1.a) $\lim\limits_{x\to\frac{\pi}{2}} (\sin x)^{\frac{1}{\cos^2 x}} = \lim\limits_{x\to\frac{\pi}{2}} e^{\frac{1}{\cos^2 x}\cdot\ln(\sin x)} \overset{(*)}{=} e^{-\frac{1}{2}}$

$(*)$ $\lim\limits_{x\to\frac{\pi}{2}} \frac{1}{\cos^2 x}\cdot\ln(\sin x) = \lim\limits_{x\to\frac{\pi}{2}} \underbrace{\frac{\ln(\sin x)}{\sin x - 1}}_{\to 1}\cdot\frac{\sin x - 1}{\cos^2 x} = \lim\limits_{x\to\frac{\pi}{2}} \frac{\sin(x)-1}{\cos^2 x} \overset{y=x+\frac{\pi}{2}}{=} \lim\limits_{y\to 0}\frac{\sin(y+\frac{\pi}{2})-1}{\cos^2(y+\frac{\pi}{2})} = \lim\limits_{y\to 0}\frac{\cos y - 1}{(-\sin(y))^2} = \lim\limits_{y\to 0} -\underbrace{\frac{(1-\cos y)}{y^2}}_{\to\frac{1}{2}}\underbrace{\frac{y^2}{\sin^2 y}}_{\to 1} = -\frac{1}{2}$

b) $\lim\limits_{x\to 2}\frac{(\sqrt[3]{x^3+4x^2+3}-x-1)(4^{x^2-2}-16)}{(\cos(x+2)-\cos 4)(\log_5(x)-\log_5(2))} = \lim\limits_{x\to 2}\frac{(\sqrt[3]{x^3+4x^2+3}-x-1)(4^{x^2-2}-16)}{(\cos(x+2)-\cos 4)(\frac{\ln x}{\ln 5}-\frac{\ln 2}{\ln 5})}\cdot\frac{\sqrt[3]{x^3+4x^2+3}^2+\sqrt[3]{x^3+4x^2+3}\cdot(x+1)+(x+1)^2}{\sqrt[3]{x^3+4x^2+3}^2+\sqrt[3]{x^3+4x^2+3}\cdot(x+1)+(x+1)^2}$

$= \lim\limits_{x\to 2}\frac{(x^3+4x^2+3-(x+1)^3)}{\cos(x+2)-\cos 4}\cdot\frac{16\cdot(4^{x^2-4}-1)}{\frac{\ln(\frac{x}{2})}{\ln 5}}\cdot\frac{1}{\underbrace{\sqrt[3]{x^3+4x^2+3}^2}_{\to 9}+\underbrace{\sqrt[3]{x^3+4x^2+3}\cdot(x+1)}_{\to 9}+\underbrace{(x+1)^2}_{\to 9}} = \frac{16}{27}\lim\limits_{x\to 2}\frac{4x^2+3-3x^2-3x-1}{\cos(x-2+4)-\cos 4}\cdot\underbrace{\frac{e^{(x^2-4)\cdot\ln 4}-1}{(x^2-4)\cdot\ln 4}}_{\to 1}\cdot\frac{(x^2-4)\cdot\ln 4}{\underbrace{\frac{\ln(\frac{x}{2})}{\frac{x}{2}-1}}_{\to 1}\cdot\frac{\frac{x}{2}-1}{\ln 5}} = \frac{16\cdot\ln 5\cdot\ln 4}{27}\lim\limits_{x\to 2}\frac{(x^2-3x+2)(x^2-4)}{\cos(x-2)\cos 4-\sin(x-2)\sin 4-\cos 4}\cdot\frac{2}{x-2}$

$= \frac{32\cdot\ln 5\cdot\ln 4}{27}\lim\limits_{x\to 2}\frac{(x-2)\cdot(x-1)(x-2)(x+2)}{\cos 4\cdot\frac{\cos(x-2)-1}{(x-2)^2}\cdot(x-2)^2-\sin 4\cdot\frac{\sin(x-2)}{x-2}\cdot(x-2)}\cdot\frac{1}{x-2} = \frac{32}{27}\cdot\ln 5\cdot\ln 4\lim\limits_{x\to 2}\frac{\cancel{(x-2)^2}\cdot\underbrace{(x-1)(x+2)}_{\to 4}}{\cos 4\cdot\underbrace{\frac{\cos(x-2)-1}{(x-2)^2}}_{\to -\frac{1}{2}}\cdot\underbrace{(x-2)}_{\to 0}-\sin 4\cdot\underbrace{\frac{\sin(x-2)}{x-2}}_{\to 1}}\cdot\frac{1}{\cancel{(x-2)^2}} = -\frac{2^7\cdot\ln 5\cdot\ln 4}{27\cdot\sin 4}$

2. $\sum\limits_{n=1}^{\infty}\frac{\sin(n)}{(4+(-1)^n)^n}$

AK $\left|\frac{\sin(n)}{(4+(-1)^n)^n}\right| \leq \left(\frac{1}{3}\right)^n$. Jelikož řada $\sum\left(\frac{1}{3}\right)^n$ konverguje, tak konverguje řada $\sum\limits_{n=1}^{\infty}\frac{\sin(n)}{(4+(-1)^n)^n}$ absolutně.

3. $\sum(-1)^n\sqrt{\frac{n+1}{n^2}}$

AK: Srovnáme s divergentní řadou $\sum\frac{1}{\sqrt{n}}$. $\lim\limits_{n\to\infty}\frac{\sqrt{\frac{n+1}{n^2}}}{\frac{1}{\sqrt{n}}} = \lim\limits_{n\to\infty}\sqrt{\frac{n^2+n}{n^2}} = 1\in(0,\infty)$, tedy řada $\sum\sqrt{\frac{n+1}{n^2}}$ diverguje.

RK: Leibniz: I. $a_n = \sqrt{\frac{n+1}{n^2}} > 0$ ✓

II. $\sqrt{\frac{n+1}{n^2}} > \sqrt{\frac{n+2}{(n+1)^2}} \iff \frac{n+1}{n^2} > \frac{n+2}{(n+1)^2}$ (Jelikož je fce $f(x)=\sqrt{x}$ rostoucí). $\quad \frac{n+1}{n^2} \overset{?}{>} \frac{n+2}{n^2+2n+1}$

$(n+1)(n^2+2n+1) \overset{?}{>} n^2(n+2)$

III. $\lim\limits_{n\to\infty}\sqrt{\frac{n+1}{n^2}} = 0$ ✓ $\Rightarrow$ řada $\sum(-1)^n\sqrt{\frac{n+1}{n^2}}$ konverguje $\quad n^3+3n^2+3n+1 > n^3+2n^2$ ✓ $\Rightarrow a_n > a_{n+1}$

$\Downarrow$

Řada $\sum(-1)^n\sqrt{\frac{n+1}{n^2}}$ konverguje relativně

4.a) ANO. Plyne z věty o derivaci součinu (fce $\ell(x)=x$ má derivaci všude).

b) Je-li $0\in(a,b)$, pak ne ($g(x)$ není def. v nule). Pokud $0\notin(a,b)$, pak ANO (z věty o derivaci podílu).

c) Nemusí platit (fce $f(x)$ nemusí být definovaná v bodech a a b.

d) Platí (f má v bodě x_0 lok. maximum $\Rightarrow f'(x_0)=0$).

5. a) Ne. Např. $5=\sum a_n = 5+0+0+0+\dots$, $1=\sum\limits_{n=1}^{\infty}\left(\frac{1}{2}\right)^n$. Pak $0<\left(\frac{1}{2}\right)^n$ ale $5>1$.

b) Ne. Stejný protipříklad jako v a)

c) NE. Je-li $\sum a_n = \sum\limits_{n=2}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ a $\sum b_n = \frac{1}{2}\cdot\sum\limits_{n=2}^{\infty}\frac{(-1)^n}{\sqrt{n}}$, pak obě řady konv. (dle Leibnizova krit.), $\sum a_n > \sum b_n$, ale $\sum(a_n\cdot b_n) = \frac{1}{2}\sum\frac{1}{n}$ (div.)

d) ANO (Arit. limit).