Mocninné řady II.

Sečtěte následující řady.

1. $\sum_{n=1}^{\infty} \frac{x^n}{n}$,
2. $\sum_{n=1}^{\infty} \frac{x^{2n-1}}{2n-1}$,
3. $\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$,
4. $\sum_{n=1}^{\infty} nx^n$,
5. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-1}}{2n-1}$,
6. $\sum_{n=1}^{\infty} \frac{x^n}{n(n+1)}$,
7. $1 + \sum_{n=1}^{\infty} \frac{\prod_{i=1}^{n}(2i-1)}{\prod_{i=1}^{n}(2i)} x^n$, zderivujte a přenásobte výrazem $(1-x)$,
8. $\sum_{n=1}^{\infty} (-1)^{n+1} n^2 x^n$,
9. $\sum_{n=1}^{\infty} n(n+1)x^n$,
10. $\sum_{n=2}^{\infty} \frac{n}{n-1} x^n$.

Řešení:

1.

$$S(x) = \sum_{n=1}^{\infty} \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + ...$$

$$S'(x) = 1 + x + x^2 + ... = \frac{1}{1-x}, \quad |x| < 1,$$

$$S(x) = \int \frac{1}{1-x} dx = -\ln(1-x) + C,$$

$$S(0) = \sum_{n=1}^{\infty} \frac{0^n}{n} = 0 = \ln(1) + C \Rightarrow C = 0 \Rightarrow S(x) = -\ln(1-x).$$

2.

$$S(x) = \sum_{n=1}^{\infty} \frac{x^{2n-1}}{2n-1} = x + \frac{x^3}{3} + \frac{x^5}{5} + ...$$
$$S'(x) = 1 + x^2 + x^4 + ... = \frac{1}{1-x^2}, \quad |x| < 1,$$
$$S(x) = \int \frac{1}{1-x^2} dx = \frac{1}{2} \int \left(\frac{1}{1-x} + \frac{1}{1+x} \right) dx = \frac{1}{2}(\ln(1+x) - \ln(1-x)) + C = \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right) + C,$$
$$S(0) = \sum_{n=1}^{\infty} \frac{0^{2n-1}}{2n-1} = 0 = \frac{1}{2} \ln(1) + C \Rightarrow C = 0 \Rightarrow S(x) = \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right).$$

3.

$$S(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + ...$$
$$S'(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + ...$$
$$S''(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + ... = S(x),$$
$$S''(x) - S(x) = 0$$
$$\lambda^2 - 1 = 0 \Rightarrow \lambda_1 = 1, \lambda_2 = -1 \Rightarrow S(x) = C_1 e^x + C_2 e^{-x}, \Rightarrow S'(x) = C_1 e^x - C_2 e^{-x},$$
$$S(0) = 1 = C_1 + C_2, \ S'(0) = 0 = C_1 - C_2 \Rightarrow C_1 = C_2 = \frac{1}{2}, \Rightarrow$$
$$S(x) = \frac{1}{2}(e^x + e^{-x}).$$

4.

$$S(x) = \sum_{n=1}^{\infty} n x^n = x + 2x^2 + 3x^3 + ...$$
$$\frac{S(x)}{x} = 1 + 2x + 3x^2 + 4x^3 + ...$$
$$\int \frac{S(x)}{x} dx = x + x^2 + x^3 + ... = \frac{x}{1-x}$$
$$\frac{S(x)}{x} = \left(\frac{x}{1-x} \right)' = \frac{1}{(1-x)^2}$$
$$S(x) = \frac{x}{(1-x)^2}.$$

5.

$$S(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-1}}{2n-1} = x - \frac{x^3}{3} + \frac{x^5}{5} - ...$$

$$S'(x) = 1 - x^2 + x^4 - ... = \frac{1}{1+x^x}$$

$S(x)=$ arctg x.

6.

$$S(x) = \sum_{n=1}^{\infty} \frac{x^n}{n(n+1)} = \frac{x}{2} + \frac{x^2}{2 \cdot 3} + \frac{x^3}{3 \cdot 4} + ...$$

$$xS(x) = \frac{x^2}{2} + \frac{x^3}{2 \cdot 3} + \frac{x^4}{3 \cdot 4} + ...$$

$$(xS(x))' = \frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3} + ...$$

$$(xS(x))'' = 1 + x + x^2 + x^3 + ... = \frac{1}{1-x}$$

$$(xS(x))' = \int \frac{1}{1-x} dx = -\ln(1-x) + C$$

$$\frac{0}{1} + \frac{0^2}{2} + \frac{0^3}{3} + ...= 0 = \ln(1-0) + C \Rightarrow C = 0$$

$$xS(x) = \int -\ln(1-x) dx \overset{p.p.}{=} -x\ln(1-x) - \int \frac{x}{1-x} dx$$

$$= -x\ln(1-x) + x + \ln(1-x) + C$$

$$\frac{0^2}{2} + \frac{0^3}{2 \cdot 3} + \frac{0^4}{3 \cdot 4} + ...= 0 = -0\ln(1-0) + 0 + \ln(1-0) + C \Rightarrow C = 0$$

$$S(x)= \frac{(1-x)\ln(1-x)}{x} + 1.$$

7.

$$S(x) = 1 + \sum_{n=1}^{\infty} \frac{\prod_{i=1}^{n}(2i-1)}{\prod_{i=1}^{n}(2i)} x^n = 1 + \frac{1}{2}x + \frac{1\cdot 3}{2\cdot 4}x^2 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3 + ...$$

$$(S(x))' = \frac{1}{2} + \frac{1\cdot 3}{2\cdot 4}2x + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}3x^2 + ...$$

$$(S(x))'(1-x) = \frac{1}{2}(1-x) + \frac{1\cdot 3}{2\cdot 4}2x(1-x) + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}3x^2(1-x) + ...$$

$$(S(x))'(1-x) = \frac{1}{2} + \left(\frac{1\cdot 3}{2\cdot 4}\cdot 2 - \frac{1}{2}\right)x + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot 3 - \frac{1\cdot 3}{2\cdot 4}\cdot 2\right)x^2 ...$$

$$= \frac{1}{2} + \frac{1}{2}\left(\frac{3}{4}\cdot 2 - 1\right)x + \frac{1\cdot 3}{2\cdot 4}\left(\frac{5}{6}\cdot 3 - 2\right)x^2 ...$$

$$\frac{1\cdot 3\cdot ...\cdot(2n-3)}{2\cdot 4\cdot ...\cdot(2n-2)}\left(\frac{2n-1}{2n}\cdot n - (n-1)\right) = \frac{1}{2}\cdot\frac{1\cdot 3\cdot ...\cdot(2n-3)}{2\cdot 4\cdot ...\cdot(2n-2)}$$

$$(S(x))'(1-x) = \frac{1}{2}\left(1 + \frac{1}{2}x + \frac{1\cdot 3}{2\cdot 4}x^2 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3 + ...\right) = \frac{1}{2}S(x)$$

$$\frac{(S(x))'}{S(x)} = \frac{1}{2(1-x)}$$

$$\ln(S(x)) = -\frac{1}{2}\ln(1-x) + C$$

$$S(x) = \frac{C}{\sqrt{1-x}}$$

$$S(0) = 1 + \frac{1}{2}0 + \frac{1\cdot 3}{2\cdot 4}0^2 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}0^3 + ... = 1 = \frac{C}{\sqrt{1-0}} \Rightarrow C = 1$$

$$S(x) = \frac{1}{\sqrt{1-x}}.$$

8.

$$S(x) = \sum_{n=1}^{\infty} (-1)^{n+1} n^2 x^n = x - 4x^2 + 9x^3 - 16x^4 + ...$$

$$\frac{S(x)}{x} = 1 - 4x + 9x^2 - 16x^3 + ...$$

$$\int \frac{S(x)}{x} dx = x - 2x^2 + 3x^3 - 4x^4 + ...$$

$$\frac{\int \frac{S(x)}{x} dx}{x} = 1 - 2x + 3x^2 - 4x^3 + ...$$

$$\int \frac{\int \frac{S(x)}{x} dx}{x} dx = x - x^2 + x^3 - x^4 + ... = \frac{x}{1+x}$$

$$\frac{\int \frac{S(x)}{x} dx}{x} = \frac{1}{(1+x)^2}$$

$$\int \frac{S(x)}{x} dx = \frac{x}{(1+x)^2}$$

$$\frac{S(x)}{x} = \frac{(1+x)^2 - 2(1+x)x}{(1+x)^4} = \frac{1-x}{(1+x)^3}$$

$$S(x) = \frac{x - x^2}{(1+x)^3}.$$

9.

$$S(x) = \sum_{n=1}^{\infty} n(n+1)x^n = 2x + 2 \cdot 3x^2 + 3 \cdot 4x^3 + ..$$

$$\frac{S(x)}{x} = 2 + 2 \cdot 3x + 3 \cdot 4x^2 + ...$$

$$\int \frac{S(x)}{x} dx = 2x + 3x^2 + 4x^3 + ...$$

$$\int \int \frac{S(x)}{x} dx dx = x^2 + x^3 + x^4 + ... = \frac{x^2}{1-x}$$

$$\int \frac{S(x)}{x} dx = \left(\frac{x^2}{1-x} \right)' = \frac{2x(1-x) + x^2}{(1-x)^2} = \frac{2x - x^2}{(1-x)^2}$$

$$\frac{S(x)}{x} = \left(\frac{2x - x^2}{(1-x)^2} \right)' = \frac{2(1-x)^3 + 2(1-x)(2x - x^2)}{(1-x)^4} = \frac{2(1 - 2x + x^2) + 4x - 2x^2}{(1-x)^3}$$

$$S(x) = \frac{2x}{(1-x)^3}.$$

10.

$$S(x) = \sum_{n=2}^{\infty} \frac{n}{n-1} x^n = 2x^2 + \frac{3}{2}x^3 + \frac{4}{3}x^4 + ...$$

$$\frac{S(x)}{x} = 2x + \frac{3}{2}x^2 + \frac{4}{3}x^3 + ...$$

$$\int \frac{S(x)}{x} dx = x^2 + \frac{1}{2}x^3 + \frac{1}{3}x^4 + ...$$

$$\frac{\int \frac{S(x)}{x} dx}{x} = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + ...$$

$$\left(\frac{\int \frac{S(x)}{x} dx}{x} \right)' = 1 + x + x^2 + ... = \frac{1}{1-x}$$

$$\frac{\int \frac{S(x)}{x} dx}{x} = \int \frac{1}{1-x} dx = -\ln(1-x)$$

$$\int \frac{S(x)}{x} dx = -x \ln(1-x)$$

$$\frac{S(x)}{x} = -\ln(1-x) + \frac{x}{1-x}$$

$$S(x) = -x \ln(1-x) + \frac{x^2}{1-x}.$$