Rovnice se separovanými proměnnými ($y'(x) = f(x)g(y)$).

1. $y' = y^2 + 1$,
2. $y' = 2xy - 6x$,
3. $y' = \frac{y}{x}$,
4. $y' = \frac{y}{2}\mathrm{cotg}x$,
5. $y' = \sqrt{1-y^2}$,
6. $y' = y^2$,
7. $y' = \sqrt{y}$,
8. $y' = -\frac{y}{x}$,
9. $2y' - 4y + 3 = 0$,
10. $(1-x)y' = 1 + y$,
11. $y' = 2\sqrt{|y|}$,
12. $y' - y\sin x = 0$,
13. $y' = \frac{x-2}{y}$,
14. $y' = \frac{y-1}{x(x-1)}$,
15. $y' = \frac{2x-1}{1+2y}$.

Řešení:

1.

$$\begin{aligned}
y' &= y^2 + 1 \\
\frac{y'}{1+y^2} &= 1 \\
\mathrm{arctg}\, y &= x + C \\
y &= \mathrm{tg}(x + C).
\end{aligned}$$

2.

$$
\begin{aligned}
y' &= 2xy - 6x = 2x(y-3) \\
\frac{y'}{y-3} &= 2x \quad (* \ y \neq 3) \\
\ln|y-3| &= x^2 + C \quad (C \in \mathbb{R}) \\
|y-3| &= e^{x^2+C} = e^{x^2} \cdot e^C \\
|y-3| &= Ce^{x^2} \quad (C > 0) \\
y - 3 &= Ce^{x^2} \quad (C \in \mathbb{R} \setminus \{0\}) \\
y =& Ce^{x^2} + 3 \vee y = 3 \quad (C \in \mathbb{R} \setminus \{0\}) \\
y =& Ce^{x^2} + 3 \quad (C \in \mathbb{R})
\end{aligned}
$$

3.

$$
\begin{aligned}
y' &= \frac{y}{x} \\
\frac{y'}{y} &= \frac{1}{x} \quad (* \ y \neq 0) \\
\ln|y| &= \ln|x| + C \\
y &= Cx.
\end{aligned}
$$

4.

$$
\begin{aligned}
y' &= \frac{y}{2}\mathrm{cotg} x \\
\frac{y'}{y} &= \frac{1}{2}\mathrm{cotg} x \quad (* \ y \neq 0) \\
\ln|y| &= \frac{1}{2}\ln|\sin x| = \ln\sqrt{|\sin x|} + C \\
y &= C\sqrt{|\sin x|}.
\end{aligned}
$$

5.

$$
\begin{aligned}
y' &= \sqrt{1-y^2} \\
\frac{1}{\sqrt{1-y^2}} &= 1 \quad (* \ y \neq \pm 1) \\
\arcsin y &= x + C \\
y &= \sin(x+C) \vee y = \pm 1.
\end{aligned}
$$

6.

$$
\begin{aligned}
y' &= y^2 \\
\frac{y'}{y^2} &= 1 \quad (* \; y \neq 0) \\
\frac{-1}{y} &= x + C \\
y &= \frac{-1}{x+C} \vee y = 0.
\end{aligned}
$$

7.

$$
\begin{aligned}
y' &= \sqrt{y} \\
\frac{y'}{\sqrt{y}} &= 1 \quad (* \; y \neq 0) \\
2\sqrt{y} &= x + C \\
y &= \left(\frac{x+C}{2}\right)^2 \vee y = 0.
\end{aligned}
$$

8.

$$
\begin{aligned}
y' &= -\frac{y}{x} \\
\frac{y'}{y} &= \frac{1}{x} \quad (* \; y \neq 0) \\
\ln|y| &= -\ln|x| + C \\
y &= \frac{C}{x}.
\end{aligned}
$$

9.

$$
\begin{aligned}
2y' - 4y + 3 &= 0 \\
\frac{2y'}{4y-3} &= 1 \quad (* \; y \neq \frac{3}{4}) \\
\frac{1}{2}\ln|4y-3| &= x + C \\
|4y-3| &= Ce^{2x} \\
y &= \frac{3+Ce^{2x}}{4}.
\end{aligned}
$$

10.

$$\begin{aligned}
(1-x)y' &= 1+y \\
\frac{y'}{1+y} &= \frac{1}{1-x} \quad (*\ y \neq -1) \\
\ln|1+y| &= -\ln|1-x| + C \\
y &= \frac{C}{1-x} - 1.
\end{aligned}$$

11.

$$\begin{aligned}
y' &= 2\sqrt{|y|} \\
\frac{y'}{2\sqrt{|y|}} &= 1 \quad (*\ y \neq 0) \\
\sqrt{|y|} &= x + C \\
|y| &= (x+C)^2 \\
y &= \pm(x+C)^2 \vee y = 0.
\end{aligned}$$

12.

$$\begin{aligned}
y' - y\sin x &= 0 \\
\frac{y'}{y} &= \sin x \quad (*\ y \neq 0) \\
\ln|y| &= -\cos x + C \\
y &= Ce^{-\cos x}.
\end{aligned}$$

13.

$$\begin{aligned}
y' &= \frac{x-2}{y} \\
yy' &= x - 2 \\
\frac{y^2}{2} &= \frac{x^2}{2} - 2x + C \\
y &= \pm\sqrt{x^2 - 4x + C}.
\end{aligned}$$

14.

$$\begin{aligned}
y' &= \frac{y-1}{x(x-1)} \\
\frac{y'}{y-1} &= \frac{1}{x(x-1)} \quad (*\ y \neq 1) \\
\ln|y-1| &= \int \frac{1}{x(x-1)}dx = \int \left(\frac{1}{x-1} - \frac{1}{x}\right)dx = \ln\left|\frac{x-1}{x}\right| + C \\
y &= C \cdot \frac{x-1}{x} + 1
\end{aligned}$$

15.

$$\begin{aligned}
y' &= \frac{2x-1}{1+2y} \\
y'(1+2y) &= 2x-1 \\
\frac{(1+2y)^2}{4} &= \frac{(2x-1)^2}{4} + C \\
y &= \frac{\pm\sqrt{(2x-1)^2+C}-1}{2}.
\end{aligned}$$

Homogenní rovnice.

1. $y + x + xy' = 0$
2. $y' = \frac{x}{y} + \frac{y}{x}$
3. $y' = \frac{x+y}{x-y}$
4. $y' = \frac{y^2-2xy-x^2}{y^2+2xy-x^2},\ y(1) = -1$
5. $y'(3x^2 - y^2) = 2xy$
6. $(xy' - y)\mathrm{arctg}\frac{y}{x} = x,\ y(1) = 0$
7. $x^2y' + xy = x^2 + y^2$
8. $y' = e^{\frac{y}{x}} + \frac{y}{x}$
9. $y' = \frac{y}{x} + \mathrm{tg}\left(\frac{y}{x}\right)$

Řešení: Použijeme substituci $u = \frac{y}{x}$, tedy $y' = u'x + u$.

1.

$$\begin{aligned} y + x + xy' &= 0 \\ y' &= \frac{-x-y}{x} \\ u'x + u &= \frac{-x-ux}{x} \\ u' &= \frac{-1-2u}{x} \\ \frac{u'}{1+2u} &= -\frac{1}{x} \\ \frac{1}{2}\ln|1+2u| &= -\ln|x| + C \\ u &= \frac{\frac{C}{x^2} - 1}{2} \\ y &= \frac{C}{x} - \frac{x}{2}. \end{aligned}$$

2.

$$
\begin{aligned}
y' &= \frac{x}{y} + \frac{y}{x} \\
u'x + u &= \frac{1}{u} + u \\
u'u &= \frac{1}{x} \\
\frac{u^2}{2} &= \ln|x| + C \\
u &= \pm\sqrt{\ln x^2 + C} \\
y &= \pm x\sqrt{\ln x^2 + C}
\end{aligned}
$$

3.

$$
\begin{aligned}
y' &= \frac{x+y}{x-y} \\
u'x + u &= \frac{x+ux}{x-ux} = \frac{1+u}{1-u} \\
u'x &= \frac{1+u^2}{1-u} \\
\frac{1-u}{1+u^2}u' &= \frac{1}{x} \\
\operatorname{arctg} u - \frac{1}{2}\ln(1+u^2) &= \ln|x| + C \\
\operatorname{arctg}\left(\frac{y}{x}\right) - \frac{1}{2}\ln\left(1 + \left(\frac{y}{x}\right)^2\right) &= \ln|x| + C \\
\operatorname{arctg}\left(\frac{y}{x}\right) &= \ln\sqrt{x^2+y^2} + C
\end{aligned}
$$

4.

$$y' = \frac{y^2 - 2xy - x^2}{y^2 + 2xy - x^2},\ y(1) = -1$$

$$u'x + u = \frac{u^2 - 2u - 1}{u^2 + 2u - 1}$$

$$u'x = \frac{-u^3 - u^2 - u - 1}{u^2 + 2u - 1}$$

$$\frac{u^2 + 2u - 1}{(u^2+1)(u+1)} u' = \frac{-1}{x} \quad (*\ u \neq -1)$$

$$\int \frac{u^2 + 2u - 1}{(u^2+1)(u+1)} du \Rightarrow \frac{Au + B}{u^2 + 1} + \frac{C}{(u+1)} = \frac{(Au+B)(u+1) + C(u^2+1)}{(u+1)(u^2+1)} \Rightarrow$$

$$A + C = 1,\ A + B = 2,\ B + C = -1 \Rightarrow A = 2, B = 0, C = -1 \Rightarrow$$

$$\int \frac{u^2 + 2u - 1}{(u^2+1)(u+1)} du = \int \frac{2u}{1+u^2} du - \int \frac{1}{u+1} du = \ln \left| \frac{1+u^2}{1+u} \right|$$

$$\ln \left| \frac{1+u^2}{1+u} \right| = -\ln|x| + C$$

$$\frac{1+u^2}{1+u} = \frac{C}{x} \vee u = -1$$

$$\frac{1 + \frac{y^2}{x^2}}{1 + \frac{y}{x}} = \frac{C}{x} \vee y = -x$$

$$y = -x \quad \text{(při dosazení počáteční podmínky).}$$

5.

$$
\begin{aligned}
y'(3x^2-y^2) &= 2xy \\
y' &= \frac{2xy}{3x^2-y^2} \\
u'x+u &= \frac{2u}{3-u^2} \\
u' &= \frac{u^3-u}{3-u^2}\cdot\frac{1}{x} \\
\frac{3-u^2}{u(u-1)(u+1)}u' &= \frac{1}{x} \qquad (*\ u\neq -1, u\neq 0, u\neq 1)
\end{aligned}
$$

$$
\int \frac{3-u^2}{u(u-1)(u+1)}du \Rightarrow \frac{A}{u}+\frac{B}{u+1}+\frac{C}{u-1} = \frac{A(u^2-1)+B(u^2-u)+C(u^2+u)}{u(u+1)(u-1)} \Rightarrow
$$

$$
A+B+C=-1, -B+C=0, -A=3 \Rightarrow A=-3, B=C=1
$$

$$
\int \frac{3-u^2}{u(u-1)(u+1)}du = \int\left(\frac{-3}{u}+\frac{1}{u+1}+\frac{1}{u-1}\right)du = \ln\frac{|u^2-1|}{|u|^3}
$$

$$
\begin{aligned}
\ln\left|\frac{u^2-1}{u^3}\right| &= \ln|x|+C \\
\frac{u^2-1}{u^3} &= Cx \vee u=0 \vee u=\pm 1 \\
\frac{\left(\frac{y}{x}\right)^2-1}{\left(\frac{y}{x}\right)^3} &= Cx \Rightarrow y^2-x^2 = Cy^3 \vee y=0 \vee y=\pm x.
\end{aligned}
$$

6.

$$
\begin{aligned}
(xy'-y)\mathrm{arctg}\frac{y}{x} &= x,\ \ y(1)=0 \\
(x(u'x+u)-ux)\mathrm{arctg}u &= x \\
u'\mathrm{arctg}u &= \frac{1}{x}
\end{aligned}
$$

$$
\int \mathrm{arctg}\ u\ du = u\ \mathrm{arctg}\ u - \int\frac{u}{1+u^2}du = u\ \mathrm{arctg}\ u - \frac{1}{2}\ln(1+u^2)
$$

$$
\begin{aligned}
u\ \mathrm{arctg}\ u - \frac{1}{2}\ln(1+u^2) &= \ln|x|+C \\
u\ \mathrm{arctg}\ u &= \ln\left(\sqrt{1+u^2}|x|\right)+C \\
\frac{y}{x}\mathrm{arctg}\left(\frac{y}{x}\right) &= \ln\sqrt{x^2+y^2}+C
\end{aligned}
$$

Po dosazení počáteční podmínky dostaneme $C=0$, tedy řešení je dáno rovnicí

$$
\frac{y}{x}\mathrm{arctg}\left(\frac{y}{x}\right) = \ln\sqrt{x^2+y^2}.
$$

7.

$$
\begin{aligned}
x^2y' + xy &= x^2 + y^2 \\
y' &= \frac{x^2 - xy + y^2}{x^2} \\
u'x + u &= 1 - u + u^2 \\
\frac{u'}{(1-u)^2} &= \frac{1}{x} \quad (*\ u \neq 1) \\
\frac{1}{1-u} &= \ln|x| + C \\
u &= \frac{-1}{\ln|x| + C} + 1 \vee u = 1 \\
y &= \frac{-x}{\ln|x| + C} + x \vee \ y = x.
\end{aligned}
$$

8.

$$
\begin{aligned}
y' &= e^{\frac{y}{x}} + \frac{y}{x} \\
u'x + u &= e^u + u \\
e^{-u}u' &= \frac{1}{x} \\
-e^{-u} &= \ln|x| + C \\
u &= -\ln(C - \ln|x|) \\
y &= -x\ln(C - \ln|x|).
\end{aligned}
$$

9.

$$
\begin{aligned}
y' &= \frac{y}{x} + \operatorname{tg}\left(\frac{y}{x}\right) \\
u'x + u &= \operatorname{tg}\ u + u \quad (*\ u \neq 0 + k\pi); k \in \mathbb{Z} \\
\operatorname{cotg}\ uu' &= \frac{1}{x} \\
\ln|\sin u| &= \ln|x| + C \\
u &= \arcsin(Cx) \\
y &= x\ \arcsin(Cx)
\end{aligned}
$$

Lineární diferenciální rovnice prvního řádu.

1. $xy' + y - e^x = 0$
2. $y' = 2y + x$
3. $y'x^2 + y - 2xy = x^2$
4. $y' + 2xy = xe^{-x^2}$
5. $x(y' - y) = (1 + x^2)e^x$
6. $(1 + x^2)y' - 2xy = (1 + x^2)^2$
7. $xy' - \frac{y}{x+1} = x,\ y(1) = 0$
8. $y' - y\mathrm{tg}x = \frac{1}{\cos x},\ y(0) = 0$

Řešení: Vyřešíme nejdříve homogenní rovnici a pak využijeme variaci konstant.

1.

$$\begin{aligned}
xy' + y &= 0 \\
y_h &= C\frac{1}{x} \\
y_p &= \frac{C(x)}{x}, y_p' = \frac{C'(x)}{x} - \frac{C(x)}{x^2} \\
xy' + y - e^x &= 0 \\
x\left(\frac{C'(x)}{x} - \frac{C(x)}{x^2}\right) + \frac{C(x)}{x} - e^x &= 0 \\
C'(x) = e^x, C(x) &= e^x \\
y = y_h + y_p &= \frac{C}{x} + \frac{e^x}{x}.
\end{aligned}$$

2.

$$
\begin{aligned}
y' &= 2y \\
y_h &= Ce^{2x} \\
y_p &= C(x)e^{2x}, y_p' = e^{2x}(2C(x) + C'(x)) \\
y' &= 2y + x \\
e^{2x}(2C(x) + C'(x)) &= 2C(x)e^{2x} + x \\
C'(x) &= xe^{-2x} \\
C(x) &= \frac{xe^{-2x}}{-2} - \int \frac{e^{-2x}}{-2} dx = -e^{-2x}\left(\frac{x}{2} + \frac{1}{4}\right) \\
y = y_h + y_p &= Ce^{2x} - \frac{x}{2} - \frac{1}{4}.
\end{aligned}
$$

3.

$$
\begin{aligned}
y'x^2 + y - 2xy &= 0 \\
y'x^2 &= y(2x-1) \\
\frac{y'}{y} &= \frac{2x-1}{x^2} \\
\ln|y| &= \ln|x^2| + \frac{1}{x} + C \\
y_h &= Cx^2e^{\frac{1}{x}} \\
y_p = C(x)x^2e^{\frac{1}{x}}, y_p' &= e^{\frac{1}{x}}[C'(x)x^2 + C(x)2x - C(x)] \\
y'x^2 + y - 2xy &= x^2 \\
e^{\frac{1}{x}}[C'(x)x^2 + C(x)2x - C(x)]x^2 + C(x)x^2e^{\frac{1}{x}}(1-2x) &= x^2 \\
e^{\frac{1}{x}}C'(x)x^4 &= x^2 \\
C'(x) &= e^{-\frac{1}{x}}\frac{1}{x^2} \\
C(x) &= \int e^{-\frac{1}{x}}\frac{1}{x^2} dx = e^{-\frac{1}{x}} \\
y = y_h + y_p &= Cx^2e^{\frac{1}{x}} + x^2.
\end{aligned}
$$

4.

$$\begin{aligned}
y' + 2xy &= 0 \\
\frac{y'}{y} &= -2x \\
\ln|y| &= -x^2 + C \\
y_h &= Ce^{-x^2} \\
y_p &= C(x)e^{-x^2}, y_p' = e^{-x^2}(C'(x) - 2xC(x)) \\
y' + 2xy &= xe^{-x^2} \\
e^{-x^2}(C'(x) - 2xC(x)) + 2xC(x)e^{-x^2} &= xe^{-x^2} \\
C'(x) &= x \\
C(x) &= \frac{x^2}{2} \\
y = y_h + y_p &= Ce^{-x^2} + \frac{x^2}{2}e^{-x^2}.
\end{aligned}$$

5.

$$\begin{aligned}
x(y' - y) &= 0 \\
y_h &= Ce^x \\
y_p = C(x)e^x, y_p' &= e^x(C'(x) + C(x)) \\
x(y' - y) &= (1 + x^2)e^x \\
x(e^x(C'(x) + C(x)) - C(x)e^x) &= (1 + x^2)e^x \\
C'(x) &= \frac{1 + x^2}{x} \\
C(x) &= \ln|x| + \frac{x^2}{2} \\
y = y_h + y_p &= Ce^x + \left(\ln|x| + \frac{x^2}{2}\right)e^x.
\end{aligned}$$

6.

$$
\begin{aligned}
(1+x^2)y' - 2xy &= 0 \\
\frac{y'}{y} &= \frac{2x}{1+x^2} \\
\ln|y| &= \ln(1+x^2) + C \\
y_h &= C(1+x^2) \\
y_p &= C(x)(1+x^2), y_p' = C'(x)(1+x^2) + C(x)2x \\
(1+x^2)y' - 2xy &= (1+x^2)^2 \\
(1+x^2)[C'(x)(1+x^2) + C(x)2x] - 2xC(x)(1+x^2) &= (1+x^2)^2 \\
C'(x) &= 1, C(x) = x \\
y &= y_h + y_p = C(1+x^2) + x(1+x^2).
\end{aligned}
$$

7.

$$
\begin{aligned}
xy' - \frac{y}{x+1} &= 0 \\
\frac{y'}{y} &= \frac{1}{x(x+1)} \\
\ln|y| &= \int \left(\frac{1}{x} - \frac{1}{x+1}\right) dx = \ln\left|\frac{x}{x+1}\right| + C \\
y_h &= C\frac{x}{x+1} \\
y_p &= C(x)\frac{x}{x+1}, y_p' = C'(x)\frac{x}{x+1} + C(x)\frac{1}{(x+1)^2} \\
xy' - \frac{y}{x+1} &= x \\
C'(x)\frac{x^2}{x+1} + C(x)\frac{x}{(x+1)^2} - C(x)\frac{x}{(x+1)^2} &= x \\
C'(x) &= \frac{x+1}{x} \\
C(x) &= x + \ln|x| \\
y &= y_h + y_p = C\frac{x}{x+1} + (x + \ln|x|)\frac{x}{x+1}.
\end{aligned}
$$

Při dosazení počáteční podmínky $y(1) = 0$ dostaneme $0 = C \cdot \frac{1}{2} + 1 \cdot \frac{1}{2}$, tedy $C = -1$ a $y = \frac{x}{x+1}(-1 + x + \ln|x|)$.

8.

$$
\begin{aligned}
y' - y\mathrm{tg}x &= 0\\
\frac{y'}{y} &= \mathrm{tg}x\\
\ln|y| &= -\ln|\cos x| + C\\
y_h &= \frac{C}{\cos x}\\
y_p &= \frac{C(x)}{\cos x}, y_p' = \frac{C'(x)}{\cos x} + C(x)\frac{\sin x}{\cos^2 x}\\
y' - y\mathrm{tg}x &= \frac{1}{\cos x}\\
\frac{C'(x)}{\cos x} + C(x)\frac{\sin x}{\cos^2 x} - \frac{C(x)}{\cos x}\mathrm{tg}x &= \frac{1}{\cos x}\\
C'(x) &= 1\\
C(x) &= x
\end{aligned}
$$

$$
y = y_h + y_p = \frac{C}{\cos x} + \frac{x}{\cos x}.
$$

Při dosazení počáteční podmínky $y(0) = 0$ dostaneme $0 = C$, tedy $y = \frac{x}{\cos x}$.

Lineární diferenciální rovnice druhého řádu.

I. U následujících rovnic najděte druhé řešení příslušné homogenní rovnice metodou snížení řádu rovnice (a) i pomocí Wronskiánu (b), je-li y_1 řešení homogenní rovnice.

1. $y'' - \frac{2y'}{x} + \frac{2y}{x^2} = 0, \quad y_1 = x$
2. $(2x+1)y'' + 4xy' - 4y = 0, \quad y_1 = e^{-2x}$
3. $y'' - \frac{x}{x-1}y' + \frac{y}{x-1} = 0, \quad y_1 = e^x$
4. $y'' - \frac{2+x}{x}y' + \frac{2+x}{x^2}y = 0, \quad y_1 = x$

II. U následujících rovnic najděte druhé řešení příslušné homogenní rovnice metodou snížení řádu rovnice nebo pomocí Wronskiánu, je-li y_1 řešení homogenní rovnice (a). Dále nalezněte jedno partikulární řešení metodou variace konstant (b).

1. $y'' + y' - 2y = x, \quad y_1 = e^x$
2. $y'' - \frac{4y'}{x} + \frac{6y}{x^2} = x, \quad y_1 = x^2$

III. U následujících rovnic najděte partikulární řešení metodou variace konstant.

1. $y'' - 2y' = e^{2x}\ln x$
2. $y'' - 2y' + y = e^x \ln x$

Řešení: U metody snížení řádu využijeme substituci $z(x) = \frac{y(x)}{y_1(x)}$, a pak substituci $w = z'$. U Wronskiánu použijeme vzorec $\left(\frac{y}{y_1}\right)' = \frac{e^{-\int p(x)dx}}{y_1^2}$, kde pracujeme s rovnicí ve tvaru $y'' + p(x)y' + q(x)y = 0$.

I. 1. a)

$$y = zy_1 = zx, y' = z + xz', y'' = 2z' + xz''$$

$$y'' - \frac{2y'}{x} + \frac{2y}{x^2} = 0$$

$$2z' + xz'' - \frac{2z + 2xz'}{x} + \frac{2xz}{x^2} = 0$$

$$xz'' = 0$$

$$z = x$$

$$y_h = zy_1 = x^2$$

b)

$$\left(\frac{y}{y_1}\right)' = \frac{e^{-\int p(x)dx}}{y_1^2} = \frac{e^{-\int \frac{-2}{x}dx}}{x^2} = 1$$
$$\frac{y}{y_1} = x$$
$$y = xy_1 = x^2$$

$$y_h = C_1 y_1 + C_2 y_2 = C_1 x + C_2 x^2.$$

2. a)

$$y = zy_1 = ze^{-2x}, y' = e^{-2x}(z' - 2z), y'' = e^{-2x}(z'' - 4z' + 4z)$$
$$(2x+1)y'' + 4xy' - 4y = 0$$
$$e^{-2x}[(2x+1)(z'' - 4z' + 4z) + 4x(z' - 2z) - 4z] = 0$$
$$e^{-2x}[(2x+1)z'' + (-4x-4)z'] = 0$$
$$(2x+1)w' - 4(x+1)w = 0$$
$$\frac{w'}{w} = \frac{4(x+1)}{2x+1}$$
$$\ln|w| = 2x + \ln|2x+1| + C$$
$$w = (2x+1)e^{2x}$$
$$z = \int (2x+1)e^{2x}dx = (x+\frac{1}{2})e^{2x} - \int e^{2x}dx = \left(x + \frac{1}{2}\right)e^{2x} - \frac{e^{2x}}{2} = xe^{2x}$$
$$y_2 = zy_1 = xe^{2x}e^{-2x} = x.$$

b)

$$\left(\frac{y}{y_1}\right)' = \frac{e^{-\int p(x)dx}}{y_1^2} = \frac{e^{-\int \frac{4x}{2x+1}dx}}{e^{-4x}} = \frac{e^{-2x+\ln|2x+1|}}{e^{-4x}} = e^{2x}(2x+1)$$
$$\frac{y}{y_1} = \int e^{2x}(2x+1)dx = xe^{2x}$$
$$y_2 = xe^{2x}y_1 = x.$$

$$y_h = C_1 y_1 + C_2 y_2 = C_1 e^{-2x} + C_2 x.$$

3. a)

$$
\begin{aligned}
y = zy_1 = ze^x, y' = e^x(z'+z), y'' &= e^x(z''+2z'+z) \\
y'' - \frac{x}{x-1}y' + \frac{y}{x-1} &= 0 \\
e^x\left[(z''+2z'+z) - \frac{x}{x-1}(z'+z) + \frac{z}{x-1}\right] &= 0 \\
z'' + z'\left(2 - \frac{x}{x-1}\right) &= 0 \\
w' + \frac{x-2}{x-1}w &= 0 \\
\frac{w'}{w} &= -\frac{x-2}{x-1} \\
\ln|w| &= -x + \ln|x-1| \\
w &= e^{-x}(x-1) \\
z = \int (e^{-x}(x-1))dx = e^{-x}(1-x) + \int e^{-x}dx &= -e^{-x}x \\
y_2 = zy_1 &= -e^{-x}xe^x = -x
\end{aligned}
$$

b)

$$
\begin{aligned}
\left(\frac{y}{y_1}\right)' &= \frac{e^{-\int p(x)dx}}{y_1^2} = \frac{e^{\int \frac{x}{x-1}dx}}{e^{2x}} = \frac{e^{x+\ln|x-1|}}{e^{2x}} = e^{-x}(x-1) \\
\frac{y}{y_1} &= \int e^{-x}(x-1) = -xe^{-x} \\
y_2 &= -xe^{-x}y_1 = -x.
\end{aligned}
$$

$$y_h = C_1y_1 + C_2y_2 = C_1e^x + C_2x.$$

4. a)

$$
\begin{aligned}
y = zy_1 = zx, y' = z + xz', y'' &= 2z' + xz'' \\
y'' - \frac{2+x}{x}y' + \frac{2+x}{x^2}y &= 0 \\
xz'' + 2z' - \frac{2+x}{x}(z+xz') + \frac{2+x}{x^2}xz &= 0 \\
xz'' - xz' &= 0 \\
z' &= e^x \\
z &= e^x \\
y = zy_1 &= xe^x.
\end{aligned}
$$

b)

$$
\begin{aligned}
\left(\frac{y}{y_1}\right)' &= \frac{e^{-\int p(x)dx}}{y_1^2} = \frac{e^{\int \frac{2+x}{x}dx}}{x^2} = \frac{e^{2\ln|x|+x}}{x^2} = e^x \\
\frac{y}{y_1} &= \int e^x dx = e^x \\
y_2 &= e^x y_1 = xe^x.
\end{aligned}
$$

$$y_h = C_1y_1 + C_2y_2 = C_1x + C_2xe^x.$$

II. 1. a)

$$
\begin{aligned}
\left(\frac{y}{y_1}\right)' &= \frac{e^{-\int p(x)dx}}{y_1^2} = \frac{e^{-\int 1dx}}{e^{2x}} = e^{-3x} \\
\frac{y}{y_1} &= \int e^{-3x}dx = \frac{e^{-3x}}{-3} \\
y &= \frac{e^{-3x}}{-3}e^x = \frac{e^{-2x}}{-3} \\
y_h &= C_1e^x + C_2e^{-2x}.
\end{aligned}
$$

b)

$$
\begin{aligned}
y_p &= C_1(x)e^x + C_2(x)e^{-2x} \\
y_p' &= (C_1(x) + C_1'(x))e^x + (-2C_2(x) + C_2'(x))e^{-2x} \\
&\quad (*\text{přidáme podmínku } C_1'(x)e^x + C_2'(x)e^{-2x}) \\
y_p' &= C_1(x)e^x - 2C_2(x)e^{-2x} \\
y_p'' &= (C_1(x) + C_1'(x))e^x - 2(C_2'(x) - 2C_2(x))e^{-2x} \\
y'' + y' - 2y &= x \\
C_1'(x)e^x - 2C_2'(x)e^{-2x} &= x \\
C_1'(x)e^x + C_2'(x)e^{-2x} &= 0 \\
3C_1'(x)e^x &= x \\
C_1(x) &= \frac{1}{3}\int xe^{-x}dx = \frac{1}{3}\left(-xe^{-x} - \int -e^{-x}dx\right) = \frac{-e^{-x}}{3}(x+1) \\
C_2(x) &= \int -C_1'(x)e^{3x}dx = \int \frac{-1}{3}xe^{2x}dx = \frac{e^{2x}}{6}(-x+\frac{1}{2}) \\
y_p &= C_1(x)e^x + C_2(x)e^{-2x} = -\frac{x+1}{3} + \frac{1}{6}(-x+\frac{1}{2}) = -\frac{x}{2} - \frac{1}{4}.
\end{aligned}
$$

$$y = y_h + y_p = C_1e^x + C_2e^{-2x} - \frac{x}{2} - \frac{1}{4}.$$

2. a)

$$
\begin{aligned}
\left(\frac{y}{y_1}\right)' &= \frac{e^{-\int p(x)dx}}{y_1^2} = \frac{e^{-\int -\frac{4}{x}dx}}{x^4} = 1 \\
\frac{y}{y_1} &= x \\
y_2 &= x^3 \\
y_h &= C_1x^2 + C_2x^3.
\end{aligned}
$$

b)

$$
\begin{aligned}
y_p &= C_1(x)x^2 + C_2(x)x^3 \\
y_p' &= C_1'(x)x^2 + 2xC_1(x) + C_2'(x)x^3 + 3x^2C_2(x) \overset{(*)}{=} 2xC_1(x) + 3x^2C_2(x) \\
C_1'(x)x^2 + C_2'(x)x^3 &= 0 \quad (*) \\
y_p'' &= 2xC_1'(x) + 2C_1(x) + 3x^2C_2'(x) + 6xC_2(x) \\
y'' - \frac{4y'}{x} + \frac{6y}{x^2} &= x \\
2xC_1'(x) + 3x^2C_2'(x) &= x \\
C_1'(x) + C_2'(x)x &= 0 \\
C_2(x) &= \int \frac{1}{x}dx = \ln|x| \\
C_1(x) &= \int -1dx = -x \\
y_p &= -x^3 + \ln|x|x^3 = x^3(\ln|x| - 1)
\end{aligned}
$$

$$
y = y_h + y_p = C_1x^2 + C_2x^3 + x^3(\ln|x| - 1).
$$

III. 1.

$$
\begin{aligned}
y_h &= C_1 + C_2 e^{2x} \\
y_p &= C_1(x) + C_2(x)e^{2x} \\
y_p' &= C_1'(x) + C_2'(x)e^{2x} + 2C_2(x)e^{2x} \overset{(*)}{=} 2C_2(x)e^{2x} \\
C_1'(x) + C_2'(x)e^{2x} &= 0 \quad (*) \\
y_p'' &= 4C_2(x)e^{2x} + 2C_2'(x)e^{2x} \\
y'' - 2y' &= e^{2x}\ln x \\
2C_2'(x)e^{2x} &= e^{2x}\ln x \\
C_2(x) &= \frac{1}{2}\int \ln x\,dx = \frac{1}{2}(\ln x - 1)x \\
C_1'(x) + C_2'(x)e^{2x} &= 0 \\
C_1(x) &= \int -\frac{1}{2}\ln x e^{2x}dx \\
y_p &= C_1(x) + C_2(x)e^{2x} = \frac{-1}{2}\int \ln x e^{2x}dx + \frac{1}{2}(\ln x - 1)xe^{2x}.
\end{aligned}
$$

2.

$$
\begin{aligned}
y_h &= e^x(C_1 + C_2 x) \\
y_p &= e^x(C_1(x) + C_2(x)x) \\
y_p' &= e^x(C_1'(x) + C_1(x) + C_2(x)x + C_2(x) + C_2'(x)x) \overset{(*)}{=} e^x(C_1(x) + C_2(x)(x+1)) \\
C_1'(x) + C_2'(x)x &= 0 \quad (*) \\
y_p'' &= e^x(C_1(x) + C_1'(x) + C_2(x)(x+1) + C_2(x) + C_2'(x)(x+1)) \\
y'' - 2y' + y &= e^x \ln x \\
C_1'(x) + C_2'(x)(x+1) &= \ln x \\
C_1'(x) + C_2'(x)x &= 0 \\
C_2(x) &= \int \ln x\,dx = x(\ln x - 1) \\
C_1(x) &= \int -x\ln x = -\left[\frac{x^2}{2}\ln x - \int \frac{x}{2}dx\right] = \frac{-x^2}{2}\left(\ln x - \frac{1}{2}\right) \\
y_p &= e^x\left[\frac{-x^2}{2}\left(\ln x - \frac{1}{2}\right) + x^2(\ln x - 1)\right] = e^x\left[\frac{x^2}{2}\ln x - \frac{3}{4}x^2\right].
\end{aligned}
$$

Homogenní rovnice s konstantními koeficienty.

1. $y'' + 4y' + 3y = 0$
2. $y'' - 2y' + y = 0$
3. $y'' + y = 0$
4. $y'' + 2y' + 5y = 0$
5. $y''' - 5y'' + 8y' - 4y = 0$
6. $y''' - 4y'' + 7y' = 0$
7. $y''' + 6y'' + 12y' + 8y = 0$

Řešení:

1.

$$\begin{aligned} y'' + 4y' + 3y &= 0 \\ \lambda^2 + 4\lambda + 3 = (\lambda+3)(\lambda+1) &= 0 \\ \lambda_1 = -3, \lambda_2 &= -1 \\ y_1 = e^{\lambda_1} = e^{-3x}, y_2 &= e^{\lambda_2} = e^{-x} \\ y = C_1y_1 + C_2y_2 &= C_1e^{-3x} + C_2e^{-x}. \end{aligned}$$

2.

$$\begin{aligned} y'' - 2y' + y &= 0 \\ \lambda^2 - 2\lambda + 1 = (\lambda-1)^2 &= 0 \\ \lambda_{1,2} &= 1 \\ y_1 = e^{\lambda_1} = e^x, y_2 &= xe^{\lambda_1} = xe^x \\ y = C_1y_1 + C_2y_2 &= C_1e^x + C_2xe^x. \end{aligned}$$

3.

$$\begin{aligned} y'' + y &= 0 \\ \lambda^2 + 1 &= 0 \\ \lambda_{1,2} &= \pm i \\ y = e^{\lambda x} &= e^{ix} = \cos x + i \sin x \Rightarrow y_1 = \sin x, y_2 = \cos x \\ y &= C_1y_1 + C_2y_2 = C_1 \sin x + C_2 \cos x. \end{aligned}$$

4.

$$\begin{aligned} y'' + 2y' + 5y &= 0 \\ \lambda^2 + 2\lambda + 5 = (\lambda+1)^2 + 4 &= 0 \\ \lambda_{1,2} &= -1 \pm 2i \\ y = e^{\lambda x} &= e^{(-1+2i)x} = e^{-x}(\cos 2x + i \sin 2x) \Rightarrow y_1 = e^{-x} \sin 2x, y_2 = e^{-x} \cos 2x \\ y &= C_1y_1 + C_2y_2 = C_1e^{-x} \sin 2x + C_2e^{-x} \cos 2x. \end{aligned}$$

Zkouška pro y_1

$$y_1' = e^{-x}(-\sin 2x + 2\cos 2x),\, y_1'' = e^{-x}(-3\sin 2x - 4\cos 2x)$$
$$y_1'' + 2y_1' + 5y_1 = e^{-x}[-3\sin x - 4\cos x + 2(-\sin 2x + 2\cos 2x) + 5(\sin 2x)] = 0.$$

5.

$$\begin{gathered}
y''' - 5y'' + 8y' - 4y = 0 \\
\lambda^3 - 5\lambda^2 + 8\lambda - 4 = (\lambda - 1)(\lambda^2 - 4\lambda + 4) = (\lambda - 1)(\lambda - 2)^2 = 0 \\
\lambda_1 = 1, \lambda_{2,3} = 2 \\
y_1 = e^x, y_2 = e^{2x}, y_3 = xe^{2x} \\
y = C_1y_1 + C_2y_2 + C_3y_3 = C_1e^x + C_2e^{2x} + C_3xe^{2x}.
\end{gathered}$$

6.

$$\begin{gathered}
y''' - 4y'' + 7y' = 0 \\
\lambda^3 - 4\lambda^2 + 7\lambda = \lambda[\lambda^2 - 4\lambda + 7] = \lambda[(\lambda - 2)^2 + 3] = 0 \\
\lambda_1 = 0, \lambda_{2,3} = 2 \pm \sqrt{3}i \\
y_1 = e^{0x} = 1, y_2 = e^{2x}\sin(\sqrt{3}x), y_3 = e^{2x}\cos(\sqrt{3}x) \\
y = C_1y_1 + C_2y_2 + C_3y_3 = C_1 + C_2e^{2x}\sin(\sqrt{3}x) + C_3e^{2x}\cos(\sqrt{3}x).
\end{gathered}$$

7.

$$\begin{gathered}
y''' + 6y'' + 12y' + 8y = 0 \\
\lambda^3 + 6\lambda^2 + 12\lambda + 8 = (\lambda + 2)^3 = 0 \\
\lambda_{1,2,3} = -2 \\
y_1 = e^{-2x}, y_2 = xe^{-2x}, y_3 = x^2e^{-2x} \\
y = C_1y_1 + C_2y_2 + C_3y_3 = e^{-2x}[C_1 + C_2x + C_3x^2].
\end{gathered}$$

Zkouška

$$\begin{aligned}
y &= e^{-2x}[C_1 + C_2x + C_3x^2] \\
y' &= e^{-2x}[-2C_1 + C_2(-2x + 1) + C_3(-2x^2 + 2x)] \\
y'' &= e^{-2x}[4C_1 + C_2(4x - 2 - 2) + C_3(4x^2 - 4x - 4x + 2)] = e^{-2x}[4C_1 + C_2(4x - 4) + C_3(4x^2 - 8x + 2)] \\
y''' &= e^{-2x}[-8C_1 + C_2(-8x + 8 + 4) + C_3(-8x^2 + 16x - 4 + 8x - 8)] = e^{-2x}[-8C_1 + C_2(-8x + 12) + C_3(-8x^2 + 24x - 12)] \\
y''' + 6y'' + 12y' + 8y &= e^{-2x}C_1(-8 + 24 - 24 + 8] + e^{-2x}C_2(-8x + 12 + 24x - 24 - 24x + 12 + 8x) + \\
&\quad + C_3(-8x^2 + 24x - 12 + 24x^2 - 48x + 12 - 24x^2 + 24x + 8x^2) = 0.
\end{aligned}$$

Homogenní rovnice s konstantními koeficienty a speciální pravou stranou.

1. $y'' + 4y' + 3y = e^{-3x}$
2. $y'' - 2y' + y = x^2 e^{-x}$
3. $y'' + y = \cos x$
4. $y'' + 2y' + 5y = \sin 2x + x$
5. $y''' - 5y'' + 8y' - 4y = e^x$
6. $y''' - 4y'' + 7y' = \sin(\sqrt{3}x)$
7. $y''' + 6y'' + 12y' + 8y = x^2$

Řešení: Jelikož jsou příslušné homogenní rovnice převzaty z předchozího cvičení, tak se budeme zabývat pouze hledáním partikulárního řešení a použijeme výsledky z předchozího cvičení.

1. Jelikož e^{-3x} je řešením příslušné homogenní rovnice a $\lambda = -3$ je jednonásobný kořen příslušného charakteristického polynomu, tak hledáme partikulární řešení ve tvaru $y_p = Cxe^{-3x}$.

$$\begin{aligned}
y_p &= Cxe^{-3x} \\
y_p' &= Ce^{-3x}(-3x + 1) \\
y_p'' &= Ce^{-3x}(9x - 6) \\
y'' + 4y' + 3y &= e^{-3x} \\
e^{-3x}C(9x - 6 + 4(-3x + 1) + 3x) &= e^{-3x} \\
C &= -\frac{1}{2} \\
y_p &= -\frac{x}{2}e^{-3x}
\end{aligned}$$

$$y = y_h + y_p = C_1 e^{-3x} + C_2 e^{-x} - \frac{x}{2}e^{-3x}.$$

2. Jelikož -1 není kořenem příslušného charakteristického polynomu, tak hledáme partikulární řešení ve tvaru $y_p = e^{-x}(ax^2 + bx + c)$.

$$\begin{aligned}
y_p &= e^{-x}(ax^2 + bx + c) \\
y_p' &= e^{-x}(-ax^2 - bx - c + 2ax + b) = e^{-x}(-ax^2 + (2a - b)x + (b - c)) \\
y_p'' &= e^{-x}(+ax^2 - (2a - b)x - (b - c) - 2ax + (2a - b)) = e^{-x}(ax^2 + (b - 4a)x + (2a - 2b + c))
\end{aligned}$$

$$
\begin{aligned}
y'' - 2y' + y &= x^2e^{-x}\\
e^{-x}[ax^2 + (b-4a)x + (2a-2b+c) - 2(-ax^2 + (2a-b)x + (b-c)) + ax^2 + bx + c] &= x^2e^{-x}\\
4ax^2 &= x^2\\
(b-4a-4a+2b+b)x = (-8a+4b)x &= 0x\\
2a-2b+c-2b+2c+c = 2a-4b+4c &= 0\\
a = \frac{1}{4}, b = \frac{1}{2}, c &= \frac{3}{8}
\end{aligned}
$$

$$
y = y_h + y_p = C_1e^x + C_2xe^x + e^{-x}\left(\frac{x^2}{4} + \frac{x}{2} + \frac{3}{8}\right).
$$

3. Jelikož i je jednonásobný kořen charakteristického polynomu, hledáme řešení ve tvaru $y_p = ax\sin x + bx\cos x$.

$$
\begin{aligned}
y_p &= ax\sin x + bx\cos x\\
y_p' &= x(a\cos x - b\sin x) + (a\sin x + b\cos x)\\
y_p'' &= x(-a\sin x - b\cos x) + (2a\cos x - 2b\sin x)\\
y'' + y &= \cos x\\
x(-a\sin x - b\cos x) + (2a\cos x - 2b\sin x) + ax\sin x + bx\cos x &= \cos x\\
a = \frac{1}{2}, b &= 0
\end{aligned}
$$

$$
y = y_h + y_p = C_1\sin x + C_2\cos x + \frac{1}{2}x\sin x.
$$

4. Jelikož $2i$ ani 0 není kořenem příslušného charakteristického polynomu, tak hledáme partikulární řešení ve tvaru $y_p = a\sin(2x) + b\cos(2x) + cx + d$.

$$
\begin{aligned}
y_p &= a\sin 2x + b\cos 2x + cx + d\\
y_p' &= -2b\sin 2x + 2a\cos 2x + c\\
y_p'' &= -4a\sin 2x - 4b\cos 2x
\end{aligned}
$$

$$
\begin{aligned}
y'' + 2y' + 5y &= \sin 2x + x\\
[-4a - 4b + 5a]\sin 2x + [-4b + 4a + 5b]\cos 2x + 5cx + (5d + 2c) &= \sin 2x + x\\
a - 4b &= 1\\
4a + b &= 0\\
5c &= 1\\
(5d + 2c) &= 0\\
a = \frac{1}{17}, b = \frac{-4}{17}, c = \frac{1}{5}, d &= \frac{-2}{25}
\end{aligned}
$$

$$
y = y_h + y_p = C_1e^{-x}\sin 2x + C_2e^{-x}\cos 2x + \frac{1}{17}\sin 2x - \frac{4}{17}\cos 2x + \frac{x}{5} - \frac{2}{25}.
$$

5. Jelikož 1 je jednonásobným kořenem příslušného charakteristického polynomu, tak hledáme partikulární řešení ve tvaru $y_p = cxe^x$.

$$\begin{aligned} y_p &= ce^x x \\ y_p' &= ce^x(x+1) \\ y_p'' &= ce^x(x+2) \\ y_p''' &= ce^x(x+3) \end{aligned}$$

$$\begin{aligned} y''' - 5y'' + 8y' - 4y &= e^x \\ ce^x(x+3-5(x+2)+8(x+1)-4x) &= e^x \\ c(3-10+8) &= 1 \\ c &= 1 \end{aligned}$$

$$y = y_h + y_p = C_1e^x + C_2e^{2x} + C_3xe^{2x} + xe^x.$$

6. Jelikož $\sqrt{3}i$ není kořenem příslušného charakteristického polynomu, tak hledáme partikulární řešení ve tvaru $y_p = a\sin(\sqrt{3}x) + b\cos(\sqrt{3}x)$.

$$\begin{aligned} y_p &= a\sin(\sqrt{3}x) + b\cos(\sqrt{3}x) \\ y_p' &= \sqrt{3}(-b\sin(\sqrt{3}x) + a\cos(\sqrt{3}x)) \\ y_p'' &= -3(a\sin(\sqrt{3}x) + b\cos(\sqrt{3}x)) \\ y_p''' &= -3\sqrt{3}(-b\sin(\sqrt{3}x) + a\cos(\sqrt{3}x)) \end{aligned}$$

$$\begin{aligned} y''' - 4y'' + 7y' &= \sin(\sqrt{3}x) \\ -3\sqrt{3}(-b\sin(\sqrt{3}x) + a\cos(\sqrt{3}x)) + 12(a\sin(\sqrt{3}x) + b\cos(\sqrt{3}x)) + 7\sqrt{3}(-b\sin(\sqrt{3}x) + a\cos(\sqrt{3}x)) &= \sin(\sqrt{3}x) \\ \sin(\sqrt{3}x)[3\sqrt{3}b + 12a - 7\sqrt{3}b] + \cos(\sqrt{3}x)[-3\sqrt{3}a + 12b + 7\sqrt{3}a] &= \sin(\sqrt{3}x) \\ 12a - 4\sqrt{3}b &= 1 \\ 4\sqrt{3}a + 12b &= 0 \\ a &= \frac{1}{16} \\ b &= \frac{-\sqrt{3}}{48} \end{aligned}$$

$$y = y_h + y_p = C_1 + C_2e^{2x}\sin(\sqrt{3}x) + C_3e^{2x}\cos(\sqrt{3}x) + \frac{1}{16}\sin(\sqrt{3}x) + \frac{-\sqrt{3}}{48}\cos(\sqrt{3}x).$$

7. Jelikož 0 není kořenem příslušného charakteristického polynomu, tak hledáme partikulární řešení ve tvaru $y_p = ax^2 + bx + c$.

$$\begin{aligned} y_p &= ax^2 + bx + c \\ y_p' &= 2ax + b \\ y_p'' &= 2a \\ y_p''' &= 0 \end{aligned}$$

$$
\begin{aligned}
y''' + 6y'' + 12y' + 8y &= x^2 \\
12a + 24ax + 12b + 8ax^2 + 8bx + 8c &= x^2 \\
8a &= 1 \\
24a + 8b &= 0 \\
12a + 12b + 8c &= 0 \\
a &= \frac{1}{8} \\
b &= \frac{-3}{8} \\
c &= \frac{3}{8}
\end{aligned}
$$

$$
y = y_h + y_p = e^{-2x}[C_1 + C_2x + C_3x^2] + \frac{x^2 - 3x + 3}{8}.
$$