Stejnoměrná konvergence.

Návod: $f_n(x) \rightrightarrows f(x)$ na I právě tehdy, když $\sigma_n = \sup\limits_{x\in I} |f_n(x) - f(x)| \to 0$.

1. $f_n(x) = x^n, \quad x \in (0,1)$,
2. $f_n(x) = x^n - x^{n+1}, \quad x \in [0,1]$,
3. $f_n(x) = x^n - x^{2n}, \quad x \in [0,1]$,
4. $f_n(x) = \frac{1}{x+n}, \quad x \in (0,\infty)$,
5. $f_n(x) = \frac{nx}{1+n+x}, \quad x \in [0,1]$,
6. $f_n(x) = \frac{x^n}{1+x^n}, \quad a)\ x \in [0,\frac{1}{2}],\ b)\ x \in [0,\infty)$,
7. $f_n(x) = \sqrt{x^2 + \frac{1}{n}}, \quad x \in \mathbb{R}$,
8. $f_n(x) = \frac{2nx}{1+n^2x^2}, \quad a)\ x \in [0,1],\ b)\ x \in [1,\infty)$,
9. $f_n(x) = n\left(\sqrt{x+\frac{1}{n}} - \sqrt{x}\right), \quad x \in (0,\infty)$,
10. $f_n(x) = \frac{\sin(nx)}{n}, \quad x \in \mathbb{R}$,
11. $f_n(x) = \sin\left(\frac{x}{n}\right), \quad x \in \mathbb{R}$,
12. $f_n(x) = e^{n(x-1)}, \quad x \in (0,1)$,
13. $f_n(x) = e^{-(x-n)^2}, \quad a)\ x \in (-l,l),\ b) x \in \mathbb{R}$,
14. $f_n(x) = \mathrm{arctg}(nx), \quad x \in (0,\infty)$,
15. $f_n(x) = x\mathrm{arctg}(nx), \quad x \in (0,\infty)$.

Řešení:

1.

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n = 0, \quad x \in (0,1)$$

$$\lim_{x\to 0+} |f_n(x) - f(x)| = \lim_{x\to 0+} x^n = 0, \quad \lim_{x\to 1-} |f_x(x) - f(x)| = \lim_{x\to 1-} x^n = 1$$

$$1 \le \sigma_n \nrightarrow 0 \Rightarrow \text{ konvergence není stejnoměrná.}$$

2.

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n - x^{n+1} = 0, \quad x \in [0,1],$$
$$\sigma_n = \sup_{x\in[0,1]} |x^n(1-x) - 0| = \sup_{x\in[0,1]} x^n(1-x)$$
$$\lim_{x\to 0+} |f_n(x) - f(x)| = \lim_{x\to 0+} x^n(1-x) = 0, \quad \lim_{x\to 1-} |f_n(x) - f(x)| = \lim_{x\to 1-} x^n(1-x) = 0$$
$$(x^n(1-x))' = nx^{n-1}(1-x) - x^n = x^{n-1}(n-(n+1)x) = 0 \Rightarrow x_1 = 0,\ x_2 = \frac{n}{n+1}$$
$$|f_n(x_2) - f(x_2)| = \left|\left(\frac{n}{n+1}\right)^n \left(1 - \frac{n}{n+1}\right) - 0\right| = \left(\frac{n}{n+1}\right)^n \cdot \frac{1}{n+1}$$
$$\lim_{n\to\infty} \sigma_n = \left(\frac{n}{n+1}\right)^n \cdot \frac{1}{n+1} = 0 \Rightarrow f_n(x) \rightrightarrows f(x).$$

3.

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n - x^{2n} = 0, \quad x \in [0,1],$$
$$\sigma_n = \sup_{x\in[0,1]} |x^n - x^{2n}|$$
$$(x^n - x^{2n})' = nx^{n-1} - 2nx^{2n-1} = x^{n-1}n(1-2x^n) = 0 \Rightarrow x_1 = 0,\ x_2 = \sqrt[n]{\frac{1}{2}}$$
$$|f_n(x_2) - f(x_2)| = \left|\frac{1}{2} - \frac{1}{4} - 0\right| = \frac{1}{4}$$
$$\frac{1}{4} \le \lim_{n\to\infty} \sigma_n \nrightarrow 0 \Rightarrow \text{ konvergence není stejnoměrná}.$$

4.

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{1}{x+n} = 0, \quad x \in (0,\infty),$$
$$\sigma_n = \sup_{x\in(0,\infty)} \left|\frac{1}{x+n}\right| = \frac{1}{n}$$
$$\lim_{n\to\infty} \sigma_n = \lim_{n\to\infty} \frac{1}{n} = 0 \Rightarrow f_n(x) \rightrightarrows f(x).$$

5.

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{nx}{1+n+x} = x, \quad x \in [0,1],$$
$$\sigma_n = \sup_{x\in[0,1]} |f_n(x) - f(x)| = \sup_{x\in[0,1]} \left|\frac{nx}{1+n+x} - x\right| = \sup_{x\in[0,1]} \left|\frac{-x-x^2}{1+n+x}\right|$$
$$= \sup_{x\in[0,1]} \frac{x+x^2}{1+n+x} \le \frac{1+1}{1+n+0} = \frac{2}{1+n}$$
$$\lim_{n\to\infty} \sigma_n \le \lim_{n\to\infty} \frac{2}{1+n} = 0 \Rightarrow f_n(x) \rightrightarrows f(x).$$

6. a)

$$f(x)=\lim_{n\to\infty} f_n(x)=\lim_{n\to\infty}\frac{x^n}{1+x^n}=0,\quad x\in[0,\frac{1}{2}],$$
$$\sigma_n=\sup_{x\in[0,\frac{1}{2}]}\left|\frac{x^n}{1+x^n}\right|\le\frac{(\frac{1}{2})^n}{1+0}=\frac{1}{2^n}$$
$$\lim_{n\to\infty}\sigma_n\le\lim_{n\to\infty}\frac{1}{2^n}=0\Rightarrow f_n(x)\rightrightarrows f(x).$$

b)

$$\begin{aligned}f(x)=\lim_{n\to\infty} f_n(x)=\lim_{n\to\infty}\frac{x^n}{1+x^n}&=0,\quad x\in[0,1),\\&=\frac{1}{2},\quad x=1,\\&=1,\quad x>1,\end{aligned}$$

$$\sigma_n=\sup_{x\in[0,\infty)}\left|\frac{x^n}{1+x^n}-f(x)\right|$$
$$\lim_{x\to1-}\left|\frac{x^n}{1+x^n}-f(x)\right|=\lim_{x\to1-}\left|\frac{x^n}{1+x^n}-0\right|=\frac{1}{2}$$
$$\frac{1}{2}\le\lim_{n\to\infty}\sigma_n\nrightarrow 0\Rightarrow \text{konvergence není stejnoměrná}.$$

7.

$$f(x)=\lim_{n\to\infty} f_n(x)=\lim_{n\to\infty}\sqrt{x^2+\frac{1}{n}}=|x|,\quad x\in\mathbb{R},$$
$$\sigma_n=\sup_{x\in\mathbb{R}}|f_n(x)-f(x)|=\sup_{x\in\mathbb{R}}\left|\sqrt{x^2+\frac{1}{n}}-|x|\right|=\sup_{x\in[0,\infty)}\left|\sqrt{x^2+\frac{1}{n}}-x\right|$$
$$=\sup_{x\in[0,\infty)}\left(\sqrt{x^2+\frac{1}{n}}-x\right)$$
$$f_n(0)-f(0)=\sqrt{0+\frac{1}{n}}-0=\frac{1}{\sqrt{n}}$$
$$\lim_{n\to\infty}\left(\sqrt{x^2+\frac{1}{n}}-x\right)=\lim_{n\to\infty}\left(\sqrt{x^2+\frac{1}{n}}-x\right)\frac{\sqrt{x^2+\frac{1}{n}}+x}{\sqrt{x^2+\frac{1}{n}}+x}=\lim_{x\to\infty}\frac{x^2+\frac{1}{n}-x^2}{x\left(\sqrt{1+\frac{1}{nx^2}}+1\right)}=0$$
$$\left(\sqrt{x^2+\frac{1}{n}}-x\right)'=\frac{x}{\sqrt{x^2+\frac{1}{n}}}-1=\frac{x-\sqrt{x^2+\frac{1}{n}}}{\sqrt{x^2+\frac{1}{n}}}\neq 0,$$
$$\lim_{n\to\infty}\sigma_n=\lim_{n\to\infty}\frac{1}{\sqrt{n}}=0\Rightarrow f_n(x)\rightrightarrows f(x).$$

8. a)

$$f(x)=\lim_{n\to\infty} f_n(x)=\lim_{n\to\infty}\frac{2nx}{1+n^2x^2}=0,\quad x\in[0,1],$$

$$\sigma_n=\sup_{x\in[0,1]}\left|\frac{2nx}{1+n^2x^2}\right|$$

$$f_n(0)=\frac{0}{1+0}=0,\quad f_n(1)=\frac{2n}{1+n^2},$$

$$\left(\frac{2nx}{1+n^2x^2}\right)'=\frac{2n(1+n^2x^2)-2xn^2\cdot 2nx}{(1+n^2x^2)^2}=\frac{2n(1-n^2x^2)}{(1+n^2x^2)^2}=0\Rightarrow x_1=\frac{1}{n}$$

$$f_n\left(\frac{1}{n}\right)=\frac{2n\frac{1}{n}}{1+n^2\frac{1}{n^2}}=1$$

$$\lim_{n\to\infty}\sigma_n=\lim_{n\to\infty}1\to 1\neq 0\Rightarrow \text{ konvergence není stejnoměrná}.$$

b)

$$f(x)=\lim_{n\to\infty} f_n(x)=\lim_{n\to\infty}\frac{2nx}{1+n^2x^2}=0,\quad x\in[1,\infty),$$

$$\sigma_n=\sup_{x\in[1,\infty)}\left|\frac{2nx}{1+n^2x^2}\right|$$

$$f_n(1)=\frac{2n}{1+n^2},\quad \lim_{x\to\infty}\frac{2nx}{1+n^2x^2}=0$$

$$x_1=\frac{1}{n}\notin[1,\infty)$$

$$\lim_{n\to\infty}\sigma_n=\lim_{n\to\infty}\frac{2n}{1+n^2}=0\Rightarrow f_n(x)\rightrightarrows f(x).$$

9.

$$f(x)=\lim_{n\to\infty} f_n(x)=\lim_{n\to\infty} n\left(\sqrt{x+\frac{1}{n}}-\sqrt{x}\right)=\lim_{n\to\infty}\frac{n(x+\frac{1}{n}-x)}{\sqrt{x+\frac{1}{n}}+\sqrt{x}}=\frac{1}{2\sqrt{x}},\quad x\in[0,\infty),$$

$$\sigma_n=\sup_{x\in[0,\infty)}\left|\frac{1}{\sqrt{x+\frac{1}{n}}+\sqrt{x}}-\frac{1}{2\sqrt{x}}\right|=\sup_{x\in[0,\infty)}\left|\frac{\sqrt{x}-\sqrt{x+\frac{1}{n}}}{2\sqrt{x}\left(\sqrt{x+\frac{1}{n}}+\sqrt{x}\right)}\right|$$

$$=\sup_{x\in[0,\infty)}\left|\frac{1-\sqrt{1+\frac{1}{nx}}}{2\left(\sqrt{x+\frac{1}{n}}+\sqrt{x}\right)}\right|\geq\lim_{x\to 0+}\left|\frac{1-\sqrt{1+\frac{1}{nx}}}{2\left(\sqrt{x+\frac{1}{n}}+\sqrt{x}\right)}\right|=\infty$$

$$\lim_{n\to\infty}\sigma_n=\lim_{n\to\infty}\infty\neq 0\Rightarrow \text{ konvergence není stejnoměrná}.$$

10.

$$f(x)=\lim_{n\to\infty}\frac{\sin(nx)}{n}=0,\quad x\in\mathbb{R},$$
$$\sigma_n=\sup_{x\in\mathbb{R}}\left|\frac{\sin(nx)}{n}\right|=\frac{1}{n}$$
$$\lim_{n\to\infty}\sigma_n=\lim_{n\to\infty}\frac{1}{n}=0\Rightarrow f_n(x)\rightrightarrows f(x).$$

11.

$$f(x)=\lim_{n\to\infty}\sin\left(\frac{x}{n}\right)=0,\quad x\in\mathbb{R},$$
$$\sigma_n=\sup_{x\in\mathbb{R}}\left|\sin\left(\frac{x}{n}\right)\right|$$
$$\left(\sin\left(\frac{x}{n}\right)\right)'=\cos\left(\frac{x}{n}\right)\cdot\frac{1}{n}=0\Rightarrow x_1=\frac{\pi+k\pi}{2}n$$
$$\sigma_n=\left|\sin\left(\frac{\pi}{2}\cdot n\cdot\frac{1}{n}\right)\right|=1$$
$$\lim_{n\to\infty}\sigma_n=\lim_{n\to\infty}1=1\neq 0\Rightarrow \text{ konvergence není stejnoměrná}.$$

12.

$$f(x)=\lim_{n\to\infty}e^{n(x-1)}=0,\quad x\in(0,1),$$
$$\sigma_n=\sup_{x\in(0,1)}\left|e^{n(x-1)}\right|$$
$$\lim_{x\to 1-}\left|e^{n(x-1)}\right|=1$$
$$1\leq\sigma_n\Rightarrow\lim_{n\to\infty}\sigma_n\neq 0\Rightarrow \text{ konvergence není stejnoměrná}.$$

13. a)

$$f(x)=\lim_{n\to\infty}f_n(x)=\lim_{n\to\infty}e^{-(x-n)^2}=0,\quad x\in(-l,l),$$
$$\sigma_n=\sup_{x\in(-l,l)}\left|e^{-(x-n)^2}\right|$$
$$\lim_{x\to -l_+}\left|e^{-(x-n)^2}\right|=e^{-(l+n)^2},\quad \lim_{x\to l_-}|e^{-(x-n)^2}=e^{-(l-n)^2}$$
$$\left(e^{-(x-n)^2}\right)'=-2e^{-(x-n)^2}(x-n)=0\Rightarrow x_1=n\notin(-l,l)\text{ pro } n \text{ dostatečně velké}$$
$$\lim_{n\to\infty}\sigma_n=\lim_{n\to\infty}e^{-(l-n)^2}=0\Rightarrow f_n(x)\rightrightarrows f(x).$$

b)

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} e^{-(x-n)^2} = 0, \quad x \in \mathbb{R},$$

$$\sigma_n = \sup_{x\in\mathbb{R}} \left| e^{-(x-n)^2} \right|$$

$$\left(e^{-(x-n)^2}\right)' = -2e^{-(x-n)^2}(x-n) = 0 \Rightarrow x_1 = n$$

$$\lim_{x\to\pm\infty} e^{-(x-n)^2} = 0, \quad \sigma_n = \left| e^{-(n-n)^2} \right| = 1$$

$$\lim_{n\to\infty} \sigma_n = \lim_{n\to\infty} 1 = 1 \neq 0 \Rightarrow \text{ konvergence není stejnoměrná}.$$

14.

$$f(x) = \lim_{n\to\infty} \operatorname{arctg}(nx) = \frac{\pi}{2}, \quad x \in (0,\infty),$$

$$\sigma_n = \sup_{x\in(0,\infty)} \left| \operatorname{arctg}(nx) - \frac{\pi}{2} \right|$$

$$\lim_{x\to 0_+} \left| \operatorname{arctg}(nx) - \frac{\pi}{2} \right| = \frac{\pi}{2}$$

$$\frac{\pi}{2} \leq \lim_{n\to\infty} \sigma_n \neq 0 \Rightarrow \text{ konvergence není stejnoměrná}.$$

15.

$$f(x) = \lim_{n\to\infty} x\operatorname{arctg}(nx) = \frac{\pi}{2}x, \quad x \in (0,\infty),$$

$$\sigma_n = \sup_{x\in(0,\infty)} \left| x\left(\operatorname{arctg}(nx) - \frac{\pi}{2}\right) \right|$$

$$\lim_{x\to 0_+} \left| x\left(\operatorname{arctg}(nx) - \frac{\pi}{2}\right) \right| = 0,$$

$$\lim_{x\to\infty} \left| x\left(\operatorname{arctg}(nx) - \frac{\pi}{2}\right) \right| = \lim_{x\to\infty} \frac{\left(\frac{\pi}{2} - \operatorname{arctg}(nx)\right)}{\frac{1}{x}} \overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{-n}{1+n^2x^2}}{-\frac{1}{x^2}} = \frac{1}{n},$$

$$\left(x\left(\operatorname{arctg}(nx) - \frac{\pi}{2}\right)\right)' = \operatorname{arctg}(nx) - \frac{\pi}{2} + \frac{nx}{1+n^2x^2} \overset{?}{=} 0$$

Jelikož rovnici $\operatorname{arctg}(nx) - \frac{\pi}{2} + \frac{nx}{1+n^2x^2} = 0$ neumíme řešit, je třeba se na tento problém podívat jinak. Platí

$$\lim_{x\to 0_+} \operatorname{arctg}(nx) - \frac{\pi}{2} + \frac{nx}{1+n^2x^2} = -\frac{\pi}{2},$$

$$\lim_{x\to\infty} \operatorname{arctg}(nx) - \frac{\pi}{2} + \frac{nx}{1+n^2x^2} = 0 \text{ a}$$

$$\left(\operatorname{arctg}(nx) - \frac{\pi}{2} + \frac{nx}{1+n^2x^2}\right)' = \frac{n}{1+n^2x^2} + \frac{n(1+n^2x^2) - 2n^3x^2}{(1+n^2x^2)^2}$$

$$= \frac{2n}{(1+n^2x^2)^2} > 0.$$

Tedy funkce $\operatorname{arctg}(nx) - \frac{\pi}{2} + \frac{nx}{1+n^2x^2}$ je rostoucí a jelikož $\lim_{x\to\infty} \operatorname{arctg}(nx) - \frac{\pi}{2} + \frac{nx}{1+n^2x^2} = 0$, tak $\operatorname{arctg}(nx) - \frac{\pi}{2} + \frac{nx}{1+n^2x^2} \neq 0$. Tedy

$$\lim_{n\to} \sigma_n = \lim_{n\to\infty} \frac{1}{n} = 0 \Rightarrow f_n(x) \rightrightarrows f(x).$$