Důležité limity:

- $\lim_{x\to 0} \frac{\sin x}{x} = 1$
- $\lim_{x\to 0} \frac{e^x-1}{x} = 1.$

Zadání

1. Vypočtěte limity:
   a) $\lim_{x\to 1} \frac{x^2-1}{2x^2-x-1}$
   b) $\lim_{x\to\infty} \left(\frac{x^3}{x^2+1} - x\right)$
   c) $\lim_{x\to 8} \frac{\sqrt{9+2x}-5}{\sqrt[3]{x}-2}$
   d) $\lim_{x\to 2} \left(\frac{1}{x^2-2x} - \frac{x}{x^2-4}\right)$
   e) $\lim_{x\to 1} \frac{x^{100}-2x+1}{x^{50}-2x+1}$
   f) $\lim_{x\to 1} \frac{x+x^2+x^3+\ldots+x^n-n}{x-1}$.
2. Vypočtěte limity:
   a) $\lim_{x\to 0} \frac{(1+x)(1+2x)\ldots(1+nx)-1}{x}$
   b) $\lim_{x\to 0} \frac{(1+mx)^n-(1+nx)^m}{x^2}$
   c) $\lim_{x\to 2} \frac{(x^2-x-2)^{20}}{(x^3-12x+16)^{10}}$.
3. Vypočtěte limity:
   a) $\lim_{x\to\infty} x^{\frac{4}{3}} \left(\sqrt[3]{x^2+1} - \sqrt[3]{x^2-1}\right)$
   b) $\lim_{x\to 16} \frac{\sqrt[4]{x}-2}{\sqrt{x}-4}$
   c) $\lim_{x\to\infty} \sqrt{x^3} \left(\sqrt{x+1} + \sqrt{x-1} - 2\sqrt{x}\right)$
   d) $\lim_{x\to a_+} \frac{\sqrt{x}-\sqrt{a}+\sqrt{x-a}}{\sqrt{x^2-a^2}}$.

Řešení

1. a) $\lim_{x\to 1} \frac{x^2-1}{2x^2-x-1} = \lim_{x\to 1} \frac{(x-1)(x+1)}{(x-1)(2x+1)} = \lim_{x\to 1} \frac{x+1}{2x+1} = \frac{2}{3}$.

   b) $\lim_{x\to\infty} \left(\frac{x^3}{x^2+1} - x\right) = \lim_{x\to\infty} \frac{x^3-x(x^2+1)}{x^2+1} = \lim_{x\to\infty} \frac{-1}{x+\frac{1}{x}} = 0$.

   c)

$$\lim_{x\to 8} \frac{\sqrt{9+2x}-5}{\sqrt[3]{x}-2} = \lim_{x\to 8} \frac{\sqrt{9+2x}-5}{\sqrt[3]{x}-2} \cdot \frac{\sqrt{9+2x}+5}{\sqrt{9+2x}+5} \cdot \frac{\sqrt[3]{x}^2+2\sqrt[3]{x}+4}{\sqrt[3]{x}^2+2\sqrt[3]{x}+4} = \lim_{x\to 8} \frac{9+2x-25}{x-8} \cdot \frac{\sqrt[3]{x}^2+2\sqrt[3]{x}+4}{\sqrt{9+2x}+5}$$
$$= \lim_{x\to 8} 2 \cdot \frac{\sqrt[3]{x}^2+2\sqrt[3]{x}+4}{\sqrt{9+2x}+5} = \frac{12}{5}.$$

   d) $\lim_{x\to 2} \left(\frac{1}{x^2-2x} - \frac{x}{x^2-4}\right) = \lim_{x\to 2} \frac{(x+2)-x^2}{x(x-2)(x+2)} = \lim_{x\to 2} \frac{-(x-2)(x+1)}{x(x-2)(x+2)} = \lim_{x\to 2} \frac{-(x+1)}{x(x+2)} = \frac{-3}{8}$.

   e) $\lim_{x\to 1} \frac{x^{100}-2x+1}{x^{50}-2x+1} = \lim_{x\to 1} \frac{(x-1)(x^{99}+x^{98}+x^{97}+\ldots+x-1}{x^{49}+x^{48}+\ldots+x-1)} = \frac{98}{48} = \frac{49}{24}$.

   f)

$$\lim_{x\to 1} \frac{x+x^2+x^3+\ldots+x^n-n}{x-1} = \lim_{x\to 1} \frac{x-1+x^2-1+x^3-1+\ldots+x^n-1}{x-1}$$
$$= \lim_{x\to 1} \frac{(x-1)+(x-1)(x+1)+\ldots+(x-1)(x^{n-1}+x^{n-2}+\ldots+1)}{x-1}$$
$$= \lim_{x\to 1} \big(1+(x+1)+(x^2+x+1)+\ldots+(x^{n-1}+x^{n-2}+\ldots+1)\big)$$
$$= \lim_{x\to 1} \sum_{i=0}^{n-1}(n-i)x^i = \sum_{i=0}^{n-1}(n-i) = \frac{n(n+1)}{2}.$$

2. a) $\lim_{x\to 0} \frac{(1+x)(1+2x)\ldots(1+nx)-1}{x} = \lim_{x\to 0} \frac{1+(x+2x+\ldots+nx)+o(x)-1}{x} = 1+2+\ldots+n = \frac{n(n+1)}{2}$.

   b)

$$\lim_{x\to 0} \frac{(1+mx)^n-(1+nx)^m}{x^2} = \lim_{x\to 0} \frac{1+\binom{n}{1}mx+\binom{n}{2}m^2x^2+\ldots+m^nx^n-\big(1+\binom{m}{1}nx+\binom{m}{2}n^2x^2+\ldots+n^mx^n\big)}{x^2}$$
$$= \lim_{x\to 0} \frac{\binom{n}{2}m^2x^2-\binom{m}{2}n^2x^2+o(x^2)}{x^2} = \binom{n}{2}m^2-\binom{m}{2}n^2 = \frac{mn}{2}(n-m).$$

   c)

$$\lim_{x\to 2} \frac{(x^2-x-2)^{20}}{(x^3-12x+16)^{10}} = \lim_{x\to 2} \frac{((x-2)(x+1))^{20}}{((x-2)(x^2+2x-8))^{10}} = \lim_{x\to 2} \frac{((x-2)(x+1))^{20}}{((x-2)(x-2)(x+4))^{10}} = \lim_{x\to 2} \frac{(x+1)^{20}}{(x+4)^{10}}$$
$$= \frac{3^{20}}{6^{10}} = \frac{3^{10}}{2^{10}}.$$

3. a)

$$\lim_{x\to\infty} x^{\frac{4}{3}}\left(\sqrt[3]{x^2+1}-\sqrt[3]{x^2-1}\right) = \lim_{x\to\infty} x^{\frac{4}{3}}\left(\sqrt[3]{x^2+1}-\sqrt[3]{x^2-1}\right) \cdot \frac{\sqrt[3]{x^2+1}^2+\sqrt[3]{x^2+1}\cdot\sqrt[3]{x^2-1}+\sqrt[3]{x^2-1}^2}{\sqrt[3]{x^2+1}^2+\sqrt[3]{x^2+1}\cdot\sqrt[3]{x^2-1}+\sqrt[3]{x^2-1}^2}$$
$$= \lim_{x\to\infty} \frac{x^{\frac{4}{3}}}{x^{\frac{4}{3}}} \cdot \frac{\big(x^2+1-(x^2-1)\big)}{\sqrt[3]{1+\frac{1}{x^2}}^2+\sqrt[3]{1+\frac{1}{x^2}}\cdot\sqrt[3]{1-\frac{1}{x^2}}+\sqrt[3]{1-\frac{1}{x^2}}^2}$$
$$= \lim_{x\to\infty} \frac{2}{\sqrt[3]{1+\frac{1}{x^2}}^2+\sqrt[3]{1+\frac{1}{x^2}}\cdot\sqrt[3]{1-\frac{1}{x^2}}+\sqrt[3]{1-\frac{1}{x^2}}^2} = \frac{2}{3}.$$

b)

$$\begin{aligned}\lim_{x\to16}\frac{\sqrt[4]{x}-2}{\sqrt{x}-4} &= \lim_{x\to16}\frac{\sqrt[4]{x}-2}{\sqrt{x}-4}\cdot\frac{\sqrt[4]{x}^3+2\sqrt[4]{x}^2+2^2\sqrt[4]{x}+2^3}{\sqrt[4]{x}^3+2\sqrt[4]{x}^2+2^2\sqrt[4]{x}+2^3}\cdot\frac{\sqrt{x}+4}{\sqrt{x}+4}\\ &= \lim_{x\to16}\frac{x-16}{x-16}\cdot\frac{\sqrt{x}+4}{\sqrt[4]{x}^3+2\sqrt[4]{x}^2+2^2\sqrt[4]{x}+2^3} = \frac{4+4}{8+8+8+8} = \frac{1}{4}.\end{aligned}$$

c)

$$\begin{aligned}\lim_{x\to\infty}\sqrt{x^3}\left(\sqrt{x+1}+\sqrt{x-1}-2\sqrt{x}\right) &= \lim_{x\to\infty}\sqrt{x^3}\left(\sqrt{x+1}+\sqrt{x-1}-2\sqrt{x}\right)\cdot\frac{\sqrt{x+1}+\sqrt{x-1}+2\sqrt{x}}{\sqrt{x+1}+\sqrt{x-1}+2\sqrt{x}}\\ &= \lim_{x\to\infty}\sqrt{x^3}\cdot\frac{(\sqrt{x+1}+\sqrt{x-1})^2-4x}{\sqrt{x+1}+\sqrt{x-1}+2\sqrt{x}}\\ &= \lim_{x\to\infty}\sqrt{x^3}\cdot\frac{x+1+2\sqrt{x+1}\sqrt{x-1}+x-1-4x}{\sqrt{x+1}+\sqrt{x-1}+2\sqrt{x}}\\ &= \lim_{x\to\infty}\sqrt{x^3}\cdot\frac{2\sqrt{x+1}\sqrt{x-1}-2x}{\sqrt{x+1}+\sqrt{x-1}+2\sqrt{x}}\cdot\frac{2\sqrt{x+1}\sqrt{x-1}+2x}{2\sqrt{x+1}\sqrt{x-1}+2x}\\ &= \lim_{x\to\infty}\sqrt{x^3}\cdot\frac{4(x+1)(x-1)-4x^2}{\sqrt{x+1}+\sqrt{x-1}+2\sqrt{x}}\cdot\frac{1}{2\sqrt{x+1}\sqrt{x-1}+2x}\\ &= \lim_{x\to\infty}\frac{-4}{\sqrt{1+\frac{1}{x}}+\sqrt{1-\frac{1}{x}}+2}\cdot\frac{1}{2\sqrt{1+\frac{1}{x}}\sqrt{1-\frac{1}{x}}+2} = -\frac{1}{4}.\end{aligned}$$

d)

$$\begin{aligned}\lim_{x\to a_+}\frac{\sqrt{x}-\sqrt{a}+\sqrt{x-a}}{\sqrt{x^2-a^2}} &= \lim_{x\to a_+}\frac{\sqrt{x}-\sqrt{a}+\sqrt{x-a}}{\sqrt{x^2-a^2}}\cdot\frac{\sqrt{x}+\sqrt{a}+\sqrt{x-a}}{\sqrt{x}+\sqrt{a}+\sqrt{x-a}} = \lim_{x\to a_+}\frac{(\sqrt{x}+\sqrt{x-a})^2-a}{\sqrt{x^2-a^2}(\sqrt{x}+\sqrt{a}+\sqrt{x-a})}\\ &= \lim_{x\to a_+}\frac{(x+2\sqrt{x}\sqrt{x-a}+x-a)-a}{\sqrt{x^2-a^2}(\sqrt{x}+\sqrt{a}+\sqrt{x-a})} = \lim_{x\to a_+}\frac{2((x-a)+\sqrt{x}\sqrt{x-a})}{\sqrt{x^2-a^2}(\sqrt{x}+\sqrt{a}+\sqrt{x-a})}\\ &= \lim_{x\to a_+}\frac{2(\sqrt{x-a}\sqrt{x-a}+\sqrt{x}\sqrt{x-a})}{\sqrt{(x-a)(x+a)}(\sqrt{x}+\sqrt{a}+\sqrt{x-a})}\\ &= \lim_{x\to a_+}\frac{2(\sqrt{x-a}+\sqrt{x})}{\sqrt{(x+a)}(\sqrt{x}+\sqrt{a}+\sqrt{x-a})} = \frac{2\sqrt{a}}{\sqrt{2a}2\sqrt{a}} = \frac{1}{\sqrt{2a}}.\end{aligned}$$