theta$. It follows by contradiction that (2) implies (1). $\square$

4.4 Ideal neutral varieties

An algebra $\mathbf{A}$ is **ideal neutral** if the commutator of ideals reduces to the intersection. This is equivalent to saying that $\mathbf{A} \vDash_{id} [x, y] \approx x \cap y$. For a variety being ideal neutral is equivalent to several other conditions but first we need some preliminary definitions.

Let $\mathbf{A}_\lambda \in \mathsf{V}$ for $\lambda \in \Lambda$ and let $\mathbf{B}$ be a subdirect product of $(\mathbf{A}_\lambda)_{\lambda\in\Lambda}$. An ideal K of $\mathbf{B}$ is a **product ideal** if there is a family $(I_\lambda)_{\lambda\in\Lambda}$ such that $I_\lambda \in \mathrm{Id}(\mathbf{A}_\lambda)$ and $K = \prod_{\lambda\in\Lambda} I_\lambda$.

We say that a variety V

- **has no skew ideals on finite subdirect products** if for any family $(\mathbf{A}_i)_{i=1}^n$ of algebras in V whenever $\mathbf{B}$ is a subdirect product of $(\mathbf{A}_i)_{i=1}^n$, every ideal of $\mathbf{B}$ is a product ideal ;
- is **ideal distributive** if for all $\mathbf{A} \in \mathsf{V}$, $\mathrm{Id}(\mathbf{A})$ is a distributive lattice.

The following is obvious:

Proposition 4.4.1. *For a subtractive variety* V *the following are equivalent:*

1. V *is ideal distributive;*
2. *for all* $\mathbf{A} \in \mathsf{V}$ *and* $\theta, \varphi, \psi \in \mathrm{Con}(\mathbf{A})$

$$0/(\theta \vee \varphi) \wedge \psi = 0/(\theta \wedge \psi) \vee (\varphi \wedge \psi).$$

Next we need some more information on ideals of subdirect products; the proof of the following is straightforward and is left to the reader.

Proposition 4.4.2. *Let* $\mathbf{A}_\lambda \in \mathsf{V}$ *for* $\lambda \in \Lambda$ *and let* $\mathbf{B}$ *be a subdirect product of* $(\mathbf{A}_\lambda)_{\lambda\in\Lambda}$. *If* π_λ *is the* λ*-th projection,* $\pi_\lambda I \in \mathrm{I}(\mathbf{A}_\lambda)$ *for any* $I \in \mathrm{Id}(\mathbf{B})$. *For* $I \in \mathrm{Id}(\mathbf{B})$, *define*

$$I_{|\lambda} = \{(a_\mu)_{\mu\in\Lambda} : a_\mu = 0 \textit{ if } \mu \neq \lambda \textit{ and } a_\lambda \in \pi_\lambda(I)\}.$$

Then $I_{|\lambda} \in \mathrm{Id}(\mathbf{B})$ *and moreover*

$$\bigcap_{\lambda\in\Lambda}(I \vee I_{|\lambda}) \subseteq \prod_{\lambda\in\Lambda} \pi_\lambda(I).$$

Proposition 4.4.3. *Let* V *be subtractive,* $\mathbf{A}_1, \ldots, \mathbf{A}_n \in \mathsf{V}$ *and* $\mathbf{A}$ *a subdirect product of* $\mathbf{A}_1, \ldots, \mathbf{A}_n$*. If* $I \in \mathrm{Id}(\mathbf{A})$*, then (*$\pi_i$ *are the projections)*

$$\prod_{i=1}^{n} \pi_i(I) \subseteq \bigcap_{i=1}^{n} (I \vee I_{|i}).$$

Hence I *is a product ideal if and only if*

$$I = \prod_{i=1}^{n} \pi_i(I) = \bigcap_{i=1}^{n} (I \vee I_{|i}).$$

Proof. The proof is by induction on n. Assume $n = 2$ and let $(a_1, a_2) \in \pi_1(I) \times \pi_2(I)$; then for some k, h, $(a_1, k) \in I$ and $(h, a_2) \in I$. Hence $(0, k) \in I_{|2}$, $(0, a_2) \in I_{|2}$ and $(0, s(a_2, k)) \in I_{|2}$. Therefore

$$(a_1, a_2) = (u(a_1, a_1, 0), u(a_2, k, s(a_2, k)) = u((a_1, a_1), (a_1, k), (0, s(a_2, k))) \in I \vee I_{|2}.$$

The inductive step is totally similar. □

Theorem 4.4.4. *[3] For a subtractive variety* V *the following are equivalent:*

1. V *is ideal distributive ;*

2. V *has no skew ideals on finite subdirect products;*

3. $\mathsf{V} \vDash_{id} [x, y] = x \cap y$*;*

4. *there are four ternary terms* $q_1, \ldots, q_4$ *such that the following identities hold in* V*:*

$$\begin{aligned} &q_i(x, y, 0) = 0 \qquad i = 1, \ldots, 4 \\ &q_1(x, y, x) = q_2(x, y, y) \\ &q_3(x, y, x) = q_4(x, y, s(x, y)) = s(x, q_1(x, y, x)); \end{aligned}$$

5. *there is a binary term* $b(x, y)$ *such that the following identities hold in* V*:*

$$b(x, x) = 0 \qquad b(0, x) = 0 \qquad b(x, 0) = x.$$

Proof. First we show that (1),(2) and (3) are equivalent. Assume (1) and let $\mathbf{B}$ be a subdirect product of $\mathbf{A}_1, \ldots, \mathbf{A}_n$ where $\mathbf{A} \in \mathsf{V}$. Then, since $\bigcap_{i=1}^n I_{|i} = \{(0,0)\}$, via Proposition 4.2.4 we get

$$\begin{aligned} I &= (I \vee \bigcap_{i=1}^n I_{|i}) \cap B \\ &= \bigcap_{i=1}^n (I \vee I_{|n}) \cap B = \prod_{i=1}^n \pi_i(I) \cap B = \prod_{i=1}^n \pi_i(I). \end{aligned}$$

So I is a product ideal and (2) holds.

Assume now (2) and suppose that (3) fails; i.e. there is an $\mathbf{A} \in \mathsf{V}$ and $I, J \in \mathrm{Id}(\mathbf{A})$ such that $[I,J] \neq I \cap J$. Then, if $H = I \cap J$, $[H,H] \neq H$. Let $\theta \in \mathrm{Con}(\mathbf{A})$ such that $[H,H] = 0/\theta$ and let f be the natural epimorphism of $\mathbf{A}$ onto $\mathbf{A}/\theta$; then

$$[f(H), f(H)]_{\mathbf{A}/\theta} = f([H,H]) = \{0\}^{\mathbf{A}/\theta} = \{0/\theta\}$$

but of course $f(H) \neq \{0/\theta\}$. Therefore we have an algebra $\mathbf{B} = \mathbf{A}/\theta \in \mathsf{V}$ and an ideal $N = f(H)$ of $\mathbf{B}$ with $[N,N] = \{0\}$ but $N \neq \{0\}$.

We will show that $K_{N,N}$ (defined in Section 4.2) is a skew ideal of

$$N' = \mathrm{Sub}_{\mathbf{B}\times\mathbf{B}}(\{(0,n) : n \in N\} \cup \{(b,b) : b \in B\})$$

which is a subdirect power of $\mathbf{B}$. Since $[N,N]_0 = [N,N] = \{0\}$ we have that if $(0,a) \in K_{N,N}$, then $a = 0$. Suppose by contradiction that

$$K_{N,N} = (I' \times J') \cap N' \tag{$*$}$$

for some $I', J' \in \mathrm{Id}(\mathbf{B})$. If $a \in J'$ and $(0,a) \in N'$, then $a = 0$. If $b \in I'$ and $(b,0) \in N'$, since $(b,b) \in N$ we get $(s(b,b), s(b,0)) = (0,b) \in N'$, hence again $b = 0$. Let now $h \in N$, $h \neq 0$; then $(h,h) \in K_{N,N}$, $(0,h) \in N'$ but $(0,h) \notin K_{N,N}$, since $h \notin [H,H]$. It follows that $(0,h) \notin I' \times J'$, hence $h \notin J'$, which in turn implies $(h,h) \notin (I' \times J') \cap N'$, contradicting $(*)$. Hence $K_{N,N}$ is a skew ideal of N'. This argument is local: we have in fact shown that if $\mathbf{A}$ is subtractive and $\mathrm{Id}(\mathbf{A}) \not\models_{id} [x,y] \approx x \cap y$, then in $\mathbf{V}(\mathbf{A})$ there is a skew ideal.

Finally (3) implies (1) by the properties of the commutator; hence (1), (2) and (3) are equivalent.

Assume (1) and let $\mathbf{F}$ be the algebra in V freely generated by $\{x, y, z\}$. In $\mathbf{F}$

$$(x) \cap ((y) \vee (s(x, y))) \subseteq ((x) \cap (y)) \vee (x \cap (s(x, y))).$$

Now x belongs to the left side, hence for some $w \in (x) \cap (y)$, we have $s(x, w) \in (x) \cap (s(x, y))$. From here we get the q_i's and their identities in the standard way so (4) holds.

Assume (4) and define $b(x, y) = q_3(x, y, x)$, then

$$\begin{aligned} b(x, 0) &= s(x, q_1(x, 0, x)) = s(x, q_2(x, 0, 0)) = s(x, 0) = x \\ b(x, x) &= q_4(x, x, 0) = 0 \\ b(0, x) &= s(0, q_1(0, y, 0)) = s(0, 0) = 0 \end{aligned}$$

and (5) holds. Finally (5) clearly implies (1) and this concludes the proof. $\square$

4.5 Ideal abelian algebras

Let $\mathbf{A}$ be any algebra; $\mathbf{A}$ is called **abelian** (see [24]) if for every term $t(x, \vec{y})$, for every $a, b, \vec{u}, \vec{v} \in A$, if $t(a, \vec{u}) = t(a, \vec{v})$ then $t(b, \vec{u}) = t(b, \vec{v})$. By Mal'cev's criterion, this is equivalent to saying that the diagonal of $D(\mathbf{A}) = \{(a, a) : a \in A\}$ is a congruence class of $\mathbf{A} \times \mathbf{A}$. In congruence modular varieties, this is equivalent to: $[1_{\mathbf{A}}, 1_{\mathbf{A}}] = 0_{\mathbf{A}}$ in $\mathrm{Con}(\mathbf{A})$ [13].

We recall that a subtractive algebra $\mathbf{A}$ is **ideal abelian** if $[A, A] = \{0\}$; a subtractive variety V is ideal abelian if it consists entirely of ideal abelian algebras. This is of course equivalent to saying that $\mathsf{V} \models_{id} [x, y] = 0$.

From Proposition 4.2.4 we get two equivalent conditions for being ideal abelian:

$$\begin{gathered} \forall\, t(x, \vec{y}) \text{ term, } \forall\, u, v, \vec{a}, \vec{b} \in A, \\ s(t(u, \vec{a}), t(u, \vec{b})) = 0 \qquad \text{if and only if} \qquad s(t(v, \vec{a}), t(v, \vec{b})) = 0 \end{gathered} \tag{TC_i}$$

$$\begin{gathered} \forall\, t(x, \vec{y}) \text{ term, } \forall\, v, \vec{a}, \vec{b} \in A, \\ s(t(0, \vec{a}), t(0, \vec{b})) = 0 \qquad \text{if and only if} \qquad s(t(v, \vec{a}), t(v, \vec{b})) = 0 \end{gathered} \tag{TC_0}$$

Let now V be a subtractive variety and let $\mathsf{IAB}(\mathsf{V})$ be the class of ideal abelian algebras in V. Then, since by (TC_i) or (TC_0) the condition of being ideal abelian is expressible by quasiequations, $\mathsf{IAB}(\mathsf{V})$ is closed under subalgebras and direct products. Moreover:

Lemma 4.5.1. *Let* $\mathbf{A}$*,* $\mathbf{B}$ *belong to a subtractive variety* V*; let* $I, J \in \mathrm{Id}(\mathbf{A})$ *and let* g *be a homomorphism from* $\mathbf{A}$ *onto* $\mathbf{B}$*. Then* $g([I, J]_{\mathbf{A}}) = [g(I), g(J)]_{\mathbf{B}}$.

Proof. Let $u \in g([I, J]_{\mathbf{A}})$; then there is a commutator term for V in $\vec{y}, \vec{z}$ and elements $\vec{a} \in A$, $\vec{b} \in I$ and $\vec{c} \in J$ with

$$u = g(t(\vec{a}, \vec{b}, \vec{c})) = t(g(\vec{a}), g(\vec{b}), g(\vec{c})) \in [g(I), g(J)]_{\mathbf{B}}.$$

The reverse inclusion is equally obvious. $\square$

Thus we get:

Corollary 4.5.2. *For every subtractive variety* V*,* $\mathsf{IAB}(\mathsf{V})$ *is a variety.*

Proof. We need only observe that if $g : \mathbf{A} \longrightarrow \mathbf{B}$ is a onto homomorphism and $U, V \in \mathrm{Id}(\mathbf{B})$, then $g^{-1}(U), g^{-1}(V) \in \mathrm{Id}(\mathbf{A})$. Then we apply Lemma 4.5.1. $\square$

A more interesting observation is the following:

Proposition 4.5.3. *[30] If* V *is subtractive then* $\mathsf{IAB}(\mathsf{V})$ *is strongly subtractive.*

Proof. According to Proposition 3.4.1 we will show that if $\mathbf{A} \in \mathsf{IAB}(\mathsf{V})$ and $I \in \mathrm{Id}(\mathbf{A})$, then I^* is a subalgebra of $\mathbf{A} \times \mathbf{A}$. First observe that if $t(\vec{x}, \vec{y})$ is an ideal term in $\vec{y}$, then the identity

$$s(t(\vec{x}, \vec{y}), t(\vec{z}, \vec{y})) \approx 0$$

holds in $\mathsf{IAB}(\mathsf{V})$, simply because the shown term is a commutator term in $\vec{x} * \vec{z}, \vec{y}$. Let f be an n-ary operation; then

$$s(f(u(x_1, y_1, 0), \ldots, u(x_n, y_n, 0)), f(\vec{y}))$$

is an ideal term in $\vec{y}$. Therefore in $\mathsf{IAB}(\mathsf{V})$

$$\begin{aligned}
0 &\approx s(s(f(u(x_1, y_1, 0), \ldots, u(x_n, y_n, 0)), f(\vec{y})), s(f(u(y_1, y_1, 0), \ldots, u(y_n, y_n, 0)), f(\vec{y})))\\
&\approx s(s(f(u(x_1, y_1, 0), \ldots, u(x_n, y_n, 0)), f(\vec{y})), s(f(\vec{y}), f(\vec{y})))\\
&\approx s(f(u(x_1, y_1, 0), \ldots, u(x_n, y_n, 0)), f(\vec{y})).
\end{aligned}$$

This means that

$$s(f(u(x_1, y_1, z_1), \ldots, u(x_n, y_n, z_n)), f(\vec{y}))$$

is an ideal term for $\mathsf{IAB}(\mathsf{V})$ in $\vec{z}$. Therefore, if $(a_i, b_i) \in I^*$ then also $(f(\vec{a}), f(\vec{b})) \in I^*$. $\square$

For an even more interesting observation we need a definition. Let $\mathbf{A}$ be an algebra and consider $K_{A,A}$; then

$$K^+_{A,A} = (K_{A,A}))\mathbf{A} \times \mathbf{A}.$$

Lemma 4.5.4. *If $\mathbf{A}$ is subtractive and ideal abelian, then $0/K^+_{A,A} = \{0\}$.*

Proof. Let $(0,d) \in K^+_{A,A}$; then (with a harmless reduction of the number of variables) there are an ideal term $t(x,y)$ in y and $(b,c) \in A \times A$, $(u,v) \in K^+_{A,A}$ such that $(0,d) = t((b,c),(u,v))$. In turn there are an ideal term $p(x,y,z)$ in y,z, $a,i,j \in A$ such that $(u,v) = p((a,a),(0,i),(j,j))$. Notice that $t(b,p(a,0,j)) = 0$ and $t(b,p(a,0,0)) = 0$, hence by (TC_i)

$$s(t(c,p(a,0,j)),t(c,p(a,0,0))) = 0.$$

But $t(c,p(a,0,0)) = 0$, hence $t(c,p(a,0,j)) = 0$ and also $t(c,p(a,i,0)) = 0$. Therefore, by Proposition 4.2.3, $d = t(c,p(a,i,j)) \in [A,A]_{\mathbf{A}} =$, i.e. $d = 0$. □

Now we can show:

Proposition 4.5.5. *[3] Let $\mathbf{A}$ be subtractive. If $\mathbf{M}_3$ is a 0-1-sublattice of $\mathrm{Id}(\mathbf{A})$, then $\mathbf{A}$ is ideal Abelian. Moreover the following are equivalent:*

1. *$\mathbf{A}$ is ideal Abelian and non trivial;*
2. *$\mathrm{Id}(\mathbf{A} \times \mathbf{A})$ has $\mathbf{M}_3$ as a 0-1-sublattice;*
3. *$\pi_1^{-1}(0)$ and $\pi_2^{-1}(0)$ have a common complement in $\mathrm{Id}(\mathbf{A} \times \mathbf{A})$;*
4. *for some subdirect product $\mathbf{S}$ of $\mathbf{A} \times \mathbf{A}$, $\mathrm{Id}(\mathbf{S})$ has an $\mathbf{M}_3$ as a 0-1-sublattice.*

Proof. Suppose that $\{(0), I, J, K, A\}$ form the $\mathbf{M}_3$ in question. Then

$$\begin{aligned}[A,A] &= [I \vee J, I \vee K] \\ &\leq [I,I] \vee [J,I] \vee [I,K] \vee [J,K] \\ &\leq I \vee (J \wedge K) = I\end{aligned}$$

Then $[A,A] \leq J \vee (I \wedge K) = J$; hence $[A,A] \leq I \wedge J = (0)$ and $\mathbf{A}$ is i-Abelian.

(3) $\longrightarrow$ (2) $\longrightarrow$ (4) are obvious. Assuming (4), since $\mathbf{S}$ is i-Abelian, also $\mathbf{A}$ is i-Abelian, being a homomorphic image of $\mathbf{S}$; hence (4) $\longrightarrow$ (1).

To prove (1) $\longrightarrow$ (2), consider $K^+_{A,A}$ in $\mathrm{I}(\mathbf{A}\times\mathbf{A})$. For any $a, b \in A$ we have $(a, a) \in K^+_{A,A}$ and $s((a, b), (a, a)) = (0, s(b, a)) \in \pi_1^{-1}(0)$, therefore $(a, b) \in K^+_{A,A} \vee \pi_1^{-1}(0)$. Also $(b, b) \in K^+_{A,A}$ and $s((a, b), (b, b)) \in \pi_2^{-1}(0)$, therefore $(a, b) \in K^+_{A,A} \vee \pi_2^{-1}(0)$. Now by the previous lemma $K^+_{A,A} \cap \pi_1^{-1}(0) = (0)$, therefore, since $\mathrm{I}(\mathbf{A} \times \mathbf{A})$ is modular, $K^+_{A,A} \cap \pi_2^{-1}(0) = (0)$. It follows that $\{(0), \pi_1^{-1}(0), \pi_2^{-1}(0), K^+_{A,A}, A \times A\}$ form an $\mathbf{M}_3$ in $\mathrm{I}(\mathbf{A} \times \mathbf{A})$. □

What is the relation between being abelian and ideal abelian for varieties? By Proposition 4.5.3 every ideal abelian variety V is strongly subtractive; in particular $\{0\}^\# \in \mathrm{Con}(\mathbf{A})$ for an $\mathbf{A} \in \mathsf{V}$. From now on we will denote $\{0\}^\#$ by $\Delta_\mathbf{A}$.

Proposition 4.5.6. *[3] Let $\mathbf{A}$ be subtractive. If $\mathbf{A}$ is ideal abelian then $\mathbf{A}/\Delta_\mathbf{A}$ is abelian. Conversely if $\Delta_\mathbf{A} \in \mathrm{Con}(\mathbf{A})$ and $\mathbf{A}/\Delta_\mathbf{A}$ is abelian, then $\mathbf{A}$ is ideal Abelian.*

Proof. To avoid too many decorations we will denote $a/\Delta_\mathbf{A}$ simply by a_Δ. Let $t(x, \vec{y})$ be a term and suppose that $t(u_\Delta, \vec{a}_\Delta) = t(u_\Delta, \vec{b}_\Delta)$; then we have $t(u, \vec{a})_\Delta = t(u, \vec{b})_\Delta$ which in turn implies $s(t(u, \vec{a}), t(u, \vec{b})) = 0$. Since $\mathbf{A}$ is ideal Abelian, for any $v \in A$, $s(t(v, \vec{a}), t(v, \vec{b})) = 0$ and with obvious steps $t(v_\Delta, \vec{a}_\Delta) = t(v_\Delta, \vec{b}_\Delta)$. On the other hand, once we know that the relation $\Delta_\mathbf{A}$ is a congruence, if $\mathbf{A}/\Delta_\mathbf{A}$ isaAbelian we can repeat the argument above almost *verbatim* to conclude that $\mathbf{A}$ is ideal abelian. □

4.6 Affine algebras

Let $\mathbf{A}$ be a subtractive algebra, witness $s(x, y)$. We define a term

$$p(x, y, z) := s(x, s(y, z))$$

and we observe that, if $p(x, y, z)$ is given, one can recover $s(x, y) = p(x, y, 0)$ and that $p(x, y, z)$ obeys the laws

$$p(x, y, y) \approx x \qquad\qquad p(x, x, 0) \approx 0$$

of 0-permutability.

An n-ary operation f on $\mathbf{A}$ will be called **affine** if for any $a_1, \dots, a_n, b_1, \dots, b_n \in A$ satisfies

$$s(f(a_1, \dots, a_n), f(b_1, \dots, b_n)) = s(f(s(a_1, b_1), \dots, s(a_n, b_n)), f(0, \dots, 0)).$$

It turns out that this is equivalent to: there is an n-ary term t_f such that, for any $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$,

$$s(f(a_1, \ldots, a_n), f(b_1, \ldots, b_n)) = t_f(s(a_1, b_1), \ldots, s(a_n, b_n)).$$

One easily sees that the composition of affine operations is still affine. Hence if every basic operation of a subtractive algebra $\mathbf{A}$ is affine, then every term operation of $\mathbf{A}$ is affine as well; in particular $s(x, y)$ commutes with itself, i.e. satisfies

$$s(s(x, x'), s(y, y')) \approx s(s(x, y), s(x', y')). \tag{*}$$

It follows that if s commutes with itself in $\mathbf{A}$ then so does in any $\mathbf{B} \in \mathbf{V}(\mathbf{A})$. By Theorem 3.4.3 if every basic operation of $\mathbf{A}$ is affine, then $\mathbf{A}$ is strongly subtractive. The converse does not hold in view of Proposition 4.6.2 below: just consider non Abelian groups. We have a Lemma whose easy verification is left to the reader.

Lemma 4.6.1. *If $s(x, y)$ commutes with itself in $\mathbf{A}$, then:*

1. $s(0, s(0, x)) = x$;
2. *if* $s(x, y) = 0$, *then* $x = y$;
3. $p_s(x, y, z)$ *defined above is a Mal'cev term for* $\mathbf{A}$.

Notice also that if s commutes with itself in $\mathbf{A}$, then $\mathbf{A}$ is 0-regular: if θ, φ are congruences of $\mathbf{A}$, $0/\theta = 0/\varphi$ and $(a, b) \in \theta$ then $s(a, b) \in 0/\theta = 0/\varphi$. So, in $\mathbf{A}/\theta$, $s(a/\varphi, b/\varphi) = 0/\varphi$ and thus $a/\varphi = b/\varphi$ by Lemma 4.6.1. So $(a, b) \in \varphi$.

The classical definition of affine algebra is as follows: an algebra $\mathbf{A}$ is **affine** if there exists an abelian group $\langle A, +, -, 0\rangle$ with the same universe of $\mathbf{A}$ and a ternary term $t(x, y, z)$ such that

- $t(x, y, z) = x - y + z$;
- for any n-ary term $f$and for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in A^n$

$$f(\mathbf{x} - \mathbf{y} + \mathbf{z}) = f(\mathbf{x}) - f(\mathbf{y}) + f(\mathbf{z}).$$

The following proposition establishes a connection between affine operations on a subtractive algebra and being affine.

Proposition 4.6.2. *[3] For an algebra* $\mathbf{A}$ *in a subtractive variety the following are equivalent:*

1. $\mathbf{A}$ *is affine witness* $p(x,y,z)$*;*
2. *every basic operation (and thus any term operation) of* $\mathbf{A}$ *is affine;*
3. $\mathbf{A}$ *is abelian;*
4. $\mathrm{D}(\mathbf{A})$ *is an ideal of* $\mathbf{A}\times\mathbf{A}$ *and* s *commutes with itself in* $\mathbf{A}$*;*
5. $\mathrm{D}(\mathbf{A})$ *is an ideal of* $\mathbf{A}\times\mathbf{A}$ *and* $\mathbf{A}$ *generates a congruence permutable variety with Mal'cev term* $p(x,y,z)$ *such that*

$$p(x,y,z)=z \qquad \textit{implies} \qquad x=y.$$

Hence $\mathbf{V}(\mathbf{A})$ *is ideal determined and strongly subtractive.*

Proof. Assume (1) and let's assume, without loss of generality, that f be a unary basic operation. Then

$$\begin{aligned}
s(f(x),f(y)) &= f(x)-f(y)+f(0)-f(0)\\
&= p(f(x),f(y),f(0))-f(0)\\
&= f(p(x,y,0))-f(0)=s(f(s(x,y)),f(0)).
\end{aligned}$$

For (2) $\longrightarrow$ (3) define a binary relation Θ on $A\times A$ by setting

$$\langle a,b\rangle\,\Theta\,\langle a',b'\rangle \qquad \text{if and only if} \qquad s(a,b)=s(a',b').$$

Θ is obviously an equivalence relation; to show that it is a congruence of $\mathbf{A}\times\mathbf{A}$, take a basic operation f (again we suppose f unary) and assume $\langle a,b\rangle\,\Theta\,\langle a',b'\rangle$:

$$\begin{aligned}
s(f(a),f(b)) &= t_f(s(a,b))=t_f(s(a',b'))\\
&= s(f(a'),f(b')).
\end{aligned}$$

Therefore

$$f^{A\times A}(\langle a,b\rangle)=\langle f(a),f(b)\rangle\,\Theta\,\langle f(a'),f(b')\rangle=f^{A\times A}(\langle a',b'\rangle).$$

Moreover $\langle a,b\rangle\,\Theta\,\langle 0,0\rangle$ if an only if $s(a,b)=0$ if and only if (via Lemma 4.6.1) $a=b$, i.e. $\mathrm{D}(\mathbf{A})=\langle 0,0\rangle/\Theta$.

For (3) $\longrightarrow$ (4), assuming $\mathrm{D}(\mathbf{A}) = \langle 0,0\rangle/\Theta$ for some $\Theta \in \mathrm{Con}(\mathbf{A}\times\mathbf{A})$ we have to show that $(*)$ holds. Let L and R be the left and the right hand side of $(*)$ respectively. In $\mathbf{A}\times\mathbf{A}$ we have

$$\langle L, R\rangle = s(\langle s(a,a'), s(a,b)\rangle, \langle s(b,b'), s(a,'b')\rangle);$$

but

$$\langle s(a,a'), s(a,b)\rangle = s(\langle a,a\rangle, \langle a',b\rangle)\,\Theta\, s(\langle b,b\rangle, \langle a',b\rangle) = \langle s(b,a'), 0\rangle$$

and

$$\langle s(b,b'), s(a',b')\rangle = s(\langle b,a'\rangle, \langle b',b'\rangle)\,\Theta\, s(\langle b,a'\rangle, \langle a',a'\rangle) = \langle s(b,a'), 0\rangle.$$

Therefore

$$\langle L, R\rangle\,\Theta\, s(\langle s(b,a'), 0\rangle, \langle s(b,a'), 0\rangle) = \langle 0, 0\rangle.$$

This implies $\langle L, R\rangle \in \mathrm{D}(\mathbf{A})$ and hence $L = R$.

Next assume (4). By Lemma 4.6.1, p is a Mal'cev term for $\mathbf{A}$ and hence for the variety it generates. Moreover, assume $p(x,y,z) = z$; then by Lemma 4.6.1

$$s(s(x, s(y,z)), s(0, s(0,z))) = s(z,z) = 0.$$

Since s commutes with itself we then have

$$0 = s(s(x,0), s(s(y,z), s(0,z))) = s(x, s(s(y,0), s(z,z))) = s(x, s(y,0)) = s(x,y).$$

Via Lemma 4.6.1(2) we finally get $x = y$.

Finally (5) $\longrightarrow$ (1) is a basic result on Abelian algebras (see [17]). $\square$

Corollary 4.6.3. *Let* V *a subtractive variety and* $\mathbf{A} \in \mathsf{V}$*. Then the following are equivalent:*

1. $\mathbf{A}$ *is ideal abelian;*
2. $\Delta_{\mathbf{A}} \in \mathrm{Con}(\mathbf{A})$ *and* $\mathbf{A}/\Delta_{\mathbf{A}}$ *is affine.*

An algebra $\mathbf{A}$ is Hamiltonian if every subalgebra of $\mathbf{A}$ is a block of some congruence of $\mathbf{A}$. A variety is Hamiltonian if it consists of Hamiltonian algebras. A subtractive algebra is Hamiltonian if and only if every subalgebra is an ideal. For subtractive varieties it is possible to strengthen the well-known Klukovits' result for Hamiltonian varieties [21]:

Proposition 4.6.4. *A subtractive variety* V *is Hamiltonian if and only if no ideal term* $t(\vec{x}, \vec{y})$ *in* $\vec{y}$ *depends on* $\vec{x}$, *i.e. there is a term* $s_t(\vec{y})$ *such that*

$$t(\vec{x}, \vec{y}) \approx s_t(\vec{y})$$

holds in V.

Proof. Let V be Hamiltonian and let $t(\vec{x}, \vec{y})$ be an ideal term in $\vec{y}$. Let $\mathbf{F}$ be the free algebra in V on $\vec{x}, \vec{y}$ and let $\mathbf{B}$ be the subalgebra of $\mathbf{F}$ generated by $\vec{y}$. Then $\mathbf{B}$ is an ideal of $\mathbf{F}$ and hence $t(\vec{x}, \vec{y}) \in \mathbf{B}$. So there exists a term $s_t(\vec{y})$ with

$$s_t(\vec{y}) = t(\vec{x}, \vec{y}).$$

The converse is obvious. □

Being abelian or Hamiltonian are unrelated properties for single algebras. In [20] Kiss and Valeriote produced an example of a (non subtractive) algebra which is abelian but not Hamiltonian. The same holds for subtractive algebras.

Example 4.6.5. [3] We construct a finite algebra $\mathbf{A}$ in which $\Delta \in \mathrm{Con}(\mathbf{A})$, $\Delta \neq 0_{\mathbf{A}}$ and $\mathbf{A}/\Delta$ is ideal abelian. By Corollary 4.6.3 $\mathbf{A}$ will not be Abelian.

Let $\mathbf{A} = \langle \{0, a, b, c, d\}, s \rangle$ where s is defined by the following table:

s	0	a	b	c	d
0	0	a	b	c	d
a	a	0	c	b	b
b	b	c	0	a	a
c	c	b	a	0	0
d	d	b	a	0	0

It is clear that $s(x, y)$ is a commutative subtraction term and it is easily seen that $\Delta \in \mathrm{Con}(\mathbf{A})$. Next we observe that $\mathbf{A}/\Delta$ is weakly isomorphic to $\mathbf{Z}_2 \times \mathbf{Z}_2$, the Klein group and it is of course Abelian. Also observe that

$$\Delta = (0)(a)(b)(cd) \qquad \theta_0 = (0a)(bcd) \qquad \theta_1 = (0b)(acd) \qquad \alpha = (0cd)(ab)$$

are congruences of $\mathbf{A}$ and, together with $1_{\mathbf{A}}$, they sit in $\mathrm{Con}(\mathbf{A})$ as an $\mathbf{M}_3$. Therefore $\mathbf{M}_3$ is a 0-1-sublattice of $\mathrm{Id}(\mathbf{A})$, hence $\mathbf{A}$ is ideal abelian.

Moreover:

- $\mathbf{A}$ is subdirectly irreducible (with monolith Δ) but not ideal irreducible.
- $\mathbf{A}$ is not Hamiltonian since $\{0, a, b, c\}$ is a subalgebra which is not the block of any congruence.
- $\mathbf{A}$ is strongly subtractive, since it is ideal abelian. But $\mathbf{A}$ generates a variety which is not ideal determined, otherwise it would be abelian, which is not.

For a variety however, being Hamiltonian has stronger consequences. In particular it is an easy exercise to show that if $\mathbf{A} \times \mathbf{A}$ is Hamiltonian then $\mathbf{A}$ is abelian; hence every Hamiltonian variety is abelian.

Theorem 4.6.6. *[3] For a subtractive variety V the following are equivalent:*

1. *V is Hamiltonian;*
2. *V is abelian and classically ideal determined;*
3. *V is ideal abelian and ideal determined;*
4. *any algebra in V is affine.*

Proof. Assume (1); since any Hamiltonian variety is abelian, V is ideal determined. Moreover the term $u(x, y, z)$ is an ideal term in y, z; hence there exists an $s_u(y, z) = u(x, y, z)$. Hence

$$s_u(y, s(x, y)) = u(x, y, s(x, y)) = x$$

which is enough to ensure classical ideal determinacy.

Now (2) trivially implies (3), so let's the latter. In this case V is abelian and congruence modular, so (4) follows from Proposition 4.6.2.

Finally assume (4) and let $\mathbf{A} \in \mathsf{V}$; let $\mathbf{B}$ be a subalgebra of $\mathbf{A}$ and $p(\vec{x}, \vec{y})$ be an ideal term in $\vec{x}, \vec{y}$. Let $\vec{a} \in A$ and $\vec{b} \in B$; since $\mathbf{A}$ is affine p is affine by Proposition 4.6.2 and thus there is a term t_p such that

$$\begin{aligned} p(\vec{a}, \vec{b}) &= s(p(\vec{a}, \vec{b}), p(\vec{a}, \vec{0})) \\ &= t_p(s(a_1, a_1), \dots, s(a_n, a_n), s(b_1, 0), \dots, s(b_m, 0)) \\ &= t_p(0, \dots, 0, b_1, \dots, b_m) \in B. \end{aligned}$$

Therefor $\mathbf{B}\mathrm{Id}(\mathbf{A})$ and $\mathbf{A}$ is Hamiltonian.

□

Corollary 4.6.7. *A subtractive ideal abelian algebra $\mathbf{A}$ generates a Hamiltonian variety if and only if $\mathbf{A}$ is affine.*

Chapter 5

Equationally definable principal ideals

5.1 Foreword

In a generic variety, dealing with the join of two congruences of an algebra is somewhat complex; in general one has to deal with longer and longer chains of relational products and, unless some form of congruence permutability is present, there might be no upper bound for the length of the chains. Finiteness does not help either; of course in a finite algebra $\mathbf{A}$ there is an upper bound for the length of the chains but it is trivial to produce an example of an algebra $\mathbf{A}$ generating a variety in which no such bound can exist. In contrast the join of two ideals is a subtractive variety is nice; essentially (Lemma 2.4.6) if $b \in A$ and $I, J \in \mathrm{Id}(\mathbf{A})$, then $b \in I \vee J$ if and only if there is an $a \in I$ with $s(b, a) \in J$. This can be easily generalized to multiple joins: for every $n \geq 2$, there is an n-ary term s_n such that for any $\mathbf{A} \in \mathsf{V}$ and $I_1, \dots, I_n \in \mathrm{Id}(\mathbf{A})$ the following holds:

$$a \in \bigvee_{i=1}^{n} I_i \text{ if and only if there are } c_i \in I_i,\ i = 2, \dots, n$$
$$\text{such that } s_n(a, c_2, \dots, c_n) \in I_1. \tag{†}$$

A further motivation is in the relation of having equationally definable principal ideals with the algebrization of the natural deduction system of logic. Take the introduction and elimination rules for implication $\to$ in classical (or intuitionistic) natural deduction

$$
\frac{\begin{array}{c}[A]\\ \vdots \\ B\end{array}}{A \to B} \qquad\qquad \frac{A \quad A \to B}{B}
$$

In an algebraic-logical setting, these may be translated as:

$$b \in (a)_{\mathbf{A}} \vee I \qquad \text{if and only if} \qquad a \to b \in I$$

where I is an ideal. In a subtractive variety, this is equivalent to:

$$b \in (a)_{\mathbf{A}} \qquad \text{if and only if} \qquad a \to b = 0.$$

This (see Theorem 5.3.1 below) means exactly that the principal ideals are equationally definable.

Also, there exist natural examples of varieties which are subtractive and have equationally definable principal ideals, but are neither ideal determined nor have equationally definable principal congruences.

We close this section with the description of some classes of algebras that will be used repeatedly in the sequel.

A **Brouwerian semilattice** [22] is an algebra $\langle A, \to, \wedge, 1\rangle$ such that for any $a, b, c \in A$

1. $\langle A, \wedge, 1\rangle$ is an upper bounded semilattice;
2. $a \to a = 1$;
3. $(a \to b) \wedge a = (b \to a) \wedge b$;
4. $(a \wedge b) \to c = a \to (b \to c)$.

If $\mathbf{A}$ is a Brouwerian semilattice and $a, b, c \in A$, then

$$c \leq a \to b \qquad \text{if and only if} \qquad a \wedge c \leq b.$$

Hence $a \to b$ is the relative pseudocomplement of a and b. The variety BS of Brouwerian semilattice is of course ideal determined. Moreover it can be shown that the class of $\to$-subreduct of BS coincides with the variety HI of Hilbert algebras and that the congruences (hence the ideals) of a Brouwerian semilattice coincide with those of its $\to$-reduct.

Since we will be dealing with partially ordered structures in which the constant is the smallest element in the ordering, we feel that the dual concepts are more suitable.

A **dual Brouwerian semilattice** is a join semilattice with dual relative pseudocomplementation, i.e. an algebra $\langle A, \vee, *, 0\rangle$ such that

$$a * b \leq c \qquad \text{if and only if} \qquad b \leq a \vee c.$$

A **dual Hilbert algebra** is the $*$-subreduct of a dual Brouwerian semilattice. The distinction between Brouwerian semilattices (Hilbert algebras) and dual Brouwerian semilattices (dual Hilbert algebras) is of course purely notational.

5.2 Definability of principal ideals

For the notion of definable principal congruences and equationally definable principal congruences we refer to the literature, mainly to [14], [8] and [7].

If K is a class of algebras, we say that K has **definable principal ideals** (DPI) if there is a first order formula $\Psi(x, y, y_1, \ldots, y_n)$ in the language of K such that for all $\mathbf{A} \in \mathsf{K}$, $a, b \in A$

$$a \in (b)_{\mathbf{A}} \qquad \text{if and only if} \qquad \mathbf{A} \vDash \exists y_1, \ldots, y_n\, \Psi(a, b, y_1, \ldots, y_n).$$

Proposition 5.2.1. *[5] Let K be a class of algebras with a constant 0.*

1. *If K has normal ideals and has definable principal congruences, then $\mathbf{K}$ has definable principal ideals.*

2. *If K is contained in an ideal determined variety and has definable principal ideals then K has definable principal congruences.*

Proof. For 1. just remember that, by normality of ideals, if $\mathbf{A} \in \mathsf{K}$ and $a, b \in A$, then

$$a \in (b)_{\mathbf{A}} \quad \text{if and only if} \quad (0, a) \in \vartheta(0, b).$$

Hence if $\Psi(x, y, z, w, z_1, \ldots, z_k)$ defines principal congruences, then

$$\Psi(0, x, 0, y, z_1, \ldots, z_k)$$

defines principal ideals in K.

For 2., assume K is contained in an ideal determined variety V and let $d_1, \dots, d_n$ be the binary terms witnessing 0-regularity for V. One easily checks that (see for instance [18])

$$\begin{aligned}(a,b) \in \vartheta_{\mathbf{A}}(c,d) \qquad & \text{if and only if} \\ & d_i(a,b) \in \langle d_1(c,d), \dots, d_n(c,d)\rangle_{\mathbf{A}} \ \text{ for } i = 1, \dots, n.\end{aligned}$$

Next using subtractivity and recalling (†) in Section 5.1, if $\Psi(x, y, z_1, \dots, z_n)$ defines principal ideals in K then the formula

$$\begin{aligned}\bigwedge_{i=1}^{m} \bigwedge_{r=1}^{m-1} (\Psi(z_r^i, d_r(z,t), y_1^{i,r}, \dots, y_k^{i,r}) \\ \wedge \bigwedge_{i=1}^{m} \varphi(s_n(d_i(x,y), z_1^i, \dots, z_{m-1}^i), d_m(z,t), y_1^{i,m}, \dots, y_k^{i,m}))\end{aligned}$$

defines principal congruences in V. □

We say that $\mathbf{K}$ has **equationally definable principal ideals in the broad sense** (EDPI$^{\#}$ for short) if there are terms p_i, q_i $i = 1, \dots, k$ such that for all $\mathbf{A} \in \mathsf{K}$, $a, b \in A$

$$\begin{aligned}& a \in (b)_{\mathbf{A}} \ \text{ if and only if } \ \exists u_1, u_2, \dots \in A \text{ s.t.} \\ & p_i(a, b, u_1, u_2, \dots) = q_i(a, b, u_1, u_2, \dots) \ \text{ for all } i = 1, \dots, k\end{aligned}$$

Let EDPC$^{\#}$ denote the corresponding notion for congruences; by adapting the proofs of Proposition 5.2.1 we get:

Theorem 5.2.2. *[5] Let K be a class of algebras with 0.*

1. *If for all $\mathbf{A} \in \mathsf{K}$, $\mathrm{N}(\mathbf{A}) = \mathrm{I}_{\mathsf{K}}(\mathbf{A})$ and if $\mathbf{K}$ has EDPC$^{\#}$, then it has EDPI$^{\#}$.*

2. *If K is contained in an ideal determined variety and K has EDPI$^{\#}$, then K has EDPC$^{\#}$.*

Therefore an ideal determined variety has the EDPI$^{\#}$ if and only if it has EDPC$^{\#}$.

In case V is 0-regular (with witness terms $d_1, \dots, d_n$), then

$$p_i(a, b, u_1, u_2, \dots) = q_i(a, b, u_1, u_2, \dots) \text{ for } 1 = 1, \dots, k$$

is equivalent to

$$d_j(p_i(a, b, u_1, u_2, \dots), q_i(a, b, u_1, u_2, \dots)) = 0 \text{ for } 1 = 1, \dots, k, \ j = 1, \dots, n.$$

This suggests the following definitions. We say that K has a **uniform implicit term** p for principal ideals (UIT) if for any $\mathbf{A} \in \mathsf{K}$ $a, b \in A$

$$a \in (b)_{\mathbf{A}} \text{ if and only if } \exists u_1, u_2, \dots \in A \text{ s.t. } \quad p(a, b, u_1, u_2, \dots) = 0.$$

Next K has a **uniform explicit term** $q(x_1, \dots, x_n, y)$ for principal ideals (UET) if q is an ideal term in y and moreover, for any $\mathbf{A} \in \mathsf{K}$

$$a \in (b)_{\mathbf{A}} \text{ if and only if } \exists u_1, \dots, u_n \in A \text{ s.t. } \quad q(u_1, \dots, u_n, b) = a.$$

A variety V has **factorable principal ideals on direct products** if, whenever $\mathbf{A}_i \in \mathsf{V}$ and $b \in \prod_{i \in I} \mathbf{A}_i$,

$$\prod_{i \in I} (b_i)_{\mathbf{A}_i} \subseteq (b)_{\mathbf{A}}.$$

Note that the inclusion $(b)_{\mathbf{A}} \subseteq \prod_{i \in I} (b_i)_{\mathbf{A}_i}$ holds in any case.

A variety V has a **test algebra for principal ideals**, if there exists an $\mathbf{A} \in \mathsf{V}$ and $a, b \in A$, such that

- $a \in (b)_{\mathbf{A}}$;
- for any $\mathbf{B} \in \mathsf{V}$ and $a', b' \in \mathbf{B}$, if $a' \in (b')_{\mathbf{B}}$ then there is a homomorphism φ of $\mathbf{A}$ into $\mathbf{B}$ such that $\varphi(a) = a'$ and $\varphi(b) = b'$.

We have:

Theorem 5.2.3. *[5] For a subtractive variety V the following are equivalent:*

1. *V has a UET.*
2. *V has a UIT.*
3. *V has EDPI$^{\#}$.*

4. V *has factorable principal ideals on direct products.*

5. V *has a test algebra for principal ideals.*

Proof. Assume (1) and let $q(x_1, \ldots, x_k, y)$ be a UET for K. Define

$$p(x_1, \ldots, x_k, x, y) = s(x, q(x_1, \ldots, x_k, y)).$$

If $a \in (b)_{\mathbf{A}}$ then there are $u_1, \ldots, u_n \in A$ such that $q(u_1, \ldots, u_n, b) = a$. Thus

$$p(u_1, \ldots, u_k, a, b) = s(a, q(u_1, \ldots, u_n, b)) = s(a, a) = 0.$$

On the other hand if for some $u_1, \ldots, u_k \in A$, $p(u_1, \ldots, u_k, a, b) = 0$, then

$$s(a, q(u_1, \ldots, u_k, b)) = 0 \in (b)_{\mathbf{A}}.$$

Since q is an ideal term in y and $q(\vec{u}, b) \in (b)_{\mathbf{A}}$ we conclude that $a \in (b)_{\mathbf{A}}$. This shows that p is a UIT for K.

That (2) implies (3) and (3) implies (4) are immediate, so let's assume (4). We consider a *subset* F of V such that for every finitely generated algebra $\mathbf{A} \in \mathsf{V}$ and for every $a, b \in A$ with $a \in (b)_{\mathbf{A}}$ there are: an algebra $\mathbf{A}' \in \mathsf{F}$, $a', b' \in A'$ with $a' \in (b')_{\mathbf{A}}$ and an isomorphism $\varphi : \mathbf{A} \longrightarrow \mathbf{A}'$ with $\varphi(a) = a'$ and $\varphi(b) = b'$. Then it is easily seen that $\mathbf{A} = \prod\{\mathbf{A}' : \mathbf{A}' \in \mathsf{F}\}$ is a test algebra for principal ideals.

Finally assume (5) and let $\mathbf{A}$ be a test algebra for principal ideals witness $a, b \in A$. Since $a \in (b)_{\mathbf{A}}$ there is an ideal term $q(x_1, \ldots, x_k, y)$ in y such that $a = q(u_1, \ldots, u_k, b)$ for some $u_1, \ldots, u_k \in A$. Then, if $\mathbf{B} \in \mathsf{V}$, $a', b' \in \mathbf{B}$ and $a' \in (b')_{\mathbf{B}}$, we get

$$a' = \varphi(a) = \varphi(q(u_1, \ldots, u_k, b)) = q(\varphi(u_1), \ldots, \varphi(u_k), b').$$

Conversely if $a' = q(\varphi(u_1), \ldots, \varphi(u_k), b')$, being q an ideal term in y we get $a' \in (b')_{\mathbf{B}}$. So q is a UET for V and (1) holds. $\square$

5.3 Equationally definable principal ideals

If in the definition of EDPI$^{\#}$ we dispose of the parameters, then we obtain the property which will be the subject of our investigations from now on. A variety V has **equationally definable principal ideals** (EDPI) if there are terms $p_i(x,y), q_i(x,y)$, $i = 1, \ldots, n$ such that for any $\mathbf{A} \in \mathsf{K}$ and $a, b \in A$

$$a \in (b)_{\mathbf{A}} \qquad \text{if and only if} \qquad p_i(a,b) = q_i(a,b), \quad i = 1, \ldots, n.$$

For subtractive varieties with EDPI we can get a strengthening of Theorem 5.2.3.

Theorem 5.3.1. *[3] [5] For a subtractive variety V the following are equivalent.*

1. *V has EDPI.*
2. *There are binary terms p_i, $i = 1, \ldots, n$ such that*

$$a \in (b)_{\mathbf{A}} \qquad \textit{if and only if} \qquad p_i(a,b) = 0 \quad i = 1, \ldots, n.$$

3. *There is a binary term $p(x,y)$ such that*

$$a \in (b)_{\mathbf{A}} \qquad \textit{if and only if} \qquad p(a,b) = 0 \quad i = 1, \ldots, n.$$

4. *For any family $(\mathbf{A}_i : i \in I)$ of algebras in V and for any subalgebra $\mathbf{B}$ of $\prod_{i \in I} \mathbf{A}$ for any $a, b \in B$,*

$$a \in (b)_{\mathbf{B}} \qquad \textit{if and only if} \qquad a_i \in (b_i)_{\mathbf{A}_i}, \ i \in I.$$

5. *There exists an $\mathbf{A} \in \mathsf{V}$ generated by two elements a and b, such that*
   (i) $a \in (b)_{\mathbf{A}}$;
   (ii) for any $\mathbf{B} \in \mathsf{V}$ and $a', b' \in \mathbf{B}$, if $a' \in (b')_{\mathbf{B}}$ then there is a homomorphism φ of $\mathbf{A}$ into $\mathbf{B}$ such that $\varphi(a) = a'$ and $\varphi(b) = b'$.
6. *There is a ternary term $p(x,y,z)$ such that $p(x,y,0) \approx 0$ holds in V and for any algebra $\mathbf{A} \in \mathsf{V}$, $a, b \in A$, $a \in (b)_{\mathbf{A}}$ if and only if $p(a,b,b) = a$.*
7. *For any $\mathbf{A} \in \mathsf{V}$ the semilattice $\mathrm{CI}(\mathbf{A})$ is a dual Brouwerian semilattice.*

Proof. The proofs of equivalences (1)-(6) go along the lines of Theorem 5.2.3 and they are left to the reader. So we need only show that (7) fits well.

For general results on Brouwerian semilattices we refer the reader to [22] or [8]. Suppose V has EDPI with witness terms $p_1, \ldots, p_n$. First we show that for all $\mathbf{A} \in \mathsf{V}$, $a, b \in A$, $I \in \mathrm{Id}(\mathbf{A})$

$$a \in (b)_{\mathbf{A}} \vee I \qquad \text{if and only if} \qquad p_i(a,b) \in I \text{ for } i = 1, \ldots, n.$$

In fact let $I = 0/\theta$ for some $\theta \in \mathrm{Con}(\mathbf{A})$; then

$$\begin{aligned} a \in (b)_{\mathbf{A}} \vee I &\text{ if and only if } a \in (b)_{\mathbf{A}} \vee 0/\theta \\ &\text{ if and only if } a/\theta \in (b/\theta)_{\mathbf{A}/\theta} \\ &\text{ if and only if } p_i(a/\theta, b/\theta) = 0/\theta \text{ for } i = 1, \ldots, n \\ &\text{ if and only if } p_i(a,b) \in I \text{ for } i = 1, \ldots, n. \end{aligned}$$

It follows that the operation $(a) * (b) = (p_1(a,b), \ldots, p_n(a,b))_{\mathbf{A}}$ is a dual relative pseudocomplementation in $\mathrm{CI}(\mathbf{A})$ for any two principal ideals of $\mathbf{A}$. But it is a general fact (see [23], Lemma 4) that, if any two elements of a generating set of a join semilattice have a dual pseudocomplement, then the semilattice is dually Brouwerian.

For the converse assume that $\mathrm{CI}(\mathbf{A})$ is dually Brouwerian for any $\mathbf{A} \in \mathsf{V}$. Let $\mathbf{F}$ be the algebra freely generated in V by $\{x, y, v_j\}_{j \in \omega}$. By hypothesis $(x)_{\mathbf{F}} * (y)_{\mathbf{F}}$ exists in $\mathrm{CI}(\mathbf{F})$, hence there are terms $r_i(x, y, v_1, v_2, \ldots)$, $i = 1, \ldots, n$ such that

$$(x)_{\mathbf{F}} * (y)_{\mathbf{F}} = \bigvee_{i=1}^{n} (r_i(x, y, v_1, v_2, \ldots))_{\mathbf{F}}.$$

Let $p_i(x,y) = r_i(x, y, x, x, \ldots)$ and assume $a \in (b)_{\mathbf{A}}$. Then there is a finitely generated subalgebra $\mathbf{B}$ of $\mathbf{A}$ such that $a \in (b)_{\mathbf{B}}$. Let φ be a homomorphism from $\mathbf{F}$ onto $\mathbf{B}$ such that $\varphi(x) = \varphi(v_j) = a$ and $\varphi(y) = b$. Then $J = \varphi^{-1}(0) \in \mathrm{Id}(\mathbf{B})$ and we have

$$\begin{aligned} a \in (b)_{\mathbf{B}} &\text{ if and only if } \varphi(x) \in (\varphi(y))_{\mathbf{B}} \\ &\text{ if and only if for some } t \in (y)_{\mathbf{F}},\ (x,t) \in ker\varphi \\ &\text{ if and only if } x \in (y)_{\mathbf{F}} \vee J \\ &\text{ if and only if } r_i(x, y, v_1, v_2, \ldots) \in J \text{ for } i = 1, \ldots, n \\ &\text{ if and only if } \varphi(r_i(x, y, v_1, v_2, \ldots) = 0 \text{ for } i = 1, \ldots, n \\ &\text{ if and only if } r_i(a, b, a, a, \ldots) = 0 \text{ for } i = 1, \ldots, n \\ &\text{ if and only if } p_i(a,b) = 0 \text{ for } i = 1, \ldots, n. \end{aligned}$$

□

Corollary 5.3.2. *Every subtractive variety* V *with EDPI is ideal distributive.*

Proof. By Proposition 2.3.1 if $\mathbf{A} \in \mathsf{V}$ then $\mathrm{Id}(\mathbf{A})$ is isomorphic with the ideal lattice of $\mathrm{CI}(\mathbf{A})$. By Theorem 5.3.1 the latter is a dual Brouwerian semilattice and it is well known that the ideal lattice of a dual Brouwerian semilattice is distributive. □

Example 5.3.3. [5] The hypothesis of subtractivity in Theorem 5.3.1 cannot be weakened to having normal ideals. In fact consider the variety S_0 of lower bounded meet semilattices and let us denote the constant again by 0. Then any semilattice term is an ideal term and one sees easily that ideals coincide with order ideals in the usual sense. The variety S_0 has then normal ideals. If I is an order ideal, then the equivalence induced by the partition in which I is the unique nontrivial block is a semilattice congruence. If $\mathbf{S} \in \mathsf{S}_0$ and $a, b \in S$, then $a \in (b)_{\mathbf{S}}$ if and only if $a \leq b$, if and only if $a \wedge b = a$, so S_0 has EDPI. However it is obvious that there is no binary term satisfying (3) of 5.3.1. In fact one can check that S_0 is an variety with EDPI and normal ideals that fails any other equivalence in Theorem 5.3.1. Note also that S_0 is not even congruence modular; it is very easy to construct a finite (hence lower bounded) meet semilattice whose congruence lattice is isomorphic with $\mathbf{N}_5$, So it cannot have EDPC.

Theorem 5.3.1 implies that any subtractive variety with EDPC has EDPI and that the converse holds if the variety is ideal determined. It follows that Boolean Algebras (with dual normal operators), Heyting Algebras, Brouwerian semilattices and Hilbert and Tarski algebras EDPI. Let us remark that if $\mathbf{A}$ is a Hilbert algebra (or a Brouwerian semilattice) $a, b \in \mathbf{A}$ and $*$ is the dual relative pseudocomplementation, then

$$a \in (b)_{\mathbf{A}} \qquad \text{if and only if} \qquad a * b = 0.$$

This means that the binary term giving relative pseudocomplementation witnesses *both* subtractivity and EDPI. In other words in a Brouwerian semilattice $\mathbf{A}$

$$(a)_{\mathbf{A}} * (b)_{\mathbf{A}} = (p(b, a))_{\mathbf{A}}.$$

Hence the set $\mathrm{PI}(\mathbf{A})$ of principal ideals of $\mathbf{A}$ is closed under $*$ and $\langle \mathrm{PI}(\mathbf{A}), *, (0)_{\mathbf{A}} \rangle$ is a dual Hilbert algebra. The following theorems show that we can go even further, in that any algebra in a subtractive variety with EDPI has a "weak structure" closely resembling a dual Hilbert algebra.

Theorem 5.3.4. *[5] Let V be subtractive and EDPI. Then there exists a binary term $x * y$ with the following properties.*

1. *For all $\mathbf{A} \in \mathsf{V}$ and $a \in A$*

$$\begin{aligned}&a * a = 0\\&a * 0 = 0\\&0 * a = a\\&b \in (a)_{\mathbf{A}} \text{ if and only if } a * b = 0.\end{aligned}$$

2. *The relation $\leq$ defined by $a \leq b$ if and only if $b * a = 0$ is reflexive and transitive. The associated equivalence relation $\approx_{\mathbf{A}}$ is a congruence of $\mathbf{A}^* = \langle A, *, 0\rangle$ and $\mathbf{A}^*/ \approx_{\mathbf{A}}$ is a dual Hilbert algebra isomorphic with $\langle \mathrm{PI}(\mathbf{A}), *, (0)_{\mathbf{A}}\rangle$.*

3. *Any principal ideal of $\mathbf{A}$ is the union of a principal ideal of $\mathbf{A}^*/ \approx_{\mathbf{A}}$ and viceversa. In fact $(a)_{\mathbf{A}} = \bigcup (a/ \approx_{\mathbf{A}})_{\mathbf{A}^*/\approx_{\mathbf{A}}}$.*

Proof. Suppose that $s(x, y)$ is the witness of subtractivity. Then, since V has EDPI, from Theorem 5.3.1(6) we get the existence of a ternary term $p(x, y, z)$ such that, for any $\mathbf{A} \in \mathsf{V}$ and $a, b \in A$

$$p(b, a, 0) = 0 \qquad\qquad p(b, a, a) = b \text{ if and only if } b \in (a)_{\mathbf{A}}.$$

Define $x * y = s(y, p(y, x, x))$. Then

$$\begin{aligned}a * a &= s(a, p(a, a, a)) = s(a, a) = 0;\\a * 0 &= s(0, p(0, a, a)) = s(0, 0) = 0;\\0 * a &= s(a, p(a, 0, 0)) = s(a, 0) = a.\end{aligned}$$

Next, if $b \in (a)_{\mathbf{A}}$, then

$$a * b = s(b, p(b, a, a)) = s(b, b) = 0.$$

Conversely, if $a*b = 0$, then $s(b, p(b, a, a)) = 0$. Since $0 \in (a)_{\mathbf{A}}$ and $p(b, a, a) \in (a)_{\mathbf{A}}$ ($p(x, y, z)$ is an ideal term in z), subtractivity yields $b \in (a)_{\mathbf{A}}$ as well. This takes care of 1.

The fact that $\leq$ is a quasi order is obvious from the fact that $*$ witness EDPI. Consider the mapping

$$a \longmapsto (a)_{\mathbf{A}}$$

from $\mathbf{A}$ to $\mathrm{PI}(\mathbf{A})$. Then $(a)_{\mathbf{A}} * (b)_{\mathbf{A}} = (a*b)_{\mathbf{A}}$, therefore the mapping is a homomorphism from $\mathbf{A}^*$ to $\langle \mathrm{PI}(\mathbf{A}), *, (0)_{\mathbf{A}}\rangle$, whose kernel coincides with $\approx_{\mathbf{A}}$. Hence 2. follows.

Finally if $b \in (a)_{\mathbf{A}}$ then $a*b=0$. This implies $(a*b)/\approx_{\mathbf{A}} = 0/\approx_{\mathbf{A}}$ and so $a/\approx_{\mathbf{A}} * b/\approx_{\mathbf{A}} = 0/\approx_{\mathbf{A}}$. But $\mathbf{A}^*/\approx_{\mathbf{A}}$ is a dual Hilbert algebra, thus it has EDPI with witness term $*$. This implies $b/\approx_{\mathbf{A}} \in (a/\approx_{\mathbf{A}})_{\mathbf{A}^*/\approx_{\mathbf{A}}}$ and so $b \in \bigcup (a/\approx_{\mathbf{A}})_{\mathbf{A}^*/\approx_{\mathbf{A}}}$.

Next if $b \in \bigcup (a/\approx_{\mathbf{A}})_{\mathbf{A}^*/\approx_{\mathbf{A}}}$, then $b \in c/\approx_{\mathbf{A}} \in (a/\approx_{\mathbf{A}})_{\mathbf{A}^*/\approx_{\mathbf{A}}}$, therefore $b \approx_{\mathbf{A}} c$ and $a/\approx_{\mathbf{A}} * c/\approx_{\mathbf{A}} = 0/\approx_{\mathbf{A}}$. But this implies $(a*c)/\approx_{\mathbf{A}} = 0/\approx_{\mathbf{A}}$ and so $a*c=0$ (since $0/\approx_{\mathbf{A}} = \{0\}$, via 1.). From $a*c=0$ and $c*b=0$ we get (via 2.) $a*b=0$ and therefore $b \in (a)_{\mathbf{A}}$. $\square$

We can also prove a converse of Theorem 5.3.4.

Theorem 5.3.5. *[5] Let V be a variety with a constant 0 and such that the following hold.*

1. *There exists a binary term $x*y$ such that for any $\mathbf{A} \in \mathsf{V}$ and $a \in A$*

$$\begin{aligned} &a*a=0\\ &a*0=0\\ &0*a=0 \Rightarrow a=0. \end{aligned}$$

2. *The relation $\approx_{\mathbf{A}}$ defined by $a \approx b$ if and only if $a*b=b*a=0$ is a congruence of $\mathbf{A}^* = \langle A, *, 0\rangle$ and $\mathbf{A}^*/\approx_{\mathbf{A}}$ has EDPI defined by $u/\approx_{\mathbf{A}} \in (v/\approx_{\mathbf{A}})_{\mathbf{A}^*/\approx_{\mathbf{A}}}$ if and only if $u/\approx_{\mathbf{A}} * v/\approx_{\mathbf{A}} = 0/\approx_{\mathbf{A}}$.*

3. *For any $a \in A$*

$$(a)_{\mathbf{A}} = \bigcup (a/\approx_{\mathbf{A}})_{\mathbf{A}^*/\approx_{\mathbf{A}}}.$$

Then V is subtractive and has EDPI: for any $\mathbf{A} \in \mathsf{V}$ and $a,b \in A$

$$a \in (b)_{\mathbf{A}} \qquad \textit{if and only if} \qquad a*b=0.$$

Proof. First let us show that V has EDPI. If $b \in (a)_{\mathbf{A}}$, then $b \in \bigcup (a/\approx_{\mathbf{A}})_{\mathbf{A}^*/\approx_{\mathbf{A}}}$ by 3. Hence there is a $c \in A$ with $b \approx_{\mathbf{A}} c$ and $c/\approx_{\mathbf{A}} \in (a/\approx_{\mathbf{A}})_{\mathbf{A}^*/\approx_{\mathbf{A}}}$, which implies $b/\approx_{\mathbf{A}} \in (a/\approx_{\mathbf{A}})_{\mathbf{A}^*/\approx_{\mathbf{A}}}$. By 2. we have $a/\approx_{\mathbf{A}} * b/\approx_{\mathbf{A}} = 0/\approx_{\mathbf{A}}$ and, since $\approx_{\mathbf{A}}$ is a congruence, $(a*b)/\approx_{\mathbf{A}} = 0/\approx_{\mathbf{A}}$. But by 1. $0/\approx_{\mathbf{A}} = \{0\}$, so $a*b=0$. Conversely if $a*b=0$, then $a/\approx_{\mathbf{A}} * b/\approx_a lgA = 0/\approx_{\mathbf{A}}$ and hence by 2. $a/\approx_{\mathbf{A}} \in (b/\approx_{\mathbf{A}})_{\mathbf{A}^*/\approx_{\mathbf{A}}}$; by 3. one gets at once that $a \in (b)_{\mathbf{A}}$.

Next observe that, by 3.,

$$\begin{aligned}(0 * a)_{\mathbf{A}} &= \bigcup((0 * a)/\approx_{\mathbf{A}})_{\mathbf{A}^*/\approx_{\mathbf{A}}} = \bigcup(0/\approx_{\mathbf{A}} * a/\approx_{\mathbf{A}})_{\mathbf{A}^*/\approx_{\mathbf{A}}} \\ &= \bigcup(a/\approx_{\mathbf{A}})_{\mathbf{A}^*/\approx_{\mathbf{A}}} = (a)_{\mathbf{A}}\end{aligned}$$

where we have used again 2. Hence if one puts $t(x, y) = y * x$, we have $t(a, a) = 0$ and $(t(a, 0))_{\mathbf{A}} = (a)_{\mathbf{A}}$. By Proposition 2.4.2 V is subtractive. □

From the previous two theorem one gets an ideal theoretic characterization of dual Hilbert algebras.

Corollary 5.3.6. *For variety V of type $\{*, 0\}$ the following are equivalent.*

1. *For all $\mathbf{A} \in \mathsf{V}$ $a, b \in A$*

$$\begin{array}{ccc} b \in (a)_{\mathbf{A}} & \textit{if and only if} & a * b = 0 \\ a * b = b * a = 0 & \textit{implies} & a = b. \end{array}$$

2. *V is a variety of dual Hilbert algebras.*

5.4 MINI algebras

This section contains an example (appearing in [5]) of a subtractive variety with EDPI that does not have EDPC.

Let M be the variety of pointed binars $\langle A, *, 0\rangle$ axiomatized by

M1 $x * 0 \approx 0$

M2 $0 * x \approx x$

M3 $(x * (y * z)) * ((x * y) * (x * z)) \approx 0$

M4 $x * (y * x) \approx 0$.

First note that M is subtractive witness $s(x, y) = y * x$. In fact $0 * x \approx x$ is an axiom and moreover

$$\begin{aligned}0 &\approx (x * (0 * x)) * ((a * 0) * (a * a)) \ \text{(by M3.)} \\ &\approx 0 * (0 * (x * x)) \approx a * a \ \text{(by M4. and M2.)}\end{aligned}$$

An algebra $\mathbf{A} \in \mathsf{M}$ is called a **MINI algebra**[1].

The variety of MINI algebras has other properties. First observe that

$$\text{if } x * y \approx 0 \text{ and } y * z \approx 0, \text{ then } x * z \approx 0. \tag{A}$$

In fact then $x * (y * z) \approx 0$ and so M2 and M3 yield

$$x * z \approx 0 * (0 * (x * z)) \approx (x * (y * z)) * ((x * y) * (x * z)) = 0.$$

Next observe that

$$(y * z) * ((x * y) * (x * z)) \approx 0. \tag{B}$$

In fact by M4. $(y * z) * (x * (y * z)) \approx 0$ and an application of M3. and (A) gives (B). Finally note that

$$\text{if } x * y \approx 0, \text{ then } (y * z) * (x * z) \approx 0. \tag{C}$$

which is simply a consequence of (B) and M2.

We shall prove that M has EDPI by showing that M satisfies (1), (2) and (3) of Theorem 5.3.4. (1) follows directly from M1 and M*2. Let $\mathbf{A} \in \mathsf{M}$ and let $\approx_{\mathbf{A}}$ be the relation on A defined by $a \approx_{\mathbf{A}} b$ iff $a * b = b * a = 0$. The relation is reflexive, symmetric by definition, and transitive by (A) and (B). Moreover note that, by M2, $0/\approx_{\mathbf{A}} = \{0\}$. Suppose that $a \approx_{\mathbf{A}} b$ and $a' \approx_{\mathbf{A}} b'$. Since $a' * b' = 0$ by (B) and M2

$$(a * a') * (a * b') = 0.$$

Similarly, since $b * a = 0$, by (C) and M2

$$(a * b') * (b * b') = 0,$$

and (A) implies

$$(a * a') * (b * b') = 0.$$

The proof that $(b * b') * (a * a') = 0$ is similar and thus we conclude that $\approx_{\mathbf{A}}$ is a congruence of $\mathbf{A}$.

Next $\langle A, *, 0 \rangle / \approx_{\mathbf{A}}$ is a dual Hilbert algebra just because of the axioms M1-M4, and the fact that, if $a/\approx_{\mathbf{A}} * b/\approx_{\mathbf{A}} = b/\approx_{\mathbf{A}} * a/\approx_{\mathbf{A}} = 0$, then $a/\approx_{\mathbf{A}} = b/\approx_{\mathbf{A}}$. Therefore (2) of Theorem 5.3.4 holds as well.

[1] Later it will be clear that a more proper name might have been *dual minimal natural implicative algebras*, but we are not masochistic enough.

Let now $b \in (a)_{\mathbf{A}}$; then there is an ideal term $t(x_1, \ldots, x_n, y)$ in y and $a_1, \ldots, a_n \in A$ with $b = t(a_1, \ldots, a_n, a)$. Since $\approx_{\mathbf{A}} \in \mathrm{Con}(\mathbf{A})$,

$$b/\approx_{\mathbf{A}} = t(a_1, \ldots, a_n, a)/\approx_{\mathbf{A}} = t(a_1/\approx_{\mathbf{A}}, \ldots, a_n/\approx_{\mathbf{A}}, a/\approx_{\mathbf{A}}) \in (a/\approx_{\mathbf{A}})$$

implying $b \in \bigcup(a/\approx_{\mathbf{A}})$. On the other hand if $b \in \bigcup(a/\approx_{\mathbf{A}})$, then there is a $c \in A$ with $c/\approx_{\mathbf{A}} \in (a/\approx_{\mathbf{A}})$ and $b/\approx_{\mathbf{A}} = c/\approx_{\mathbf{A}}$. This implies $a/\approx_{\mathbf{A}} * c/\approx_{\mathbf{A}} = 0/\approx_{\mathbf{A}}$, and so $a * c = 0$, and $c * b = 0$. An application of (A) yields $a * b = 0$. Since the term $(y * x) * x$ is clearly an ideal term in y

$$b = 0 * b = (a * b) * b \in (a)_{\mathbf{A}}.$$

Therefore $(a)_{\mathbf{A}} = \bigcup(a/\approx_{\mathbf{A}})_{\mathbf{A}/\approx_{\mathbf{A}}}$.

Thus the variety of MINI algebras has EDPI. It does not have equationally definable principal congruences though, since it is not congruence distributive. In fact consider the 6-element algebra $\mathbf{A} = \langle\{0, a, b, c, d, e\}, *, 0\rangle$ where

$$b * a = \begin{cases} a & \text{if } b = 0 \\ 0 & \text{otherwise.} \end{cases}$$

One checks that $\mathbf{A} \in \mathsf{M}$ and that a congruence of $\mathbf{A}$ is simply any partition to which $\{0\}$ belongs. With this in mind one easily sees that $\mathrm{Con}(\mathbf{A})$ is not even modular.

Theorems 5.3.4 and 5.3.5 have a very interesting corollary. If V is subtractive and has EDPI, then there must be a binary term $x * y$ witnessing *both* subtractivity and EDPI. Moreover if $\mathbf{A} \in \mathsf{V}$, then $\langle A, *, 0\rangle/\approx_{\mathbf{A}}$ is a dual Hilbert algebra. This and the obvious fact that $0/\approx_{\mathbf{A}} = \{0\}$ imply the following. If $t \approx 0$ is an equation satisfied by any dual Hilbert algebra, then such equation must also hold in V. For instance

$$(x * (y * z)) * ((x * y) * (x * z)) \approx 0$$

holds in V and this does not appear immediately derivable from subtractivity or EDPI. Moreover any ideal term $t(\vec{x}, y)$ in y is **compatible** with the MINI algebra structure, in the sense that for any $\mathbf{A} \in \mathsf{V}$ and any $a, \vec{b} \in A$

$$a * t(\vec{b}, a) = 0.$$

This observation allows us to propose MINI algebras as a paradigm for subtractive EDPI varieties.

Corollary 5.4.1. *For a pointed variety* V *the following are equivalent.*

1. V *is subtractive and has EDPI.*

2. V *has a binary term* $x * y$ *such that for any* $\mathbf{A} \in \mathsf{V}$*,* $\mathbf{A}^* = \langle A, *, 0 \rangle$ *is a MINI algebra and any ideal term of* V *is compatible in the above sense.*

3. V *has a binary term* $x * y$ *such that for any* $\mathbf{A} \in \mathsf{V}$*,* $\mathbf{A}^* = \langle A, *, 0 \rangle$ *is a MINI algebra and for any* $a \in A$

$$(a)_{\mathbf{A}} = (a)_{\mathbf{A}^*}.$$

Proof. (1) implies (2) follows from the observation above and Theorem 5.3.4.

If we assume (2), then the right-to-left inclusion in (3) is trivial, while the other follows directly from the fact that any ideal term is compatible. Finally (3) implies (1) is again obvious, since MINI algebras are subtractive and have EDPI. □

Remark 5.4.2. (1) If $\approx_{\mathbf{A}}$ happens to be a congruence on any algebra in V, then any algebra in V is term-equivalent to a variety of MINI algebras with operations, where the operations are compatible in the usual way (i.e. preserve any MINI algebra congruence).
(2) If $\approx_{\mathbf{A}} = 0_{\mathbf{A}}$ for any $\mathbf{A} \in \mathsf{V}$, then the variety is *congruence orderable* in the sense of [19]. In this case V is also ideal-determined and hence *Fregean* in the sense of [19]. It follows that V is termwise equivalent to a variety of dual Hilbert algebras with compatible operations [1].

5.5 Meet and join generator terms

A class K has an n**-system of principal ideal intersection terms** if there are binary terms $q_1, \ldots, q_n$ such that for any $\mathbf{A} \in \mathsf{K}$ and $a, b \in A$,

$$(a)_{\mathbf{A}} \cap (b)_{\mathbf{A}} = \bigvee_{i=1}^{n} (q_i(a, b))_{\mathbf{A}}.$$

Theorem 5.5.1. *[5] For a subtractive variety* V *the following are equivalent.*

1. V *has an* n*-system of principal ideal intersection terms.*

2. V *is ideal distributive and the compact ideals of any algebra in* V *are closed under intersections.*

Proof. Assume (1) and let $q_1, \ldots, q_n$ be an n-system of principal ideal intersection terms for V. Note that $q_i(x, y)$ is a commutator term in x, y by definition, so for any $\mathbf{A} \in \mathsf{V}$ and $a, b \in A$ $[a, b]_{\mathbf{A}} = (a)_{\mathbf{A}} \cap (b)_{\mathbf{A}}$. Therefore, by Theorem 4.4.4, V is ideal distributive. This fact and the principal ideal intersection terms yield

$$\bigvee_{j=1}^{m} (a_j)_{\mathbf{A}} \cap \bigvee_{l=1}^{k} (b_l)_{\mathbf{A}} = \bigvee_{j=1}^{m} \bigvee_{l=1}^{k} \bigvee_{i=1}^{n} (q_i(a_j, b_l))_{\mathbf{A}},$$

so (2) holds.

Assume now (2) and let $\mathbf{F}$ be the algebra in V freely generated by $x, y, v_1, v_2, \ldots$. Since the compact ideal are closed under intersections we have that

$$(x)_{\mathbf{F}} \cap (y)_{\mathbf{F}} = \bigvee_{i=1}^{n} (t_i(x, y, v_{i_1}, \ldots, v_{i_k}))_{\mathbf{F}}.$$

Define $q_i(x, y) = t_i(x, y, x, \ldots, x)$ for $i = 1, \ldots, n$. Suppose that $\mathbf{A} \in \mathsf{V}$ is finitely generated and let $a, b \in A$. Then there is a homomorphism f of $\mathbf{F}$ onto $\mathbf{A}$ such that $f(x) = f(v_{i_j}) = a$ and $f(y) = b$. Now

$c \in (a)_{\mathbf{A}} \cap (b)_{\mathbf{A}}$ if and only if $c \in (f(x))_{\mathbf{A}} \cap (f(y))_{\mathbf{A}}$
if and only if $c \in [f(x), f(y)]_{\mathbf{F}}$ if and only if $c \in f([x, y]_{\mathbf{F}})$
if and only if $c \in f((x)_{\mathbf{F}} \cap (y)_{\mathbf{F}})$ if and only if $c \in f(\bigvee_{i=1}^{n} (t_i(x, y, v_{i_1}, \ldots, v_{i_k}))_{\mathbf{F}})$
if and only if there is an ideal term t such that
$c = f(t(u_1, \ldots, u_n, t_1(x, y, v_{1_1}, \ldots, v_{1_k}), \ldots, t_n(x, y, v_{n_1}, \ldots, v_{n_k})))$
if and only if $c = t(f(u_1), \ldots, f(u_n), t_1(a, b, a, \ldots, a), \ldots, t_n(a, b, a, \ldots, a))$
if and only if $c = t(f(u_1), \ldots, f(u_n), q_1(a, b), \ldots, q_n(a, b))$
if and only if $c \in \bigvee_{i=1}^{n} (q_i(a, b))_{\mathbf{A}}$.

So the conclusion holds if $\mathbf{A}$ is finitely generated. However, if $c \in (a)_{\mathbf{A}} \cap (b)_{\mathbf{A}}$ then there is a finitely generated subalgebra $\mathbf{B}$ of $\mathbf{A}$ such that $c \in (a)_{\mathbf{B}} \cap (b)_{\mathbf{B}}$. Therefore the conclusion holds in general and $q_1, \ldots, q_n$ is an n-system of principal ideal intersection terms for V. $\square$

The case $n = 1$ in the definition of n-system of principal ideal intersection terms deserves a special name: the binary term witnessing that is called a **meet generator** for V and is denoted by $\sqcap$. Then, for any $\mathbf{A} \in \mathsf{V}$ and $a, b \in A$

$$(a)_{\mathbf{A}} \cap (b)_{\mathbf{A}} = (a \sqcap b)_{\mathbf{A}}.$$

Just looking at the proof of Theorem 5.5.1 one sees that a subtractive variety has a meet generator term if and only if it is ideal distributive and the meet of two principal ideals is principal.

If a subtractive EDPI variety V has a meet generator term $\sqcap$, then the principal ideals are closed under both intersection and dual relative pseudocomplementation. It follows that, for any $\mathbf{A} \in \mathsf{V}$, $\langle \mathrm{PI}(\mathbf{A}), *, \cap, (0)_{\mathbf{A}} \rangle$ is a $*, \cap$-subreduct of a dual Brouwerian semilattice. Moreover, via the meet generator term and distributivity of ideals, the compact ideals themselves are closed under intersection, hence $\langle \mathrm{CI}(\mathbf{A}), *, \vee, \cap, (0)_{\mathbf{A}} \rangle$ is a dual relatively pseudocomplemented lattice. It is of some interest to note that a partial converse holds as well, but we need some facts first.

Since the join of two compact ideals is always compact we investigate only joins of principal ideals. A **join generator** for a pointed variety V is a binary term $x \sqcup y$ such that for any $\mathbf{A} \in \mathsf{V}$ and $a, b \in A$

$$(a)_{\mathbf{A}} \vee (b)_{\mathbf{A}} = (a \sqcup b)_{\mathbf{A}}.$$

We do not need subtractivity to obtain a characterization in this case.

Proposition 5.5.2. *[5] Let V be a pointed variety; then the following are equivalent.*

1. *The join of two principal ideals is principal.*
2. *Every compact ideal is principal.*
3. *There are a binary term $\sqcup$ and two ternary terms r and t such that*

$$\begin{aligned}
&0 \sqcup 0 \approx 0 \\
&r(x, y, 0) \approx t(x, y, 0) \approx 0 \\
&r(x, y, x \sqcup y) \approx x \\
&t(x, y, x \sqcup y) \approx y.
\end{aligned}$$

4. *V has a join generator term.*

Proof. (1) implies (2) trivially. We can show that (2) implies (3) with a standard Mal'cev-type argument. Let $\mathbf{F}$ be the algebra freely generated in V by x and y. Since $(x)_{\mathbf{F}} \vee (y)_{\mathbf{F}}$ is compact, by hypothesis

$$(x)_{\mathbf{F}} \vee (y)_{\mathbf{F}} = (x \sqcup y)_{\mathbf{F}}.$$

Since $(0)_{\mathbf{F}} = \{0\}$, this implies $0 \sqcup 0 = 0$. Moreover since $x, y \in (x \sqcup y)_{\mathbf{F}}$ we get the terms r and t simply by ideal generation.

Assume now (3), let $\mathbf{A} \in \mathsf{V}$ and $a, b \in A$; from

$$a = r(a, b, a \sqcup b) \qquad b = t(a, b, a \sqcup b)$$

we obtain $a, b \in (a \sqcup b)_{\mathbf{A}}$. Conversely, from $0 \sqcup 0 = 0$ we have that $a \sqcup b \in (a)_{\mathbf{A}} \vee (b)_{\mathbf{A}}$, hence (4) holds. Finally (4) obviously implies (1) $\square$

If a subtractive variety has EDPI and a join generator term we can obtain a stronger characterization theorem.

Theorem 5.5.3. *[5] Let V a subtractive EDPI variety in which the join of two principal ideals is principal. Then there are binary terms $*$ and $\sqcup$ such that the following hold.*

1. *For all $\mathbf{A} \in \mathsf{V}$ and $a, b, c \in A$*

$$\begin{array}{ll} a * a = 0 & (c * a) * ((c * b) * (c * (a \sqcup b))) = 0 \\ a * 0 = 0 & (a \sqcup b) * b = (a \sqcup b) * a = 0 \\ 0 * a = a & \\ b \in (a)_{\mathbf{A}} \textit{ if and only if } a * b = 0 & \end{array}$$

2. *The relation $\leq$ defined by $a \leq b$ if and only if $b * a = 0$ is reflexive and transitive. The associated equivalence relation $\approx_{\mathbf{A}}$ is a congruence of $\mathbf{A}^{\sqcup} = \langle A, *, \sqcup, 0 \rangle$ and $\mathbf{A}^{\sqcup}/ \approx_{\mathbf{A}}$ is a dual Brouwerian semilattice isomorphic with $\langle \mathrm{PI}(\mathbf{A}), *, \vee, (0)_{\mathbf{A}} \rangle$.*

3. *Any principal ideal of $\mathbf{A}$ is the union of a principal ideal of $\mathbf{A}^{\sqcup}/ \approx_{\mathbf{A}}$ and viceversa. In fact $(a)_{\mathbf{A}} = \bigcup (a/ \approx_{\mathbf{A}})_{\mathbf{A}^{\sqcup}/\approx_{\mathbf{A}}}$.*

Proof. By hypothesis V has a binary term $*$ witnessing subtractivity and EDPI and a join generator term $\sqcup$. Then (1) follows from Theorem 5.3.4 and the fact that any compact ideal is principal. Therefore the principal ideals of $\mathbf{A}$ form a dual Brouwerian semilattice. Next (2) follows from the fact that the term witnessing EDPI in a dual Brouwerian semilattice is the same as the one for its dual Hilbert algebra reduct. Finally (3) holds since ideals of a dual Brouwerian semilattice coincide with ideals of its dual Hilbert algebra reduct. $\square$

The previous theorem has a converse, whose proof is similar to the one of Theorem 5.3.5.

By a well known result of Blok and Pigozzi ([7], Theorem 3.5) any congruence permutable variety with equationally definable principal congruences satisfies the hypotheses of Theorem 5.5.3 . In the same paper (Section 4) they also gave a method for constructing varieties of this kind. Later we shall see a non congruence permutable variety satisfying the hypotheses of Theorem 5.5.3.

The paradigm of a subtractive variety with EDPI and a join generator term is the variety B of algebras $\mathbf{A} = \langle A, *, \sqcup, 0\rangle$ such that

1. $\langle A, *, 0\rangle$ is a MINI algebra ;
2. $\sqcup$ is a binary operation satisfying

$$\begin{aligned} &0 \sqcup 0 = 0 \\ &(a \sqcup b) * b = (a \sqcup b) * a = 0. \end{aligned}$$

An ideal term of B is **compatible** if it is compatible with the underlying MINI algebra structure. We have the following corollary, whose proof is left to the reader.

Corollary 5.5.4. *For a pointed variety V the following are equivalent.*

1. *V is subtractive, has EDPI and a join generator term.*
2. *V has binary terms $*$ and $\sqcup$ such that for any $\mathbf{A} \in \mathsf{V}$, $\mathbf{A}^{\sqcup} = \langle A, *, \sqcup, 0\rangle \in \mathsf{B}$ and any ideal term of V is compatible in the above sense.*

3. V *has binary terms* $*$ *and* $\sqcup$ *such that for any* $\mathbf{A} \in \mathsf{V}$, $\mathbf{A}^{\sqcup} = \langle A, *, \sqcup, 0\rangle \in \mathsf{B}$ *and for any* $a \in A$
$$(a)_{\mathbf{A}} = (a)_{\mathbf{A}^{\sqcup}}.$$

There is also another connection with [7]. A *weak Brouwerian semilattice with filter preserving operations* (WBSO) is a pointed algebra $\mathbf{A}$ such that the principal congruences form a dual Brouwerian semilattice and moreover there are three binary terms $*$, $\sqcup$ and $d(x, y)$ such that for any $a, b \in A$

$$\begin{aligned}
&\vartheta_{\mathbf{A}}(0, a * b) = \vartheta_{\mathbf{A}}(0, a) * \vartheta_{\mathbf{A}}(0, b)\\
&\vartheta_{\mathbf{A}}(0, a \sqcup b) = \vartheta_{\mathbf{A}}(0, a) \vee \vartheta_{\mathbf{A}}(0, b)\\
&\vartheta_{\mathbf{A}}(a, b) = \vartheta_{\mathbf{A}}(0, d(a, b)).
\end{aligned}$$

WBSO varieties have been investigated at length in [7]. It is clear that any WBSO variety has equationally definable principal congruences. A WBSO$^{\#}$ variety is a subtractive WBSO variety. It is shown in [1] that in this case the term $*$ can be chosen to witness subtractivity as well and therefore also EDPI. It follows then that the term $\sqcup$ is a join generator term for the WBSO$^{\#}$ and hence any WBSO$^{\#}$ variety satisfies the hypotheses of Theorem 5.5.3. In fact WBSO$^{\#}$ varieties turn out to be the ideal determined varieties satisfying Corollary 5.5.4 above (see also [1]).

Theorem 5.5.5. *[5] For a pointed variety* V *the following are equivalent.*

1. V *is ideal determined, has EDPI and a join generator term.*

2. V *is WBSO*$^{\#}$ *variety.*

Proof. Any WBSO$^{\#}$ variety is ideal determined, has EDPI and a join generator term by definition. Conversely assume that V satisfies 1. Then Corollary 5.5.4 and ideal-determinacy give the binary terms $*$ and $\sqcup$ satisfying the first two equations above. Thus we have only to prove the existence of the term $d(x, y)$.

Since V is ideal-determined, it is 0-regular with witness terms, say, $d_1, \dots, d_n$. Let $\mathbf{F}$ be the free algebra over two generators; then ideal determinacy implies that
$$\vartheta_{\mathbf{F}}(x, y) = \vartheta_{\mathbf{F}}(0, d_1(x, y)) \vee \ldots \vee \vartheta_{\mathbf{F}}(0, d_n(x, y))$$
and of course
$$0/\vartheta_{\mathbf{F}}(x, y) = \langle d_1(x, y), \dots, d_n(x, y)\rangle_{\mathbf{F}}.$$

Let $\sqcup$ be the join generator for V and let

$$d(x,y) = d_1(x,y) \sqcup \cdots \sqcup d_n(x,y)$$

(where we associate from left to right). One checks at once that

$$0/\vartheta_{\mathbf{F}}(x,y) = (d(x,y))_{\mathbf{F}} = 0/\vartheta_{\mathbf{F}}(0, d(x,y))$$

and 0-regularity gives again

$$\vartheta_{\mathbf{F}}(x,y) = \vartheta_{\mathbf{F}}(0, d(x,y)).$$

Now V is ideal determined and has EDPI, so it has equationally definable principal congruences. Hence V has the congruence extension property and therefore the condition

$$\vartheta_{\mathbf{A}}(a,b) = \vartheta_{\mathbf{A}}(0, d(a,b))$$

holds for any $\mathbf{A} \in \mathsf{V}$ and $a, b \in A$. □

Let's observe the following:

Proposition 5.5.6. *[5] Suppose V is a subtractive variety with a join generator term in which the compact ideals of any algebra in V form a dual relatively pseudocomplemented lattice. Then V has EDPI and also a meet generator term.*

Proof. First V has EDPI by Theorem 5.3.1. Next we observe that the equation

$$(x * (x * y)) \vee (y * (y * x)) \approx x \wedge y$$

holds in any dual relatively pseudocomplemented lattice (see for instance [26]). Let then $\sqcup$ be the join generator for V and define

$$x \sqcap y = (x * (x * y)) \sqcup (y * (y * x)).$$

Now let $\mathbf{A} \in \mathsf{V}$ and $a, b \in A$. Since V has EDPI and a join generator term then Theorem 5.5.3 applies, thus

$$(a \sqcap b)_{\mathbf{A}} = [(a)_{\mathbf{A}} * ((a)_{\mathbf{A}} * (b)_{\mathbf{A}})] \vee [(a)_{\mathbf{A}} * ((a)_{\mathbf{A}} * (b)_{\mathbf{A}})] = (a)_{\mathbf{A}} \cap (b)_{\mathbf{A}},$$

where we have used the fact that the compact ideals form a dual relatively pseudocomplemented lattice. Hence $x \sqcap y$ is a meet generator for V. □

If a subtractive variety with EDPI has both a join generator and a meet generator, then we can get a further refinement of our results. The reader will be able at this point to prove the following theorems.

Theorem 5.5.7. *[5] Let* V *a subtractive variety with EDPI in which the join and the meet of two principal ideals is principal. Then there are binary terms* $*$, $\sqcup$ *and* $\sqcap$ *such that the following hold.*

1. *For all* $\mathbf{A} \in \mathsf{V}$ *and* $a, b, c \in A$

$$\begin{array}{ll} a * a = 0 & (c * a) * ((c * b) * (c * (a \sqcup b))) = 0 \\ a * 0 = 0 & (a \sqcup b) * b = (a \sqcup b) * a = 0 \\ 0 * a = a & (a * c) * ((b * c) * ((a \sqcap b) * c)) = 0 \\ b \in (a)_{\mathbf{A}} \text{ iff } a * b = 0 & a * (a \sqcap b) = b * (a \sqcap b) = 0. \end{array}$$

2. *The relation* $\leq$ *defined by* $a \leq b$ *iff* $b * a = 0$ *is reflexive and transitive. The associated equivalence relation* $\approx_{\mathbf{A}}$ *is a congruence of* $\mathbf{A}^{\sqcap} = \langle A, *, \sqcup, \sqcap, 0 \rangle$ *and* $\mathbf{A}^{\sqcap}/\approx_{\mathbf{A}}$ *is a relatively pseudocomplemented lattice isomorphic with* $\langle \mathrm{PI}(\mathbf{A}), *, \vee, \cap, (0)_{\mathbf{A}} \rangle$.

3. *Any principal ideal of* $\mathbf{A}$ *is the union of a principal ideal of* $\mathbf{A}^{\sqcap}/\approx_{\mathbf{A}}$ *and viceversa. In fact* $(a)_{\mathbf{A}} = \bigcup (a/\approx_{\mathbf{A}})_{\mathbf{A}^{\sqcap}/\approx_{\mathbf{A}}}$.

Theorem 5.5.8. *[5] Let* V *be a variety with a constant* 0 *and such that the following hold.*

1. *there exist binary terms* $*$, $\sqcup$ *and* $\sqcap$ *such that for any* $\mathbf{A} \in \mathsf{V}$ *and* $a \in A$

$$\begin{array}{ll} a * a = 0 & (c * a) * ((c * b) * (c * (a \sqcup b))) = 0 \\ a * 0 = 0 & (a \sqcup b) * b = (a \sqcup b) * a = 0. \\ 0 * a = 0 \text{ implies } a = 0 & (a * c) * ((b * c) * ((a \sqcap b) * c)) = 0 \\ & a * (a \sqcap b) = b * (a \sqcap b) = 0. \end{array}$$

2. *The relation* $\approx_{\mathbf{A}}$ *defined by* $a \approx b$ *iff* $a * b = b * a = 0$ *is a congruence of* $\mathbf{A}^{\sqcap} = \langle A, *, \sqcup, \sqcap, 0 \rangle$ *and* $\mathbf{A}^{\sqcap}/\approx_{\mathbf{A}}$ *has EDPI with witness term* $x * y$.

3. *For any* $a \in A$

$$(a)_{\mathbf{A}} = \bigcup (a/\approx_{\mathbf{A}})_{\mathbf{A}^{\sqcap}/\approx_{\mathbf{A}}}.$$

Then V *is a subtractive variety with EDPI, witness term* $x * y$*, having* $\sqcup$ *as a join generator term and* $\sqcap$ *as a meet generator term.*

According to [7] (Section 4) any pointed *congruence permutable relative Stone variety* (such as Boolean algebras) satisfies the hypotheses of Theorem 5.5.7. As we will see later there are subtractive non congruence permutable varieties satisfying the same hypotheses.

The paradigm of a subtractive variety with EDPI with a join generator and a meet generator term is the variety of L of algebras $\langle A, *, \sqcup, \sqcap, 0\rangle$ where

1. $\langle A, *, 0\rangle$ is a MINI algebra;

2. $\sqcup$ is binary and satisfies

$$\begin{aligned} &0 \sqcup 0 = 0 \\ &(a \sqcup b) * b = (a \sqcup b) * a = 0; \end{aligned}$$

3. $\sqcap$ is binary and satisfies

$$\begin{aligned} &(a * c) * ((b * c) * ((a \sqcap b) * c)) = 0 \\ &a * (a \sqcap b) = b * (a \sqcap b) = 0. \end{aligned}$$

The proof of the following corollary is left to the reader.

Corollary 5.5.9. *For a pointed variety* V *the following are equivalent.*

1. V *is subtractive, has EDPI, a join generator term and a meet generator term.*

2. V *has binary terms* $*$, $\sqcup$ *and* $\sqcap$ *such that for any* $\mathbf{A} \in \mathsf{V}$, $\mathbf{A}^{\sqcap} = \langle A, *, \sqcup, \sqcap, 0\rangle \in \mathsf{L}$ *and any ideal term of* V *is compatible with the underlying MINI algebra structure.*

3. V *has binary terms* $*$, $\sqcup$ *and* $\sqcap$ *such that for any* $\mathbf{A} \in \mathsf{V}$, $\mathbf{A}^{\sqcap} = \langle A, *, \sqcup, \sqcap, 0\rangle \in \mathsf{L}$ *and for any* $a \in A$

$$(a)_{\mathbf{A}} = (a)_{\mathbf{A}^{\sqcap}}.$$

5.6 Pseudocomplemented semilattices

This example appears in [5]. The variety PS of type $\langle \wedge, *, 0 \rangle$ is defined by the following identities

1. a set of identities defining meet semilattices;
2. $x \wedge (x \wedge y)^* = x \wedge y^*$;
3. $x \wedge 0^* = x$;
4. $0^{**} = 0$.

Note that by 3. $1 = 0^*$ is the top element in the semilattice ordering. It is easy to see that the variety PS is a subtractive variety with witness term $x \wedge y^*$. Moreover if $\mathbf{L} \in \mathsf{PS}$ and $a \in L$, then a^* is the *pseudocomplement* of a, i.e. for any $b \in L$

$$b \leq a^* \qquad \text{if and only if} \qquad a \wedge b = 0.$$

An algebra $\mathbf{L} \in \mathsf{PS}$ is called a **pseudocomplemented semilattice**. Pseudocomplemented semilattices are well known structures. For the properties below and for any other claim we will make we refer the reader to [16], Chapter I.6 and to the extensive bibliography therein. For any $\mathbf{L} \in \mathsf{PS}$ one can define a binary operation $a \oplus b$ by

$$a \oplus b = (a^* \wedge b^*)^*.$$

The *skeleton* of $\mathbf{L}$ is the set $\mathrm{S}(\mathbf{L}) = \{a^* : a \in L\}$. It is well known that $a \in \mathrm{S}(\mathbf{L})$ if and only if $a^{**} = a$ and that $\mathbf{S}(\mathbf{L}) = \langle \mathrm{S}(\mathbf{L}), \oplus, \wedge, {}^*, 0, 1 \rangle$ is a Boolean algebra. The following properties of pseudocomplemented semilattices are either obvious or have been proved in [4] 4.2. If $a, b, c \in L$, then

1. $a \leq a^{**}$;
2. $a^* = a^{***}$;
3. $a \leq b$ implies $b^* \leq a^*$;
4. $(a \wedge b)^{**} = a^{**} \wedge b^{**}$;
5. $(a \oplus b)^{**} = a \oplus b$;

6. $a^{**} \wedge (b \oplus c) = (a \wedge b) \oplus (a \wedge c)$.

Let $\approx_{\mathbf{A}}$ be the relation on $\mathbf{L}$ defined by

$$a \approx_{\mathbf{A}} b \qquad \text{if and only if} \qquad a \wedge b^* = a^* \wedge b = 0.$$

The relation is clearly symmetric an reflexive. Moreover if $a \approx_{\mathbf{A}} b$ and $b \approx_{\mathbf{A}} c$, then $a^* \wedge b = b \wedge c^* = 0$. Since $0^{**} = 0$ we have that $(a \oplus b)^* = (a^* \wedge b)^{**} = 0$ and similarly $(b \oplus c^*)^* = 0$. Hence

$$\begin{aligned} 0 &= ((a \oplus b^*) \oplus (b \oplus c)^*)^{**} \\ &= ((a \oplus b^*) \wedge (b \oplus c^*))^* \qquad \text{(by 2. and 3.)} \\ &= ((a \oplus b^*) \wedge b) \oplus ((a \oplus b^*) \wedge c^*)^* \qquad \text{(by 4. and 5.)} \\ &= (((a \oplus b^*) \wedge b) \oplus (a \wedge c^*) \oplus (b^* \wedge c^*))^*. \end{aligned}$$

Now

$$\begin{aligned} (a \oplus b^*) \wedge b \leq (a \oplus b^*) \wedge b^{**} &= (a \wedge b^{**}) \oplus (b^* \wedge b^{**}) \\ &= (a \wedge b^{**}) \oplus 0 = a^{**} \wedge b^{**} \leq a^{**} \leq a \oplus c^* \end{aligned}$$

and of course $a \wedge c^*$, $b^* \wedge c^* \leq a \oplus c^*$. Hence

$$((a \oplus b^*) \wedge b) \oplus (a \wedge c^*) \oplus (b^* \wedge c^*) \leq a \oplus c^*$$

and so

$$\begin{aligned} 0 &= (((a \oplus b^*) \wedge b) \oplus (a \wedge c^*) \oplus (b^* \wedge c^*))^* \\ &\geq (a \oplus c^*)^* \geq a^* \wedge c^{**} \geq a^* \wedge c. \end{aligned}$$

We conclude that $a^* \wedge c = 0$ and by a symmetrical argument $a \wedge c^* = 0$ as well. Hence $a \approx_{\mathbf{A}} c$ and $\approx_{\mathbf{A}}$ is transitive.

Suppose now that $a \approx_{\mathbf{A}} b$ and $a \approx_{\mathbf{A}} d$. Then

$$\begin{aligned} (a \wedge c^*) \wedge (b \wedge d^*)^* &\leq (a \wedge b^*)^{**} \wedge (b^* \oplus d) \\ &= (a \wedge c^* \wedge b^*) \oplus (a \wedge c^* \wedge d) = 0. \end{aligned}$$

By symmetry $(a \wedge c^*)^* \wedge (b \wedge d^*) = 0$ and so $\approx_{\mathbf{A}}$ is a congruence on $\langle L, x \wedge y^*, 0\rangle$. The proof that $\langle L, x \wedge y^*, 0\rangle / \approx_{\mathbf{A}}$ is a dual Hilbert algebra and that 3. of Theorem 5.3.5 holds is just routine calculation, using arguments similar to the ones above, so we conclude that PS is a subtractive variety with EDPI. Moreover for any $\mathbf{L} \in \mathsf{PS}$ and $a \in L$:

1. $(a)_{\mathbf{L}} = \{b : b \leq a^{**}\}$.

2. The meet of two principal ideals is principal

$$(a)_{\mathbf{L}} \wedge (b)_{\mathbf{L}} = (a^{**} \wedge b^{**}).$$

3. The join of two principal ideals is principal

$$(a)_{\mathbf{L}} \vee (b)_{\mathbf{L}} = (a \oplus b)_{\mathbf{L}}$$

   and thus any compact ideal is principal.

4. The Brouwerian semilattice $\mathrm{CI}(\mathbf{L})$ of compact (i.e. principal) ideals is a Boolean algebra isomorphic with $\mathbf{S}(\mathbf{L})$ via $(a)_{\mathbf{L}} \longmapsto a^{**}$.

Hence the variety of pseudocomplemented semilattices is a subtractive variety with EDPI with both a meet generator term $(x^{**} \wedge y^{**})$ and a join generator term $(x \oplus y)$. The variety PS it is not congruence permutable since it is not congruence modular. It is left as an exercise to show that for any $\mathbf{S} \in \mathsf{S}_0$ (cf. Example 5.3.3) there is an $\mathbf{L} \in \mathsf{PS}$ with $\mathrm{Con}(\mathbf{L}) \cong \mathrm{Con}(\mathbf{S})$. Likewise, PS cannot have equationally definable principal congruences, since it is not congruence distributive.

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