1. a) $\lim\limits_{x\to 0}(\cos x)^{\operatorname{tg}(\frac{\pi}{2}-x^2)}\overset{(*)}{=}e^{-\frac{1}{2}}$

$$(*)\ \lim_{x\to 0}\operatorname{tg}\left(\tfrac{\pi}{2}-x^2\right)\cdot\ln(\cos x)=\lim_{x\to 0}\frac{\overbrace{\sin\left(\frac{\pi}{2}-x^2\right)}^{\to 1}}{\cos\left(\frac{\pi}{2}-x^2\right)}\cdot\overbrace{\frac{\ln(\cos x)}{\cos x-1}}^{\to 1}\cdot(\cos x-1)=\lim_{x\to 0}\frac{1\cdot(\cos x-1)}{\underbrace{\cos\left(\frac{\pi}{2}\right)}_{0}\cdot\cos(x^2)+\underbrace{\sin\left(\frac{\pi}{2}\right)}_{1}\cdot\sin(x^2)}=\lim_{x\to 0}-\overbrace{\frac{(1-\cos x)}{x^2}}^{\to\frac{1}{2}}\cdot\overbrace{\frac{x^2}{\sin(x^2)}}^{\to 1}=-\frac{1}{2}$$

1. b) $$\lim_{x\to a}\frac{\overbrace{\sin(x^2)-\sin(2a)}^{\to\sin(a^2)-\sin(2a)}}{\log_7(a+1)-\log_7(x-1)}\cdot\frac{b^{x^2-2xa+3}-b^{-a^2+3}}{\sqrt[3]{2a^3+2a^2x+2ax^2+2x^3}-2a}=(\sin a^2-\sin 2a)\cdot\lim_{x\to a}\frac{b^{-a^2+3}\cdot\left(b^{x^2-2ax+a^2}-1\right)}{\log_7\left(\frac{a+1}{x-1}\right)}\cdot\frac{1}{\sqrt[3]{2a^3+2a^2x+2ax^2+2x^3}-2a}\cdot\frac{\overbrace{\sqrt[3]{2a^3+2a^2x+2ax^2+2x^3}^{\,2}+\sqrt[3]{2a^3+2a^2x+2ax^2+2x^3}\cdot 2a+4a^2}^{\to 3\cdot 4a^2}}{\sqrt[3]{2a^3+2a^2x+2ax^2+2x^3}^{\,2}+\sqrt[3]{2a^3+2a^2x+2ax^2+2x^3}\cdot 2a+4a^2}$$

$$=(\sin a^2-\sin 2a)\cdot\lim_{x\to a}\frac{b^{-a^2+3}\cdot\left(e^{(x^2-2ax+a^2)\cdot\ln b}-1\right)}{\frac{\ln\left(\frac{a+1}{x-1}\right)}{\ln 7}}\cdot\frac{12a^2}{(2a^3+2a^2x+2ax^2+2x^3)-8a^3}=(\sin a^2-\sin 2a)\cdot b^{3-a^2}\cdot\ln 7\cdot 12a^2\cdot\lim_{x\to a}\overbrace{\frac{e^{(x^2-2ax+a^2)\cdot\ln b}-1}{(x^2-2ax+a^2)\cdot\ln b}}^{\to 1}\cdot\overbrace{\frac{\left(\frac{a+1}{x+1}\right)-1}{\ln\left(\frac{a+1}{x+1}\right)}}^{\to 1}\cdot\frac{1}{\left(\frac{a+1}{x+1}\right)-1}\cdot\frac{(x-a)^2\cdot\ln b}{2(-3a^3+a^2x+ax^2+x^3)}$$

$$(\sin a^2-\sin 2a)\cdot b^{3-a^2}\cdot\ln 7\cdot 12a^2\cdot\frac{\ln b}{2}\cdot\lim_{x\to a}\frac{\overbrace{x+1}^{\to a+1}}{(a-x)}\cdot\frac{(x-a)^{\cancel{2}}}{\cancel{(x-a)}\underbrace{(x^2+2ax+3a^2)}_{\to 6a^2}}=-(\sin a^2-\sin 2a)\cdot b^{3-a^2}\cdot\ln 7\cdot\cancel{12a^2}\cdot\frac{\ln b}{\cancel{2}}\cdot\frac{a+1}{\cancel{6a^2}}=\underline{-(a+1)\cdot(\sin a^2-\sin 2a)\cdot b^{3-a^2}\cdot\ln 7\cdot\ln b}$$

2. $$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{(2n+2)!}{(n+1)^{2n+2}}\cdot\frac{n^{2n}}{(2n)!}=\lim_{n\to\infty}\frac{(2n+2)(2n+1)}{(n+1)^2}\left(\frac{n}{n+1}\right)^{2n}=\lim_{n\to\infty}\frac{\cancel{n^2}}{\cancel{n^2}}\cdot\frac{(2+\overbrace{\tfrac{2}{n}}^{\to 0})(2+\overbrace{\tfrac{1}{n}}^{\to 0})}{(1+\underbrace{\tfrac{1}{n}}_{\to 0})^2}\cdot e^{2n\cdot\ln\left(\frac{n}{n+1}\right)}=\lim_{n\to\infty}4\cdot e^{2n\cdot\overbrace{\frac{\ln\left(\frac{n}{n+1}\right)}{\frac{n}{n+1}-1}}^{\to 1}\cdot\left(\frac{n-n-1}{n+1}\right)}=4\cdot e^{-2}=\left(\frac{2}{e}\right)^2<1\Rightarrow \text{řada}\sum_{n=1}^{\infty}\frac{(2n)!}{n^{2n}}\ \underline{\text{konverguje}}.$$

3. (AK): Řadu $\sum\sin(\sqrt{n+1}-\sqrt{n})$ srovnáme s divergentní řadou $\sum\frac{1}{\sqrt{n}}$. $$\lim_{n\to\infty}\frac{\sin(\sqrt{n+1}-\sqrt{n})}{\frac{1}{\sqrt{n}}}=\lim_{n\to\infty}\overbrace{\frac{\sin(\sqrt{n+1}-\sqrt{n})}{\sqrt{n+1}-\sqrt{n}}}^{\to 1}\cdot\frac{(\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})}{\sqrt{n+1}+\sqrt{n}}\cdot\frac{1}{\frac{1}{\sqrt{n}}}=\lim_{n\to\infty}\frac{(n+1-n)}{\cancel{\sqrt{n}}\underbrace{\left(\sqrt{1+\tfrac{1}{n}}+1\right)}_{\to 2}}\cdot\cancel{\sqrt{n}}=\frac{1}{2}\in(0,\infty).$$

Řada $\sum\sin(\sqrt{n+1}-\sqrt{n})$ tedy <u>diverguje</u>.

(RK)–Leibniz: I. $a_n=\sin(\sqrt{n+1}-\sqrt{n})>0$, j' $>\sqrt{n}$ a $\sqrt{n+1}<\sqrt{n}+1\Rightarrow(\sqrt{n+1}-\sqrt{n})\in(0,1]$. ✓

II. $a_n\overset{?}{>}a_{n+1}$: $\sin(x)$ je rostoucí fce na $(0,1]\Rightarrow\sin(\sqrt{n+1}-\sqrt{n})>\sin(\sqrt{n+2}-\sqrt{n+1})\Leftrightarrow(\sqrt{n+1}-\sqrt{n})>(\sqrt{n+2}-\sqrt{n+1})$. Označme $f(x)=\sqrt{x+1}-\sqrt{x}$, pak

$f'(x)=\frac{1}{2}\cdot\frac{1}{\sqrt{x+1}}-\frac{1}{2}\cdot\frac{1}{\sqrt{x}}=\frac{1}{2}\left(\frac{1}{\sqrt{x+1}}-\frac{1}{\sqrt{x}}\right)<0\Rightarrow f(x)$ je klesající fce $\Rightarrow(\sqrt{n+1}-\sqrt{n})=f(n)>f(n+1)=\sqrt{n+2}-\sqrt{n+1}\Rightarrow a_n>a_{n+1}$. ✓

III. NPK: $\lim\limits_{n\to\infty}\sin(\sqrt{n+1}-\sqrt{n})=\lim\limits_{n\to\infty}\sin\Big(\overbrace{\frac{n+1-n}{\sqrt{n+1}+\sqrt{n}}}^{\to 0}\Big)=0$. ✓

$\Rightarrow$ Řada $\sum(-1)^n\cdot\sin(\sqrt{n+1}-\sqrt{n})$ <u>konverguje</u>.

Závěr: Řada $\sum(-1)^n\sin(\sqrt{n+1}-\sqrt{n})$ konverguje <u>relativně</u>.

4. Jelikož funkce $f(x)=x\cdot\sin\left(\frac{1}{x}\right)$ není definovaná v bodě $x=0$, tak v tomto bodě není spojitá a nemá v tomto bodě derivaci.

5. a) Platí. Důsledek věty o limitě součtu (pro posloupnosti).

b) Neplatí. $\sum n$ a $\sum -n$ jsou div. řady, ale $\sum(n-n)=\sum 0$ je konvergentní řada.

c) Neplatí, viz příklad v bodě b)

d) Platí, přímý důsledek tvrzení a).