1.a) $\lim_{x\to 0}\left(\left(\frac{\cos x}{\cos 2x}\right)^{\frac{1}{x^2}}\right)=\lim_{x\to 0} e^{\frac{1}{x}\cdot\ln\left(\frac{\cos x}{\cos 2x}\right)}\overset{(*)}{=} e^{\frac{3}{2}}$

$(*)\ \lim_{x\to 0}\frac{1}{x^2}\cdot\ln\left(\frac{\cos x}{\cos 2x}\right)=\lim_{x\to 0}\frac{1}{x^2}\cdot\frac{\ln\left(\frac{\cos x}{\cos 2x}\right)}{\frac{\cos x}{\cos 2x}-1}\cdot\left(\frac{\cos x}{\cos 2x}-1\right)=\lim_{x\to 0}\frac{1}{x^2}\cdot\frac{\cos x-\cos 2x}{\cos 2x}=\lim_{x\to 0}\frac{1}{x^2}\cdot\frac{\cos x-(\cos^2 x-\sin^2 x)}{\cos 2x}=$

$=\lim_{x\to 0}\frac{\cos x}{\cos 2x}\cdot\frac{(1-\cos x)}{x^2}+\lim_{x\to 0}\frac{\sin^2 x}{x^2}\cdot\frac{1}{\cos 2x}=\frac{3}{2}$

1.b) $\lim_{x\to a}\frac{(\ln(x^2+a)-\ln(a^2+x))(\operatorname{tg}(x)-\operatorname{tg}(a))}{(\sqrt{a^2+x^2}-\sqrt{2ax})(6^{2a+1}-6^{2x+1})\cdot\sin(a^2+x^2)}=\lim_{x\to a}\frac{\ln\left(\frac{x^2+a}{a^2+x}\right)\cdot(\operatorname{tg}(x)-\operatorname{tg}(a))}{(\sqrt{a^2+x^2}-\sqrt{2ax})(6^{2a+1}-6^{2x+1})\cdot\sin(2a^2)}=\frac{1}{\sin(2a^2)}\lim_{x\to a}\frac{\ln\left(\frac{x^2+a}{a^2+x}\right)}{\frac{x^2+a}{a^2+x}-1}\cdot\left(\frac{x^3+a}{a^2+x}-1\right)\cdot\frac{(\operatorname{tg}(x)-\operatorname{tg}(a))}{(\sqrt{a^2+x^2}-\sqrt{2ax})(6^{2a+1}-6^{2x+1})}$

$\frac{1}{\sin(2a^2)}\lim_{x\to a}\frac{\left(\frac{x^3+a}{a^2+x}-1\right)\cdot(\operatorname{tg}(x)-\operatorname{tg}(a))\cdot(\sqrt{a^2+x^2}+\sqrt{2ax})}{(6^{2a+1}-6^{2x+1})(\sqrt{a^2+x^2}-\sqrt{2ax})\cdot(\sqrt{a^2+x^2}+\sqrt{2ax})}=\frac{1}{\sin 2a^2}\lim_{x\to a}\frac{x^3+a-(a^2+x)}{a^2+x}\left(\frac{\sin x}{\cos x}-\frac{\sin a}{\cos a}\right)\cdot\frac{\sqrt{a^2+x^2}+\sqrt{2ax}}{6^{2x+1}(6^{2a-2x}-1)(a^2+x^2-2ax)}=$

$\frac{1\cdot 2\sqrt{2}\cdot a}{\sin 2a^2\cdot(a^2+a)\,6^{2a+1}}\lim_{x\to a}\left(x^2-a^2-(x-a)\right)\cdot\frac{\sin x\cdot\cos a-\sin a\cdot\cos x}{\cos x\cdot\cos a}\cdot\frac{1}{e^{2(a-x)\cdot\ln 6}-1}\cdot\frac{1}{(x-a)^2}=\frac{1\cdot 2\sqrt{2}\cdot a}{\sin 2a^2\cdot(a^2+a)\,6^{2a+1}}\lim_{x\to a}\frac{(x-a)(x+a-1)}{(x-a)^2}\cdot\frac{\sin(x-a)}{\cos x\cdot\cos a}\cdot\frac{1}{\frac{e^{2(a-x)\cdot\ln 6}-1}{2(a-x)\cdot\ln 6}}\cdot 2(a-x)\cdot\ln 6$

$\frac{1\cdot 2\sqrt{2}\cdot a\cdot(2a+1)}{\sin 2a^2\cdot(a^2+a)\,6^{2a+1}\cdot\cos^2 a\cdot\ln 6}\lim_{x\to a}\frac{\sin(x-a)}{x-a}\cdot\frac{1}{2(a-x)}\Rightarrow$ neexistuje, jelikož $\lim_{x\to a+}\frac{1}{2(a-x)}=-\infty$ a $\lim_{x\to a-}\frac{1}{2(a-x)}=+\infty$.

2. $\sum_{n=1}^{\infty}\left(\frac{1}{mn^2+n\cdot\sin\frac{m^2\pi}{4}}\right)$; $m\in\mathbb{N}$. Srovnáme s konvergentní řadou $\sum\frac{1}{n^2}$.

$\lim_{n\to\infty}\frac{\frac{1}{n^2}}{\frac{1}{mn^2+n\cdot\sin\frac{m^2\pi}{4}}}=\lim_{n\to\infty}\frac{n^2}{n^2}\left(m+\frac{1}{n}\sin\frac{m^2\pi}{4}\right)=m\in(0,\infty)$. Tedy řada $\sum_{n=1}^{\infty}\left(\frac{1}{mn^2+n\cdot\sin\frac{m^2\pi}{4}}\right)$ konverguje.

3. $\sum_{n=1}^{\infty}(-1)^n\left(n^{\frac{1}{n^2+1}}-1\right)$

AK: Řadu $\sum\left(n^{\frac{1}{n^2+1}}-1\right)$ srovnáme s řadou $\sum\frac{\ln n}{n^2}$ (ta konverguje, což ukážeme později).

$\lim_{n\to\infty}\frac{n^{\frac{1}{n^2+1}}-1}{\frac{\ln n}{n^2}}=\lim_{n\to\infty}\frac{e^{\frac{1}{n^2+1}\cdot\ln n}-1}{\frac{1}{n^2+1}\cdot\ln n}\cdot\frac{\frac{1}{n^2+1}\cdot\ln n}{\frac{\ln n}{n^2}}=\lim_{n\to\infty}\frac{n^2}{n^2+1}=1\in(0,\infty)$. Tedy řada $\sum\left(n^{\frac{1}{n^2+1}}-1\right)$ konverguje $\Leftrightarrow$ konverguje řada $\sum\frac{\ln n}{n^2}$.

Jelikož $\frac{\ln n}{\sqrt{n}}=0\Rightarrow\exists n_0\in\mathbb{N}\ \forall n\geq n_0:\ \ln n<\sqrt{n}\Rightarrow\frac{\ln n}{n^2}<\frac{\sqrt{n}}{n^2}=\frac{1}{n^{\frac{3}{2}}}$.

Jelikož je řada $\sum\frac{1}{n^{\frac{3}{2}}}$ konvergentní, tak konverguje i řada $\sum\frac{\ln n}{n^2}$, tedy řada $\sum_{n=1}^{\infty}(-1)^n\left(n^{\frac{1}{n^2+1}}-1\right)$ konverguje absolutně.

Další řešení: AK řady $\sum\left(n^{\frac{1}{n^2+1}}-1\right)$ s použitím Raabeova krit. (vzhledem k pracnosti výpočtu to nepovažuji za optimální nápad).

$\lim_{n\to\infty}n\left(\frac{a_n}{a_{n+1}}-1\right)=\lim_{n\to\infty}n\left(\frac{n^{\frac{1}{n^2+1}}-1}{(n+1)^{\frac{1}{n^2+2n+2}}-1}-1\right)=\lim_{n\to\infty}n\left(\frac{e^{\frac{\ln n}{n^2+1}}-1}{e^{\frac{\ln(n+1)}{n^2+2n+2}}-1}-1\right)=\lim_{n\to\infty}n\left(\frac{e^{\frac{\ln n}{n^2+1}}-1-\left(e^{\frac{\ln(n+1)}{n^2+2n+2}}-1\right)}{e^{\frac{\ln(n+1)}{n^2+2n+2}}-1}\right)=\lim_{n\to\infty}\frac{n\cdot\left(e^{\frac{\ln n}{n^2+1}}-e^{\frac{\ln(n+1)}{n^2+2n+2}}\right)}{e^{\frac{\ln(n+1)}{n^2+2n+2}}-1}=$

$\lim_{n\to\infty}n\cdot e^{\frac{\ln(n+1)}{n^2+2n+2}}\cdot\frac{\left(e^{\left(\frac{\ln n}{n^2+1}-\frac{\ln(n+1)}{n^2+2n+2}\right)}-1\right)}{\left(e^{\frac{\ln(n+1)}{n^2+2n+2}}-1\right)\cdot\frac{\ln(n+1)}{n^2+2n+2}}=\lim_{n\to\infty}\frac{n(n^2+2n+2)}{\ln(n+1)}\cdot\frac{\left(e^{\left(\frac{\ln n}{n^2+1}-\frac{\ln(n+1)}{n^2+2n+2}\right)}-1\right)}{\frac{\ln n}{n^2+1}-\frac{\ln(n+1)}{n^2+2n+2}}\cdot\left(\frac{\ln n}{n^2+1}-\frac{\ln(n+1)}{n^2+2n+2}\right)=\lim_{n\to\infty}\frac{n(n^2+2n+2)}{\ln(n+1)}\cdot\frac{\ln n(n^2+2n+2)-\ln(n+1)(n^2+1)}{(n^2+1)(n^2+2n+2)}=$

$\lim_{n\to\infty}\frac{n(n^2+2n+2)}{\ln(n+1)}\cdot\frac{(\ln n-\ln(n+1))(n^2+1)+\ln n(2n+1)}{(n^2+1)(n^2+2n+2)}=\lim_{n\to\infty}\frac{n(n^2+2n+2)}{\ln(n+1)}\cdot\frac{\ln\left(\frac{n}{n-1}\right)(n^2+1)}{(n^2+1)(n^2+2n+2)}+\lim_{n\to\infty}\frac{n(n^2+2n+2)\cdot\ln n\cdot(2n+1)}{\ln(n+1)\cdot(n^2+1)\cdot(n^2+2n+2)}$

$=\lim_{n\to\infty}\frac{n^5+o(n^5)}{\ln(n+1)\cdot(n^4+o(n^4))}\cdot\frac{\ln\left(\frac{n}{n+1}\right)}{\frac{n}{n+1}-1}\left(\frac{n}{n+1}-1\right)+\lim_{n\to\infty}\frac{n^4\left(1+\frac{2}{n}+\frac{2}{n^2}\right)\left(2+\frac{1}{n}\right)}{n^4\left(1+\frac{1}{n^2}\right)\left(1+\frac{2}{n}+\frac{2}{n^2}\right)}\cdot\frac{\ln(n)}{\ln(n+1)}=\lim_{n\to\infty}\frac{(n^5+o(n^5))\cdot\frac{-1}{n+1}}{(n^4+o(n^4))\cdot\ln(n+1)}+2=\lim_{n\to\infty}\frac{-n^5+o(n^5)}{(n^5+o(n^5))}\cdot\frac{1}{\ln(n+1)}+2=2>1\Rightarrow$ řada konv. absolutně

4. $f'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{|h|\cdot h-0}{h}=\lim_{h\to 0}|h|=0\Rightarrow f'(0)=0$

Další řešení: $f(x)=|x|\cdot x=\begin{cases}x^2, & x\geq 0\\ -x^2, & x\leq 0\end{cases}\Rightarrow f'(x)=\begin{cases}2x, & x\geq 0\\ -2x, & x\leq 0\end{cases}$ (v nule máme pouze derivace zprava resp. zleva)

Jelikož $f'(0)_+=2\cdot 0=0=-2\cdot 0=f'(0)_-\Rightarrow f'(0)=0$