

Projectivity of modules and matrix-extensible radicals

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Abstract

Let R be an associative ring with unity and let I be a two-sided ideal of R contained in the Jacobson radical $\mathcal{J}(R)$ of R . For a right module M over the ring R we call the right module M/MI ($M/M\mathcal{J}(R)$) *the ideal (the radical) factor* of M .

We start with the notions of *an ideal supplement* and *an ideal projectivity* and as a main result we state their relation to two important conditions: the projectivity of a finitely generated flat module with the projective ideal factor and the finite generation of a (countably generated) projective module with the finitely generated ideal factor. Our observations slightly extend previous results presented in [3] and [4]. Related results were published in [5] and [6].

The latter property for radical factors is known as Lazard's conjecture and it holds true for rings with polynomial identities, with the ascending chain condition on one-sided annihilators and with one-sided Krull dimension. A counterexample for Lazard's conjecture in a form of a semilocal ring with two elements satisfying certain relations was provided in [1] and studied in details in [2]. On the other hand, it is easy to see that a projective module with a finitely generated ideal factor is finitely generated if the ideal is nilpotent and it was shown ([3]) that a finitely generated ideal-projective module is projective if the ideal is contained in the prime radical of the ring. By taking look at idempotent-lifting in matrix rings, we observe that these are true for any matrix-extensible radical ideal containing only nilpotent elements. This might lead to a question whether it holds true for the Koethe's nil radical, i.e. do the conditions give approximations of a positive solution of Koethe's problem?

Keywords

projective modules, matrix-extensible radicals, nilpotent, lifting idempotents, finitely generated

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