Mocninné řady I.

Určete poloměr konvergence mocninné řady a konvergenci v krajních bodech konvergenčního intervalu.

1. $\sum_{n=1}^{\infty} \frac{x^n}{n^p}$,
2. $\sum_{n=1}^{\infty} \frac{3^n+(-2)^n}{n}(x+1)^n$,
3. $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}x^n$,
4. $\sum_{n=1}^{\infty} \alpha^{n^2} x^n, \quad 0<\alpha<1$,
5. $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n^2} x^n$,
6. $\sum_{n=1}^{\infty} \frac{n!}{a^{n^2}}x^n, \quad (a>1)$,
7. $\sum_{n=1}^{\infty} \left(\frac{a^n}{n}+\frac{b^n}{n^2}\right) x^n, \quad a>0, b>0$,
8. $\sum_{n=1}^{\infty} \frac{x^n}{a^n+b^n}, \quad a>0, b>0$,
9. $\sum_{n=1}^{\infty} \frac{3^{-\sqrt{n}}x^n}{\sqrt{n^2+1}}$,
10. $\sum_{n=1}^{\infty} \left(1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}\right) x^n$,
11. $\sum_{n=1}^{\infty} \frac{[3+(-1)^n]^n}{n}x^n$.

Řešení:

1.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{(n+1)^p}}{\frac{x^n}{n^p}}\right| = \lim_{n\to\infty}|x|\frac{n^p}{(n+1)^p} = |x| < 1 \Rightarrow R=1,\ x\in(-1,1).$$

$$x=-1: \sum_{n=1}^{\infty}\frac{(-1)^n}{n^p},\quad \text{řada konverguje pro } p>0 \text{ dle Leibnizova kritéria.}$$

$$x=1: \sum_{n=1}^{\infty}\frac{1}{n^p},\ \text{tedy řada konverguje pro } p>1.$$

$$p\le 0: x\in(-1,1),\quad 0<p\le 1: x\in[-1,1),\quad p>1: x\in[-1,1].$$

2.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{3^{n+1}+(-2)^{n+1}}{n+1}(x+1)^{n+1}}{\frac{3^n+(-2)^n}{n}(x+1)^n}\right| = \lim_{n\to\infty}\frac{(3^{n+1}+(-2)^{n+1})n}{(3^n+(-2)^n)(n+1)}|x+1|$$

$$= \lim_{n\to\infty}\frac{3^n}{3^n}\cdot\frac{3-2\left(\frac{-2}{3}\right)^n}{1+\left(\frac{-2}{3}\right)^n}\cdot\frac{n}{n+1}|x+1| = 3|x+1| < 1 \Rightarrow R = \frac{1}{3},\ x\in(-\frac{4}{3},-\frac{2}{3}).$$

$$x=-\frac{4}{3}: \sum_{n=1}^{\infty}\frac{3^n+(-2)^n}{n}\left(\frac{-1}{3}\right)^n = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}+\sum_{n=1}^{\infty}\frac{\left(\frac{2}{3}\right)^n}{n} \Rightarrow \text{řada konverguje},$$

jelikož řada $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ konverguje dle Leibnizova kritéria a $\sum_{n=1}^{\infty}\frac{\left(\frac{2}{3}\right)^n}{n} \leq \sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n < \infty.$

$$x=\frac{-2}{3}: \sum_{n=1}^{\infty}\frac{3^n+(-2)^n}{n}\left(\frac{1}{3}\right)^n = \sum_{n=1}^{\infty}\frac{1}{n}+\sum_{n=1}^{\infty}\frac{\left(\frac{2}{3}\right)^n}{n} \Rightarrow \text{řada diverguje},$$

jelikož řada $\sum_{n=1}^{\infty}\frac{1}{n}$diverguje a $\sum_{n=1}^{\infty}\frac{\left(\frac{2}{3}\right)^n}{n}$ konverguje.

$\Rightarrow$ Mocninná řada konverguje pro $x\in[-\frac{4}{3},-\frac{2}{3})$.

3.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{((n+1)!)^2}{(2n+2)!}x^{n+1}}{\frac{(n!)^2}{(2n)!}x^n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2}{(2n+2)(2n+1)}x\right| = \frac{|x|}{4} < 1 \Rightarrow R=4,\ x\in(-4,4).$$

Jelikož $a_n = \frac{(n!)^2}{(2n)!}(\pm 4)^n \geq \pm\frac{(n!)^2}{(2^n n!)^2}(4)^n = 1$ nesplňuje nutnou podmínku konvergence ani pro $x=4$ ani pro $x=-4$, tak mocninná řada konverguje pro $x\in(-4,4)$.

4.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\alpha^{(n+1)^2}x^{n+1}}{\alpha^{n^2}x^n}\right| = \lim_{n\to\infty}\alpha^{2n+1}|x| = 0 < 1 \Rightarrow R=\infty,\ x\in\mathbb{R}.$$

5. I.

$$\lim_{n\to\infty}\sqrt[n]{a_n} = \lim_{n\to\infty}\sqrt[n]{\left(1+\frac{1}{n}\right)^{n^2}|x|^n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n|x| = |x|e < 1 \Rightarrow R=\frac{1}{e},\ x\in(-\frac{1}{e},\frac{1}{e}).$$

II.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\left(1+\frac{1}{n+1}\right)^{(n+1)^2} x^{n+1}}{\left(1+\frac{1}{n}\right)^{n^2} x^n}\right| = \lim_{n\to\infty}\left|\frac{\left(\frac{n+2}{n+1}\right)^{n^2+2n+1} x}{\left(\frac{n+1}{n}\right)^{n^2}}\right|$$

$$= |x| \lim_{n\to\infty}\left(\frac{n(n+2)}{(n+1)^2}\right)^{n^2} \cdot \left(\frac{n+2}{n+1}\right)^{2n+1}$$

$$= |x| \lim_{n\to\infty}\left(1-\frac{1}{n^2+2n+1}\right)^{n^2} \cdot \left(1+\frac{1}{n+1}\right)^{2n+1} = |x|e < 1 \Rightarrow R = \frac{1}{e}, x \in (-\frac{1}{e}, \frac{1}{e}),$$

$$x = \frac{1}{e} : \sum_{n=1}^{\infty} \frac{\left(1+\frac{1}{n}\right)^{n^2}}{e^n},$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)^{n^2}}{e^n} = \lim_{n\to\infty}\left(\frac{\left(1+\frac{1}{n}\right)^n}{e}\right)^n = \lim_{n\to\infty} e^{n\ln\left(\frac{\left(1+\frac{1}{n}\right)^n}{e}\right)}$$

$$= \lim_{n\to\infty} e^{\frac{n\ln\left(1+\frac{1}{n}\right)-1}{\frac{1}{n}}} = e^{\star} = e^{\frac{-1}{2}} \neq 0 \ \Rightarrow \text{řada diverguje, totéž platí pro } x = -\tfrac{1}{e}.$$

$$\star = \lim_{n\to\infty}\frac{n\ln\left(1+\frac{1}{n}\right)-1}{\frac{1}{n}} \overset{L'H}{=} \lim_{n\to\infty}\frac{\ln\left(1+\frac{1}{n}\right)+\frac{n}{1+\frac{1}{n}}\left(-\frac{1}{n^2}\right)}{-\frac{1}{n^2}} = \lim_{n\to\infty}\frac{\ln\left(1+\frac{1}{n}\right)-\frac{1}{n+1}}{-\frac{1}{n^2}}$$

$$\overset{L'H}{=} \lim_{n\to\infty}\frac{-\frac{1}{n(1+n)}+\frac{1}{(n+1)^2}}{\frac{2}{n^3}} = \lim_{n\to\infty}\frac{(n-(n+1))n^3}{2n(n+1)^2} = -\frac{1}{2}$$

$\Rightarrow$ mocninná řada konverguje pro $x \in (-\frac{1}{e}, \frac{1}{e})$.

6.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n+1)!}{a^{(n+1)^2}} x^{n+1}}{\frac{n!}{a^{n^2}} x^n}\right| = \lim_{n\to\infty}\left|\frac{n+1}{a^{2n+1}} x\right| = 0, \Rightarrow R = \infty,\ x \in \mathbb{R}.$$

7. Označme $c = \max\{a, b\}$, pak

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\left(\frac{a^{n+1}}{n+1}+\frac{b^{n+1}}{(n+1)^2}\right)x^{n+1}}{\left(\frac{a^n}{n}+\frac{b^n}{n^2}\right)x^n}\right| = \left|\frac{\frac{a}{n+1}\left(\frac{a}{c}\right)^n+\frac{b}{(n+1)^2}\left(\frac{b}{c}\right)^n}{\frac{1}{n}\left(\frac{a}{c}\right)^n+\frac{1}{n^2}\left(\frac{b}{c}\right)^n}x\right|$$

$$= c|x| < 1 \Rightarrow \ x \in (-\frac{1}{c}, \frac{1}{c}) = (-\frac{1}{\max\{a,b\}}, \frac{1}{\max\{a,b\}}),$$

$$x = -\frac{1}{c} = \begin{cases} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} + \frac{\left(\frac{b}{a}\right)^n(-1)^n}{n^2}, & a \geq b \\ \sum_{n=1}^{\infty} \frac{(-1)^n\left(\frac{a}{b}\right)^n}{n} + \frac{(-1)^n}{n^2}, & a < b \end{cases} \quad \Rightarrow \ \text{konverguje dle Leibnizova kritéria.}$$

$$x = \frac{1}{c} = \begin{cases} \sum_{n=1}^{\infty} \frac{1}{n} + \frac{\left(\frac{b}{a}\right)^n}{n^2}, & a \geq b \Rightarrow \text{divergence} \\ \sum_{n=1}^{\infty} \frac{\left(\frac{a}{b}\right)^n}{n} + \frac{1}{n^2}, & a < b \Rightarrow \text{konvergence} \end{cases}$$

$$\Rightarrow \text{mocninná řada konverguje pro } x \in [-\frac{1}{a}, \frac{1}{a}), \quad a \geq b \text{ a } x \in [-\frac{1}{b}, \frac{1}{b}], \quad b > a.$$

8. Označme $c = \max\{a, b\}$, pak

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{a^{n+1}+b^{n+1}}}{\frac{x^n}{a^n+b^n}}\right| = \lim_{n\to\infty}\left|\frac{\left(\frac{a}{c}\right)^n+\left(\frac{b}{c}\right)^n}{a\left(\frac{a}{c}\right)^n+b\left(\frac{b}{c}\right)^n}x\right|$$

$$= \frac{1}{c}|x| < 1 \Rightarrow \ x \in (-c, c) = (-\max\{a,b\}, \max\{a,b\}),$$

$$x = \pm c: \lim_{n\to\infty}|a_n| = \lim_{n\to\infty}\frac{c^n}{a^n+b^n} = 1 \neq 0 \Rightarrow \ \text{řada nekonverguje}$$

$$\Rightarrow \text{mocninná řada konverguje pro } x \in (-\max\{a,b\}, \max\{a,b\}).$$

9.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{3^{-\sqrt{n+1}}}{\sqrt{(n+1)^2+1}}x^{n+1}}{\frac{3^{-\sqrt{n}}}{\sqrt{n^2+1}}x^n}\right| = \lim_{n\to\infty}\left|\frac{\sqrt{\frac{n^2+1}{n^2+2n+2}}}{3^{\sqrt{n+1}-\sqrt{n}}}x\right| = |x| < 1 \Rightarrow R = 1, x \in (-1,1),$$

$$x = \pm 1: \sum_{n=1}^{\infty}\frac{3^{-\sqrt{n}}}{\sqrt{n^2+1}}(\pm 1)^n \ \text{konverguje}, \ \text{jelikož} \ \left|\frac{3^{-\sqrt{n}}}{\sqrt{n^2+1}}\right| < \frac{1}{3^{\sqrt{n}}n} < \frac{1}{\sqrt{n}n} = \frac{1}{n^{\frac{3}{2}}}$$

$$\Rightarrow \text{mocninná řada konverguje pro } x \in [-1, 1].$$

10.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\left(1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n+1}\right)x^{n+1}}{\left(1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}\right)x^n}\right| = \lim_{n\to\infty}\left(1+\frac{\frac{1}{n+1}}{\left(1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}\right)}\right)|x|$$

$$= |x| < 1 \Rightarrow R = 1, \ x \in (-1, 1),$$

$$x = \pm 1: \lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(1+\frac{1}{2}+...+\frac{1}{n}\right)(\pm 1)^n \neq 0$$

$$\Rightarrow \text{mocninná řada konverguje pro } x \in (-1, 1).$$

11.

$$\sum_{n=1}^{\infty}\frac{[3+(-1)^n]^n}{n}x^n=\sum_{n=1}^{\infty}\frac{[3+(-1)]^{2n-1}}{2n-1}x^{2n-1}+\sum_{n=1}^{\infty}\frac{[3+1]^{2n}}{2n}x^{2n}=\sum_{n=1}^{\infty}\frac{(2x)^{2n-1}}{2n-1}+\sum_{n=1}^{\infty}\frac{(4x)^{2n}}{2n}\Rightarrow$$

První řada konverguje pro $x\in(-\frac{1}{2},\frac{1}{2}),$ druhá pro $x\in(-\frac{1}{4},\frac{1}{4}),$

tedy původní řada konverguje pro $x\in(-\frac{1}{4},\frac{1}{4})$,

$$x=\pm\frac{1}{4}:\sum_{n=1}^{\infty}\frac{[3+(-1)^n]^n(\pm 1)^n}{4^n n}=\pm\sum_{n=1}^{\infty}\frac{\left(\frac{1}{2}\right)^{2n-1}}{2n-1}+\sum_{n=1}^{\infty}\frac{1}{2n}.$$

Jelikož první řada konverguje a druhá diverguje, tak původní řada diverguje

$\Rightarrow$ mocninná řada konverguje pro $x\in(-\frac{1}{4},\frac{1}{4})$.