

Rovnice se separovanými proměnnými ( $y'(x) = f(x)g(y)$ ).

1.  $y' = y^2 + 1$ ,

2.  $y' = 2xy - 6x$ ,

3.  $y' = \frac{y}{x}$ ,

4.  $y' = \frac{y}{2} \cot gx$ ,

5.  $y' = \sqrt{1 - y^2}$ ,

6.  $y' = y^2$ ,

7.  $y' = \sqrt{y}$ ,

8.  $y' = -\frac{y}{x}$ ,

9.  $2y' - 4y + 3 = 0$ ,

10.  $(1 - x)y' = 1 + y$ ,

11.  $y' = 2\sqrt{|y|}$ ,

12.  $y' - y \sin x = 0$ ,

13.  $y' = \frac{x-2}{y}$ ,

14.  $y' = \frac{y-1}{x(x-1)}$ ,

15.  $y' = \frac{2x-1}{1+2y}$ .

Řešení:

1.

$$\begin{aligned}y' &= y^2 + 1 \\ \frac{y'}{1 + y^2} &= 1 \\ \arctg y &= x + C \\ y &= \operatorname{tg}(x + C).\end{aligned}$$

2.

$$\begin{aligned}y' &= 2xy - 6x = 2x(y - 3) \\ \frac{y'}{y - 3} &= 2x \quad (* y \neq 3) \\ \ln |y - 3| &= x^2 + C \quad (C \in \mathbb{R}) \\ |y - 3| &= e^{x^2 + C} = e^{x^2} \cdot e^C \\ |y - 3| &= Ce^{x^2} \quad (C > 0) \\ y - 3 &= Ce^{x^2} \quad (C \in \mathbb{R} \setminus \{0\}) \\ y &= Ce^{x^2} + 3 \vee y = 3 \quad (C \in \mathbb{R} \setminus \{0\}) \\ y &= Ce^{x^2} + 3 \quad (C \in \mathbb{R})\end{aligned}$$

3.

$$\begin{aligned}y' &= \frac{y}{x} \\ \frac{y'}{y} &= \frac{1}{x} \quad (* y \neq 0) \\ \ln |y| &= \ln |x| + C \\ y &= Cx.\end{aligned}$$

4.

$$\begin{aligned}y' &= \frac{y}{2} \cot x \\ \frac{y'}{y} &= \frac{1}{2} \cot x \quad (* y \neq 0) \\ \ln |y| &= \frac{1}{2} \ln |\sin x| = \ln \sqrt{|\sin x|} + C \\ y &= C\sqrt{|\sin x|}.\end{aligned}$$

5.

$$\begin{aligned}y' &= \sqrt{1 - y^2} \\ \frac{1}{\sqrt{1 - y^2}} &= 1 \quad (* y \neq \pm 1) \\ \arcsin y &= x + C \\ y &= \sin(x + C) \vee y = \pm 1.\end{aligned}$$

6.

$$\begin{aligned}y' &= y^2 \\ \frac{y'}{y^2} &= 1 \quad (* y \neq 0) \\ \frac{-1}{y} &= x + C \\ y &= \frac{-1}{x + C} \vee y = 0.\end{aligned}$$

7.

$$\begin{aligned}y' &= \sqrt{y} \\ \frac{y'}{\sqrt{y}} &= 1 \quad (* y \neq 0) \\ 2\sqrt{y} &= x + C \\ y &= \left(\frac{x + C}{2}\right)^2 \vee y = 0.\end{aligned}$$

8.

$$\begin{aligned}y' &= -\frac{y}{x} \\ \frac{y'}{y} &= -\frac{1}{x} \quad (* y \neq 0) \\ \ln |y| &= -\ln |x| + C \\ y &= \frac{C}{x}.\end{aligned}$$

9.

$$\begin{aligned}2y' - 4y + 3 &= 0 \\ \frac{2y'}{4y - 3} &= 1 \quad (* y \neq \frac{3}{4}) \\ \frac{1}{2} \ln |4y - 3| &= x + C \\ |4y - 3| &= Ce^{2x} \\ y &= \frac{3 + Ce^{2x}}{4}.\end{aligned}$$

10.

$$(1-x)y' = 1+y$$

$$\frac{y'}{1+y} = \frac{1}{1-x} \quad (* y \neq -1)$$

$$\ln|1+y| = -\ln|1-x| + C$$

$$y = \frac{C}{1-x} - 1.$$

11.

$$y' = 2\sqrt{|y|}$$

$$\frac{y'}{2\sqrt{|y|}} = 1 \quad (* y \neq 0)$$

$$\sqrt{|y|}\operatorname{sgn}(y) = x + C \quad (y > 0 \Rightarrow x > -C, y < 0 \Rightarrow x < -C)$$

$$|y| = (x + C)^2$$

$$y = \pm(x + C)^2 \vee y = 0.$$

12.

$$y' - y \sin x = 0$$

$$\frac{y'}{y} = \sin x \quad (* y \neq 0)$$

$$\ln|y| = -\cos x + C$$

$$y = Ce^{-\cos x}.$$

13.

$$y' = \frac{x-2}{y}$$

$$yy' = x-2$$

$$\frac{y^2}{2} = \frac{x^2}{2} - 2x + C$$

$$y = \pm\sqrt{x^2 - 4x + C}.$$

14.

$$y' = \frac{y-1}{x(x-1)}$$

$$\frac{y'}{y-1} = \frac{1}{x(x-1)} \quad (* y \neq 1)$$

$$\ln|y-1| = \int \frac{1}{x(x-1)} dx = \int \left( \frac{1}{x-1} - \frac{1}{x} \right) dx = \ln \left| \frac{x-1}{x} \right| + C$$

$$y = C \cdot \frac{x-1}{x} + 1$$

15.

$$\begin{aligned}y' &= \frac{2x-1}{1+2y} \\y'(1+2y) &= 2x-1 \\ \frac{(1+2y)^2}{4} &= \frac{(2x-1)^2}{4} + C \\ y &= \frac{\pm\sqrt{(2x-1)^2 + C} - 1}{2}.\end{aligned}$$

## Homogenní rovnice.

1.  $y + x + xy' = 0$

2.  $y' = \frac{x}{y} + \frac{y}{x}$

3.  $y' = \frac{x+y}{x-y}$

4.  $y' = \frac{y^2 - 2xy - x^2}{y^2 + 2xy - x^2}$ ,  $y(1) = -1$

5.  $y'(3x^2 - y^2) = 2xy$

6.  $(xy' - y)\operatorname{arctg}\frac{y}{x} = x$

7.  $x^2y' + xy = x^2 + y^2$

8.  $y' = e^{\frac{y}{x}} + \frac{y}{x}$

9.  $y' = \frac{y}{x} + \operatorname{tg}\left(\frac{y}{x}\right)$

Řešení: Použijeme substituci  $u = \frac{y}{x}$ , tedy  $y' = u'x + u$ .

1.

$$y + x + xy' = 0$$

$$y' = \frac{-x - y}{x}$$

$$u'x + u = \frac{-x - ux}{x}$$

$$u' = \frac{-1 - 2u}{x}$$

$$\frac{u'}{1 + 2u} = -\frac{1}{x}$$

$$\frac{1}{2} \ln|1 + 2u| = -\ln|x| + C$$

$$u = \frac{\frac{C}{x^2} - 1}{2}$$

$$y = \frac{C}{x} - \frac{x}{2}.$$

2.

$$\begin{aligned}y' &= \frac{x}{y} + \frac{y}{x} \\u'x + u &= \frac{1}{u} + u \\u'u &= \frac{1}{x} \\\frac{u^2}{2} &= \ln|x| + C \\u &= \pm\sqrt{\ln x^2 + C} \\y &= \pm x\sqrt{\ln x^2 + C}\end{aligned}$$

3.

$$\begin{aligned}y' &= \frac{x+y}{x-y} \\u'x + u &= \frac{x+ux}{x-ux} = \frac{1+u}{1-u} \\u'x &= \frac{1+u^2}{1-u} \\\frac{1-u}{1+u^2}u' &= \frac{1}{x} \\\arctg u - \frac{1}{2}\ln(1+u^2) &= \ln|x| + C \\\arctg\left(\frac{y}{x}\right) - \frac{1}{2}\ln\left(1 + \left(\frac{y}{x}\right)^2\right) &= \ln|x| + C \\\arctg\left(\frac{y}{x}\right) &= \ln\sqrt{x^2 + y^2} + C\end{aligned}$$

4.

$$y' = \frac{y^2 - 2xy - x^2}{y^2 + 2xy - x^2}, \quad y(1) = -1$$

$$u'x + u = \frac{u^2 - 2u - 1}{u^2 + 2u - 1}$$

$$u'x = \frac{-u^3 - u^2 - u - 1}{u^2 + 2u - 1}$$

$$\frac{u^2 + 2u - 1}{(u^2 + 1)(u + 1)} u' = \frac{-1}{x} \quad (* u \neq -1)$$

$$\int \frac{u^2 + 2u - 1}{(u^2 + 1)(u + 1)} du \Rightarrow \frac{Au + B}{u^2 + 1} + \frac{C}{u + 1} = \frac{(Au + B)(u + 1) + C(u^2 + 1)}{(u + 1)(u^2 + 1)} \Rightarrow$$

$$A + C = 1, \quad A + B = 2, \quad B + C = -1 \Rightarrow A = 2, B = 0, C = -1 \Rightarrow$$

$$\int \frac{u^2 + 2u - 1}{(u^2 + 1)(u + 1)} du = \int \frac{2u}{1 + u^2} du - \int \frac{1}{u + 1} du = \ln \left| \frac{1 + u^2}{1 + u} \right|$$

$$\ln \left| \frac{1 + u^2}{1 + u} \right| = -\ln |x| + C$$

$$\frac{1 + u^2}{1 + u} = \frac{C}{x} \vee u = -1$$

$$\frac{1 + \frac{y^2}{x^2}}{1 + \frac{y}{x}} = \frac{C}{x} \vee y = -x$$

$y = -x$  (při dosazení počáteční podmínky).



5.

$$\begin{aligned}
 y'(3x^2 - y^2) &= 2xy \\
 y' &= \frac{2xy}{3x^2 - y^2} \\
 u'x + u &= \frac{2u}{3 - u^2} \\
 u' &= \frac{u^3 - u}{3 - u^2} \cdot \frac{1}{x} \\
 \frac{3 - u^2}{u(u-1)(u+1)} u' &= \frac{1}{x} \quad (* u \neq -1, u \neq 0, u \neq 1) \\
 \int \frac{3 - u^2}{u(u-1)(u+1)} du &\Rightarrow \frac{A}{u} + \frac{B}{u+1} + \frac{C}{u-1} = \frac{A(u^2 - 1) + B(u^2 - u) + C(u^2 + u)}{u(u+1)(u-1)} \Rightarrow \\
 A + B + C &= -1, -B + C = 0, -A = 3 \Rightarrow A = -3, B = C = 1 \\
 \int \frac{3 - u^2}{u(u-1)(u+1)} du &= \int \left( \frac{-3}{u} + \frac{1}{u+1} + \frac{1}{u-1} \right) du = \ln \frac{|u^2 - 1|}{|u|^3} \\
 \ln \left| \frac{u^2 - 1}{u^3} \right| &= \ln |x| + C \\
 \frac{u^2 - 1}{u^3} &= Cx \vee u = 0 \vee u = \pm 1 \\
 \frac{\left(\frac{y}{x}\right)^2 - 1}{\left(\frac{y}{x}\right)^3} &= Cx \Rightarrow y^2 - x^2 = Cy^3 \vee y = 0 \vee y = \pm x.
 \end{aligned}$$

6.

$$\begin{aligned}
 (xy' - y) \operatorname{arctg} \frac{y}{x} &= x, \quad y(1) = 0 \\
 (x(u'x + u) - ux) \operatorname{arctg} u &= x \\
 u' \operatorname{arctg} u &= \frac{1}{x} \\
 \int \operatorname{arctg} u \, du &= u \operatorname{arctg} u - \int \frac{u}{1 + u^2} du = u \operatorname{arctg} u - \frac{1}{2} \ln(1 + u^2) \\
 u \operatorname{arctg} u - \frac{1}{2} \ln(1 + u^2) &= \ln |x| + C \\
 u \operatorname{arctg} u &= \ln \left( \sqrt{1 + u^2} |x| \right) + C \\
 \frac{y}{x} \operatorname{arctg} \left( \frac{y}{x} \right) &= \ln \sqrt{x^2 + y^2} + C
 \end{aligned}$$

7.

$$\begin{aligned}x^2 y' + xy &= x^2 + y^2 \\y' &= \frac{x^2 - xy + y^2}{x^2} \\u'x + u &= 1 - u + u^2 \\\frac{u'}{(1-u)^2} &= \frac{1}{x} \quad (* u \neq 1) \\\frac{1}{1-u} &= \ln|x| + C \\u &= \frac{-1}{\ln|x| + C} + 1 \vee u = 1 \\y &= \frac{-x}{\ln|x| + C} + x \vee y = x.\end{aligned}$$

8.

$$\begin{aligned}y' &= e^{\frac{y}{x}} + \frac{y}{x} \\u'x + u &= e^u + u \\e^{-u} u' &= \frac{1}{x} \\-e^{-u} &= \ln|x| + C \\u &= -\ln(C - \ln|x|) \\y &= -x \ln(C - \ln|x|).\end{aligned}$$

9.

$$\begin{aligned}y' &= \frac{y}{x} + \operatorname{tg}\left(\frac{y}{x}\right) \\u'x + u &= \operatorname{tg} u + u \quad (* u \neq 0 + k\pi); k \in \mathbb{Z} \\\cotg uu' &= \frac{1}{x} \\\ln|\sin u| &= \ln|x| + C \\u &= \arcsin(Cx) \\y &= x \arcsin(Cx)\end{aligned}$$

## Lineární diferenciální rovnice prvního řádu.

1.  $xy' + y - e^x = 0$
2.  $y' = 2y + x$
3.  $y'x^2 + y - 2xy = x^2$
4.  $y' + 2xy = xe^{-x^2}$
5.  $x(y' - y) = (1 + x^2)e^x$
6.  $(1 + x^2)y' - 2xy = (1 + x^2)^2$
7.  $xy' - \frac{y}{x+1} = x, y(1) = 0$
8.  $y' - y \operatorname{tg} x = \frac{1}{\cos x}, y(0) = 0$

Řešení: Vyřešíme nejdříve homogenní rovnici a pak využijeme variaci konstant.

1.

$$\begin{aligned}xy' + y &= 0 \\y_h &= C \frac{1}{x} \\y_p &= \frac{C(x)}{x}, y'_p = \frac{C'(x)}{x} - \frac{C(x)}{x^2} \\xy' + y - e^x &= 0 \\x \left( \frac{C'(x)}{x} - \frac{C(x)}{x^2} \right) + \frac{C(x)}{x} - e^x &= 0 \\C'(x) = e^x, C(x) &= e^x \\y = y_h + y_p &= \frac{C}{x} + \frac{e^x}{x}.\end{aligned}$$

2.

$$y' = 2y$$

$$y_h = Ce^{2x}$$

$$y_p = C(x)e^{2x}, y'_p = e^{2x}(2C(x) + C'(x))$$

$$y' = 2y + x$$

$$e^{2x}(2C(x) + C'(x)) = 2C(x)e^{2x} + x$$

$$C'(x) = xe^{-2x}$$

$$C(x) = \frac{xe^{-2x}}{-2} - \int \frac{e^{-2x}}{-2} dx = -e^{-2x} \left( \frac{x}{2} + \frac{1}{4} \right)$$

$$y = y_h + y_p = Ce^{2x} - \frac{x}{2} - \frac{1}{4}.$$

3.

$$y'x^2 + y - 2xy = 0$$

$$y'x^2 = y(2x - 1)$$

$$\frac{y'}{y} = \frac{2x - 1}{x^2}$$

$$\ln |y| = \ln |x^2| + \frac{1}{x} + C$$

$$y_h = Cx^2e^{\frac{1}{x}}$$

$$y_p = C(x)x^2e^{\frac{1}{x}}, y'_p = e^{\frac{1}{x}}[C'(x)x^2 + C(x)2x - C(x)]$$

$$y'x^2 + y - 2xy = x^2$$

$$e^{\frac{1}{x}}[C'(x)x^2 + C(x)2x - C(x)]x^2 + C(x)x^2e^{\frac{1}{x}}(1 - 2x) = x^2$$

$$e^{\frac{1}{x}}C'(x)x^4 = x^2$$

$$C'(x) = e^{-\frac{1}{x}} \frac{1}{x^2}$$

$$C(x) = \int e^{-\frac{1}{x}} \frac{1}{x^2} dx = e^{-\frac{1}{x}}$$

$$y = y_h + y_p = Cx^2e^{\frac{1}{x}} + x^2.$$

4.

$$y' + 2xy = 0$$

$$\frac{y'}{y} = -2x$$

$$\ln |y| = -x^2 + C$$

$$y_h = Ce^{-x^2}$$

$$y_p = C(x)e^{-x^2}, y'_p = e^{-x^2}(C'(x) - 2xC(x))$$

$$y' + 2xy = xe^{-x^2}$$

$$e^{-x^2}(C'(x) - 2xC(x)) + 2xC(x)e^{-x^2} = xe^{-x^2}$$

$$C'(x) = x$$

$$C(x) = \frac{x^2}{2}$$

$$y = y_h + y_p = Ce^{-x^2} + \frac{x^2}{2}e^{-x^2}.$$

5.

$$x(y' - y) = 0$$

$$y_h = Ce^x$$

$$y_p = C(x)e^x, y'_p = e^x(C'(x) + C(x))$$

$$x(y' - y) = (1 + x^2)e^x$$

$$x(e^x(C'(x) + C(x)) - C(x)e^x) = (1 + x^2)e^x$$

$$C'(x) = \frac{1 + x^2}{x}$$

$$C(x) = \ln |x| + \frac{x^2}{2}$$

$$y = y_h + y_p = Ce^x + \left(\ln |x| + \frac{x^2}{2}\right)e^x.$$

6.

$$\begin{aligned}
 (1+x^2)y' - 2xy &= 0 \\
 \frac{y'}{y} &= \frac{2x}{1+x^2} \\
 \ln|y| &= \ln(1+x^2) + C \\
 y_h &= C(1+x^2) \\
 y_p &= C(x)(1+x^2), y'_p = C'(x)(1+x^2) + C(x)2x \\
 (1+x^2)y' - 2xy &= (1+x^2)^2 \\
 (1+x^2)[C'(x)(1+x^2) + C(x)2x] - 2xC(x)(1+x^2) &= (1+x^2)^2 \\
 C'(x) &= 1, C(x) = x \\
 y &= y_h + y_p = C(1+x^2) + x(1+x^2).
 \end{aligned}$$

7.

$$\begin{aligned}
 xy' - \frac{y}{x+1} &= 0 \\
 \frac{y'}{y} &= \frac{1}{x(x+1)} \\
 \ln|y| &= \int \left( \frac{1}{x} - \frac{1}{x+1} \right) dx = \ln \left| \frac{x}{x+1} \right| + C \\
 y_h &= C \frac{x}{x+1} \\
 y_p &= C(x) \frac{x}{x+1}, y'_p = C'(x) \frac{x}{x+1} + C(x) \frac{1}{(x+1)^2} \\
 xy' - \frac{y}{x+1} &= x \\
 C'(x) \frac{x^2}{x+1} + C(x) \frac{x}{(x+1)^2} - C(x) \frac{x}{(x+1)^2} &= x \\
 C'(x) &= \frac{x+1}{x} \\
 C(x) &= x + \ln|x| \\
 y &= y_h + y_p = C \frac{x}{x+1} + (x + \ln|x|) \frac{x}{x+1}.
 \end{aligned}$$

Při dosazení počáteční podmínky  $y(1) = 0$  dostaneme  $0 = C \cdot \frac{1}{2} + 1 \cdot \frac{1}{2}$ , tedy  $C = -1$  a  $y = \frac{x}{x+1}(-1 + x + \ln|x|)$ .

8.

$$y' - y \operatorname{tg} x = 0$$

$$\frac{y'}{y} = \operatorname{tg} x$$

$$\ln |y| = -\ln |\cos x| + C$$

$$y_h = \frac{C}{\cos x}$$

$$y_p = \frac{C(x)}{\cos x}, y_p' = \frac{C'(x)}{\cos x} + C(x) \frac{\sin x}{\cos^2 x}$$

$$y' - y \operatorname{tg} x = \frac{1}{\cos x}$$

$$\frac{C'(x)}{\cos x} + C(x) \frac{\sin x}{\cos^2 x} - \frac{C(x)}{\cos x} \operatorname{tg} x = \frac{1}{\cos x}$$

$$C'(x) = 1$$

$$C(x) = x$$

$$y = y_h + y_p = \frac{C}{\cos x} + \frac{x}{\cos x}.$$

Při dosazení počáteční podmínky  $y(0) = 0$  dostaneme  $0 = C$ , tedy  $y = \frac{x}{\cos x}$ .

## Lineární diferenciální rovnice druhého řádu.

I. U následujících rovnic najděte druhé řešení příslušné homogenní rovnice metodou snížení řádu rovnice (a) i pomocí Wronskiánu (b), je-li  $y_1$  řešení homogenní rovnice.

1.  $y'' - \frac{2y'}{x} + \frac{2y}{x^2} = 0$ ,  $y_1 = x$
2.  $(2x + 1)y'' + 4xy' - 4y = 0$ ,  $y_1 = e^{-2x}$
3.  $y'' - \frac{x}{x-1}y' + \frac{y}{x-1} = 0$ ,  $y_1 = e^x$
4.  $y'' - \frac{2+x}{x}y' + \frac{2+x}{x^2}y = 0$ ,  $y_1 = x$

II. U následujících rovnic najděte druhé řešení příslušné homogenní rovnice metodou snížení řádu rovnice nebo pomocí Wronskiánu, je-li  $y_1$  řešení homogenní rovnice (a). Dále nalezněte jedno partikulární řešení metodou variace konstant (b).

1.  $y'' + y' - 2y = x$ ,  $y_1 = e^x$
2.  $y'' - \frac{4y'}{x} + \frac{6y}{x^2} = x$ ,  $y_1 = x^2$

III. U následujících rovnic najděte partikulární řešení metodou variace konstant.

1.  $y'' - 2y' = e^{2x} \ln x$
2.  $y'' - 2y' + y = e^x \ln x$

Řešení: U metody snížení řádu využijeme substituci  $z(x) = \frac{y(x)}{y_1(x)}$ , a pak substituci  $w = z'$ . U Wronskiánu použijeme vzorec  $\left(\frac{y}{y_1}\right)' = \frac{e^{-\int p(x)dx}}{y_1^2}$ , kde pracujeme s rovnicí ve tvaru  $y'' + p(x)y' + q(x)y = 0$ .

I. 1. a)

$$\begin{aligned}y &= zy_1 = zx, y' = z + xz', y'' = 2z' + xz'' \\y'' - \frac{2y'}{x} + \frac{2y}{x^2} &= 0 \\2z' + xz'' - \frac{2z + 2xz'}{x} + \frac{2xz}{x^2} &= 0 \\xz'' &= 0 \\z &= x \\y_h &= zy_1 = x^2\end{aligned}$$



b)

$$\begin{aligned}\left(\frac{y}{y_1}\right)' &= \frac{e^{-\int p(x)dx}}{y_1^2} = \frac{e^{-\int \frac{-2}{x} dx}}{x^2} = 1 \\ \frac{y}{y_1} &= x \\ y &= xy_1 = x^2\end{aligned}$$

$$y_h = C_1 y_1 + C_2 y_2 = C_1 x + C_2 x^2.$$

2. a)

$$\begin{aligned}y &= zy_1 = ze^{-2x}, y' = e^{-2x}(z' - 2z), y'' = e^{-2x}(z'' - 4z' + 4z) \\ (2x+1)y'' + 4xy' - 4y &= 0 \\ e^{-2x}[(2x+1)(z'' - 4z' + 4z) + 4x(z' - 2z) - 4z] &= 0 \\ e^{-2x}[(2x+1)z'' + (-4x-4)z'] &= 0 \\ (2x+1)w' - 4(x+1)w &= 0 \\ \frac{w'}{w} &= \frac{4(x+1)}{2x+1} \\ \ln|w| &= 2x + \ln|2x+1| + C \\ w &= (2x+1)e^{2x} \\ z &= \int (2x+1)e^{2x} dx = \left(x + \frac{1}{2}\right)e^{2x} - \int e^{2x} dx = \left(x + \frac{1}{2}\right)e^{2x} - \frac{e^{2x}}{2} = xe^{2x} \\ y_2 &= zy_1 = xe^{2x}e^{-2x} = x.\end{aligned}$$

b)

$$\begin{aligned}\left(\frac{y}{y_1}\right)' &= \frac{e^{-\int p(x)dx}}{y_1^2} = \frac{e^{-\int \frac{4x}{2x+1} dx}}{e^{-4x}} = \frac{e^{-2x + \ln|2x+1|}}{e^{-4x}} = e^{2x}(2x+1) \\ \frac{y}{y_1} &= \int e^{2x}(2x+1) dx = xe^{2x} \\ y_2 &= xe^{2x}y_1 = x.\end{aligned}$$

$$y_h = C_1 y_1 + C_2 y_2 = C_1 e^{-2x} + C_2 x.$$

3. a)

$$\begin{aligned}
 y &= zy_1 = ze^x, y' = e^x(z' + z), y'' = e^x(z'' + 2z' + z) \\
 y'' - \frac{x}{x-1}y' + \frac{y}{x-1} &= 0 \\
 e^x \left[ (z'' + 2z' + z) - \frac{x}{x-1}(z' + z) + \frac{z}{x-1} \right] &= 0 \\
 z'' + z' \left( 2 - \frac{x}{x-1} \right) &= 0 \\
 w' + \frac{x-2}{x-1}w &= 0 \\
 \frac{w'}{w} &= -\frac{x-2}{x-1} \\
 \ln|w| &= -x + \ln|x-1| \\
 w &= e^{-x}(x-1) \\
 z &= \int (e^{-x}(x-1))dx = e^{-x}(1-x) + \int e^{-x}dx = -e^{-x}x \\
 y_2 = zy_1 &= -e^{-x}xe^x = -x
 \end{aligned}$$

b)

$$\begin{aligned}
 \left( \frac{y}{y_1} \right)' &= \frac{e^{-\int p(x)dx}}{y_1^2} = \frac{e^{\int \frac{x}{x-1}dx}}{e^{2x}} = \frac{e^{x+\ln|x-1|}}{e^{2x}} = e^{-x}(x-1) \\
 \frac{y}{y_1} &= \int e^{-x}(x-1) = -xe^{-x} \\
 y_2 &= -xe^{-x}y_1 = -x.
 \end{aligned}$$

$$y_h = C_1y_1 + C_2y_2 = C_1e^x + C_2x.$$

4. a)

$$\begin{aligned}
 y &= zy_1 = zx, y' = z + xz', y'' = 2z' + xz'' \\
 y'' - \frac{2+x}{x}y' + \frac{2+x}{x^2}y &= 0 \\
 xz'' + 2z' - \frac{2+x}{x}(z + xz') + \frac{2+x}{x^2}xz &= 0 \\
 xz'' - xz' &= 0 \\
 z' &= e^x \\
 z &= e^x \\
 y = zy_1 &= xe^x.
 \end{aligned}$$

b)

$$\begin{aligned}\left(\frac{y}{y_1}\right)' &= \frac{e^{-\int p(x)dx}}{y_1^2} = \frac{e^{\int \frac{2+x}{x} dx}}{x^2} = \frac{e^{2 \ln|x|+x}}{x^2} = e^x \\ \frac{y}{y_1} &= \int e^x dx = e^x \\ y_2 &= e^x y_1 = x e^x.\end{aligned}$$

$$y_h = C_1 y_1 + C_2 y_2 = C_1 x + C_2 x e^x.$$

II. 1. a)

$$\begin{aligned}\left(\frac{y}{y_1}\right)' &= \frac{e^{-\int p(x)dx}}{y_1^2} = \frac{e^{-\int 1 dx}}{e^{2x}} = e^{-3x} \\ \frac{y}{y_1} &= \int e^{-3x} dx = \frac{e^{-3x}}{-3} \\ y &= \frac{e^{-3x}}{-3} e^x = \frac{e^{-2x}}{-3} \\ y_h &= C_1 e^x + C_2 e^{-2x}.\end{aligned}$$

b)

$$\begin{aligned}y_p &= C_1(x)e^x + C_2(x)e^{-2x} \\ y_p' &= (C_1(x) + C_1'(x))e^x + (-2C_2(x) + C_2'(x))e^{-2x} \\ &\quad \text{(*přidáme podmínku } C_1'(x)e^x + C_2'(x)e^{-2x}) \\ y_p' &= C_1(x)e^x - 2C_2(x)e^{-2x} \\ y_p'' &= (C_1(x) + C_1'(x))e^x - 2(C_2'(x) - 2C_2(x))e^{-2x} \\ y'' + y' - 2y &= x \\ C_1'(x)e^x - 2C_2'(x)e^{-2x} &= x \\ C_1'(x)e^x + C_2'(x)e^{-2x} &= 0 \\ 3C_1'(x)e^x &= x \\ C_1(x) &= \frac{1}{3} \int x e^{-x} dx = \frac{1}{3} \left( -x e^{-x} - \int -e^{-x} dx \right) = \frac{-e^{-x}}{3} (x + 1) \\ C_2(x) &= \int -C_1'(x) e^{3x} dx = \int \frac{-1}{3} x e^{2x} dx = \frac{e^{2x}}{6} \left( -x + \frac{1}{2} \right) \\ y_p &= C_1(x)e^x + C_2(x)e^{-2x} = -\frac{x+1}{3} + \frac{1}{6} \left( -x + \frac{1}{2} \right) = -\frac{x}{2} - \frac{1}{4} \\ y &= y_h + y_p = C_1 e^x + C_2 e^{-2x} - \frac{x}{2} - \frac{1}{4}.\end{aligned}$$

2. a)

$$\begin{aligned}\left(\frac{y}{y_1}\right)' &= \frac{e^{-\int p(x)dx}}{y_1^2} = \frac{e^{-\int -\frac{4}{x}dx}}{x^4} = 1 \\ \frac{y}{y_1} &= x \\ y_2 &= x^3 \\ y_h &= C_1x^2 + C_2x^3.\end{aligned}$$

b)

$$\begin{aligned}y_p &= C_1(x)x^2 + C_2(x)x^3 \\ y_p' &= C_1'(x)x^2 + 2xC_1(x) + C_2'(x)x^3 + 3x^2C_2(x) \stackrel{(*)}{=} 2xC_1(x) + 3x^2C_2(x) \\ C_1'(x)x^2 + C_2'(x)x^3 &= 0 \quad (*) \\ y_p'' &= 2xC_1'(x) + 2C_1(x) + 3x^2C_2'(x) + 6xC_2(x) \\ y'' - \frac{4y'}{x} + \frac{6y}{x^2} &= x \\ 2xC_1'(x) + 3x^2C_2'(x) &= x \\ C_1'(x) + C_2'(x)x &= 0 \\ C_2(x) &= \int \frac{1}{x}dx = \ln|x| \\ C_1(x) &= \int -1dx = -x \\ y_p &= -x^3 + \ln|x|x^3 = x^3(\ln|x| - 1) \\ y &= y_h + y_p = C_1x^2 + C_2x^3 + x^3(\ln|x| - 1).\end{aligned}$$

III. 1.

$$y_h = C_1 + C_2 e^{2x}$$

$$y_p = C_1(x) + C_2(x)e^{2x}$$

$$y'_p = C'_1(x) + C'_2(x)e^{2x} + 2C_2(x)e^{2x} \stackrel{(*)}{=} 2C_2(x)e^{2x}$$

$$C'_1(x) + C'_2(x)e^{2x} = 0 \quad (*)$$

$$y''_p = 4C_2(x)e^{2x} + 2C'_2(x)e^{2x}$$

$$y'' - 2y' = e^{2x} \ln x$$

$$2C'_2(x)e^{2x} = e^{2x} \ln x$$

$$C_2(x) = \frac{1}{2} \int \ln x dx = \frac{1}{2}(\ln x - 1)x$$

$$C'_1(x) + C'_2(x)e^{2x} = 0$$

$$C_1(x) = \int -\frac{1}{2} \ln x e^{2x} dx$$

$$y_p = C_1(x) + C_2(x)e^{2x} = \frac{-1}{2} \int \ln x e^{2x} dx + \frac{1}{2}(\ln x - 1)x e^{2x}.$$

2.

$$y_h = e^x(C_1 + C_2 x)$$

$$y_p = e^x(C_1(x) + C_2(x)x)$$

$$y'_p = e^x(C'_1(x) + C_1(x) + C_2(x)x + C_2(x) + C'_2(x)x) \stackrel{(*)}{=} e^x(C_1(x) + C_2(x)(x+1))$$

$$C'_1(x) + C'_2(x)x = 0 \quad (*)$$

$$y''_p = e^x(C_1(x) + C'_1(x) + C_2(x)(x+1) + C_2(x) + C'_2(x)(x+1))$$

$$y'' - 2y' + y = e^x \ln x$$

$$C'_1(x) + C'_2(x)(x+1) = \ln x$$

$$C'_1(x) + C'_2(x)x = 0$$

$$C_2(x) = \int \ln x dx = x(\ln x - 1)$$

$$C_1(x) = \int -x \ln x = - \left[ \frac{x^2}{2} \ln x - \int \frac{x}{2} dx \right] = \frac{-x^2}{2} \left( \ln x - \frac{1}{2} \right)$$

$$y_p = e^x \left[ \frac{-x^2}{2} \left( \ln x - \frac{1}{2} \right) + x^2(\ln x - 1) \right] = e^x \left[ \frac{x^2}{2} \ln x - \frac{3}{4}x^2 \right].$$

Homogenní rovnice s konstantními koeficienty.

1.  $y'' + 4y' + 3y = 0$

2.  $y'' - 2y' + y = 0$

3.  $y'' + y = 0$

4.  $y'' + 2y' + 5y = 0$

5.  $y''' - 5y'' + 8y' - 4y = 0$

6.  $y''' - 4y'' + 7y' = 0$

7.  $y''' + 6y'' + 12y' + 8y = 0$

Řešení:

1.

$$y'' + 4y' + 3y = 0$$

$$\lambda^2 + 4\lambda + 3 = (\lambda + 3)(\lambda + 1) = 0$$

$$\lambda_1 = -3, \lambda_2 = -1$$

$$y_1 = e^{\lambda_1} = e^{-3x}, y_2 = e^{\lambda_2} = e^{-x}$$

$$y = C_1 y_1 + C_2 y_2 = C_1 e^{-3x} + C_2 e^{-x}.$$

2.

$$y'' - 2y' + y = 0$$

$$\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2 = 0$$

$$\lambda_{1,2} = 1$$

$$y_1 = e^{\lambda_1} = e^x, y_2 = x e^{\lambda_1} = x e^x$$

$$y = C_1 y_1 + C_2 y_2 = C_1 e^x + C_2 x e^x.$$

3.

$$y'' + y = 0$$

$$\lambda^2 + 1 = 0$$

$$\lambda_{1,2} = \pm i$$

$$y = e^{\lambda x} = e^{ix} = \cos x + i \sin x \Rightarrow y_1 = \sin x, y_2 = \cos x$$

$$y = C_1 y_1 + C_2 y_2 = C_1 \sin x + C_2 \cos x.$$

4.

$$y'' + 2y' + 5y = 0$$

$$\lambda^2 + 2\lambda + 5 = (\lambda + 1)^2 + 4 = 0$$

$$\lambda_{1,2} = -1 \pm 2i$$

$$y = e^{\lambda x} = e^{(-1+2i)x} = e^{-x}(\cos 2x + i \sin 2x) \Rightarrow y_1 = e^{-x} \sin 2x, y_2 = e^{-x} \cos 2x$$

$$y = C_1 y_1 + C_2 y_2 = C_1 e^{-x} \sin 2x + C_2 e^{-x} \cos 2x.$$

Zkouška pro  $y_1$

$$\begin{aligned}y_1' &= e^{-x}(-\sin 2x + 2 \cos 2x), y_1'' = e^{-x}(-3 \sin 2x - 4 \cos 2x) \\y_1'' + 2y_1' + 5y_1 &= e^{-x}[-3 \sin x - 4 \cos x + 2(-\sin 2x + 2 \cos 2x) + 5(\sin 2x)] = 0.\end{aligned}$$

5.

$$\begin{aligned}y''' - 5y'' + 8y' - 4y &= 0 \\ \lambda^3 - 5\lambda^2 + 8\lambda - 4 &= (\lambda - 1)(\lambda^2 - 4\lambda + 4) = (\lambda - 1)(\lambda - 2)^2 = 0 \\ \lambda_1 &= 1, \lambda_{2,3} = 2 \\ y_1 &= e^x, y_2 = e^{2x}, y_3 = xe^{2x} \\ y &= C_1y_1 + C_2y_2 + C_3y_3 = C_1e^x + C_2e^{2x} + C_3xe^{2x}.\end{aligned}$$

6.

$$\begin{aligned}y''' - 4y'' + 7y' &= 0 \\ \lambda^3 - 4\lambda^2 + 7\lambda &= \lambda[\lambda^2 - 4\lambda + 7] = \lambda[(\lambda - 2)^2 + 3] = 0 \\ \lambda_1 &= 0, \lambda_{2,3} = 2 \pm \sqrt{3}i \\ y_1 &= e^{0x} = 1, y_2 = e^{2x} \sin(\sqrt{3}x), y_3 = e^{2x} \cos(\sqrt{3}x) \\ y &= C_1y_1 + C_2y_2 + C_3y_3 = C_1 + C_2e^{2x} \sin(\sqrt{3}x) + C_3e^{2x} \cos(\sqrt{3}x).\end{aligned}$$

7.

$$\begin{aligned}y''' + 6y'' + 12y' + 8y &= 0 \\ \lambda^3 + 6\lambda^2 + 12\lambda + 8 &= (\lambda + 2)^3 = 0 \\ \lambda_{1,2,3} &= -2 \\ y_1 &= e^{-2x}, y_2 = xe^{-2x}, y_3 = x^2e^{-2x} \\ y &= C_1y_1 + C_2y_2 + C_3y_3 = e^{-2x}[C_1 + C_2x + C_3x^2].\end{aligned}$$

Zkouška

$$\begin{aligned}y &= e^{-2x}[C_1 + C_2x + C_3x^2] \\ y' &= e^{-2x}[-2C_1 + C_2(-2x + 1) + C_3(-2x^2 + 2x)] \\ y'' &= e^{-2x}[4C_1 + C_2(4x - 2 - 2) + C_3(4x^2 - 4x - 4x + 2)] = e^{-2x}[4C_1 + C_2(4x - 4) + C_3(4x^2 - 8x + 2)] \\ y''' &= e^{-2x}[-8C_1 + C_2(-8x + 8 + 4) + C_3(-8x^2 + 16x - 4 + 8x - 8)] = e^{-2x}[-8C_1 + C_2(-8x + 12) + C_3(-8x^2 + 24x - 12)] \\ y''' + 6y'' + 12y' + 8y &= e^{-2x}C_1(-8 + 24 - 24 + 8) + e^{-2x}C_2(-8x + 12 + 24x - 24 - 24x + 12 + 8x) + \\ &\quad + C_3(-8x^2 + 24x - 12 + 24x^2 - 48x + 12 - 24x^2 + 24x + 8x^2) = 0.\end{aligned}$$

Homogenní rovnice s konstantními koeficienty a speciální pravou stranou.

1.  $y'' + 4y' + 3y = e^{-3x}$
2.  $y'' - 2y' + y = x^2e^{-x}$
3.  $y'' + y = \cos x$
4.  $y'' + 2y' + 5y = \sin 2x + x$
5.  $y''' - 5y'' + 8y' - 4y = e^x$
6.  $y''' - 4y'' + 7y' = \sin(\sqrt{3}x)$
7.  $y''' + 6y'' + 12y' + 8y = x^2$

Řešení: Jelikož jsou příslušné homogenní rovnice převzaty z předchozího cvičení, tak se budeme zabývat pouze hledáním partikulárního řešení a použijeme výsledky z předchozího cvičení.

1. Jelikož  $e^{-3x}$  je řešením příslušné homogenní rovnice a  $\lambda = -3$  je jednonásobný kořen příslušného charakteristického polynomu, tak hledáme partikulární řešení ve tvaru  $y_p = Cxe^{-3x}$ .

$$\begin{aligned}y_p &= Cxe^{-3x} \\y_p' &= Ce^{-3x}(-3x + 1) \\y_p'' &= Ce^{-3x}(9x - 6) \\y'' + 4y' + 3y &= e^{-3x} \\e^{-3x}C(9x - 6 + 4(-3x + 1) + 3x) &= e^{-3x} \\C &= -\frac{1}{2} \\y_p &= -\frac{x}{2}e^{-3x} \\y &= y_h + y_p = C_1e^{-3x} + C_2e^{-x} - \frac{x}{2}e^{-3x}.\end{aligned}$$

2. Jelikož  $-1$  není kořenem příslušného charakteristického polynomu, tak hledáme partikulární řešení ve tvaru  $y_p = e^{-x}(ax^2 + bx + c)$ .

$$\begin{aligned}y_p &= e^{-x}(ax^2 + bx + c) \\y_p' &= e^{-x}(-ax^2 - bx - c + 2ax + b) = e^{-x}(-ax^2 + (2a - b)x + (b - c)) \\y_p'' &= e^{-x}(+ax^2 - (2a - b)x - (b - c) - 2ax + (2a - b)) = e^{-x}(ax^2 + (b - 4a)x + (2a - 2b + c))\end{aligned}$$



$$\begin{aligned}
y'' - 2y' + y &= x^2 e^{-x} \\
e^{-x}[ax^2 + (b - 4a)x + (2a - 2b + c) - 2(-ax^2 + (2a - b)x + (b - c)) + ax^2 + bx + c] &= x^2 e^{-x} \\
4ax^2 &= x^2 \\
(b - 4a - 4a + 2b + b)x &= (-8a + 4b)x = 0x \\
2a - 2b + c - 2b + 2c + c &= 2a - 4b + 4c = 0 \\
a = \frac{1}{4}, b = \frac{1}{2}, c = \frac{3}{8}
\end{aligned}$$

$$y = y_h + y_p = C_1 e^x + C_2 x e^x + e^{-x} \left( \frac{x^2}{4} + \frac{x}{2} + \frac{3}{8} \right).$$

3. Jelikož  $i$  je jednonásobný kořen charakteristického polynomu, hledáme řešení ve tvaru  $y_p = ax \sin x + bx \cos x$ .

$$\begin{aligned}
y_p &= ax \sin x + bx \cos x \\
y'_p &= x(a \cos x - b \sin x) + (a \sin x + b \cos x) \\
y''_p &= x(-a \sin x - b \cos x) + (2a \cos x - 2b \sin x) \\
y'' + y &= \cos x \\
x(-a \sin x - b \cos x) + (2a \cos x - 2b \sin x) + ax \sin x + bx \cos x &= \cos x \\
a = \frac{1}{2}, b = 0
\end{aligned}$$

$$y = y_h + y_p = C_1 \sin x + C_2 \cos x + \frac{1}{2} x \sin x.$$

4. Jelikož  $2i$  ani  $0$  není kořenem příslušného charakteristického polynomu, tak hledáme partikulární řešení ve tvaru  $y_p = a \sin(2x) + b \cos(2x) + cx + d$ .

$$\begin{aligned}
y_p &= a \sin 2x + b \cos 2x + cx + d \\
y'_p &= -2b \sin 2x + 2a \cos 2x + c \\
y''_p &= -4a \sin 2x - 4b \cos 2x
\end{aligned}$$

$$\begin{aligned}
y'' + 2y' + 5y &= \sin 2x + x \\
[-4a - 4b + 5a] \sin 2x + [-4b + 4a + 5b] \cos 2x + 5cx + (5d + 2c) &= \sin 2x + x \\
a - 4b &= 1 \\
4a + b &= 0 \\
5c &= 1 \\
(5d + 2c) &= 0 \\
a = \frac{1}{17}, b = \frac{-4}{17}, c = \frac{1}{5}, d = \frac{-2}{25}
\end{aligned}$$

$$y = y_h + y_p = C_1 e^{-x} \sin 2x + C_2 e^{-x} \cos 2x + \frac{1}{17} \sin 2x - \frac{4}{17} \cos 2x + \frac{x}{5} - \frac{2}{25}.$$

5. Jelikož 1 je jednonásobným kořenem příslušného charakteristického polynomu, tak hledáme partikulární řešení ve tvaru  $y_p = cxe^x$ .

$$\begin{aligned}y_p &= ce^x x \\y'_p &= ce^x(x+1) \\y''_p &= ce^x(x+2) \\y'''_p &= ce^x(x+3)\end{aligned}$$

$$\begin{aligned}y''' - 5y'' + 8y' - 4y &= e^x \\ce^x(x+3) - 5ce^x(x+2) + 8ce^x(x+1) - 4ce^x x &= e^x \\c(3 - 10 + 8) &= 1 \\c &= 1\end{aligned}$$

$$y = y_h + y_p = C_1 e^x + C_2 e^{2x} + C_3 x e^{2x} + x e^x.$$

6. Jelikož  $\sqrt{3}i$  není kořenem příslušného charakteristického polynomu, tak hledáme partikulární řešení ve tvaru  $y_p = a \sin(\sqrt{3}x) + b \cos(\sqrt{3}x)$ .

$$\begin{aligned}y_p &= a \sin(\sqrt{3}x) + b \cos(\sqrt{3}x) \\y'_p &= \sqrt{3}(-b \sin(\sqrt{3}x) + a \cos(\sqrt{3}x)) \\y''_p &= -3(a \sin(\sqrt{3}x) + b \cos(\sqrt{3}x)) \\y'''_p &= -3\sqrt{3}(-b \sin(\sqrt{3}x) + a \cos(\sqrt{3}x))\end{aligned}$$

$$\begin{aligned}y''' - 4y'' + 7y' &= \sin(\sqrt{3}x) \\-3\sqrt{3}(-b \sin(\sqrt{3}x) + a \cos(\sqrt{3}x)) + 12(a \sin(\sqrt{3}x) + b \cos(\sqrt{3}x)) + 7\sqrt{3}(-b \sin(\sqrt{3}x) + a \cos(\sqrt{3}x)) &= \sin(\sqrt{3}x) \\ \sin(\sqrt{3}x)[3\sqrt{3}b + 12a - 7\sqrt{3}b] + \cos(\sqrt{3}x)[-3\sqrt{3}a + 12b + 7\sqrt{3}a] &= \sin(\sqrt{3}x) \\12a - 4\sqrt{3}b &= 1 \\4\sqrt{3}a + 12b &= 0 \\a &= \frac{1}{16} \\b &= \frac{-\sqrt{3}}{48}\end{aligned}$$

$$y = y_h + y_p = C_1 + C_2 e^{2x} \sin(\sqrt{3}x) + C_3 e^{2x} \cos(\sqrt{3}x) + \frac{1}{16} \sin(\sqrt{3}x) + \frac{-\sqrt{3}}{48} \cos(\sqrt{3}x).$$

7. Jelikož 0 není kořenem příslušného charakteristického polynomu, tak hledáme partikulární řešení ve tvaru  $y_p = ax^2 + bx + c$ .

$$\begin{aligned}y_p &= ax^2 + bx + c \\y'_p &= 2ax + b \\y''_p &= 2a \\y'''_p &= 0\end{aligned}$$

$$y''' + 6y'' + 12y' + 8y = x^2$$

$$12a + 24ax + 12b + 8ax^2 + 8bx + 8c = x^2$$

$$8a = 1$$

$$24a + 8b = 0$$

$$12a + 12b + 8c = 0$$

$$a = \frac{1}{8}$$

$$b = -\frac{3}{8}$$

$$c = \frac{3}{8}$$

$$y = y_h + y_p = e^{-2x}[C_1 + C_2x + C_3x^2] + \frac{x^2 - 3x + 3}{8}.$$