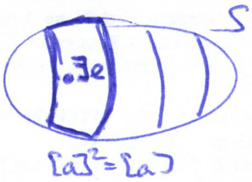
Lallement's lemma:

S regular, $\alpha \in Con(S)$, $[a]_\alpha \in E_{S/\alpha} \implies \exists e \in E_S$ s.t. $[a]_\alpha = [e]_\alpha$ & $e \leq_{\mathcal{H}} a$



Proof: assume $[a]^2 = [a]$, i.e., $a^2 \alpha a$

fix $b \in V(a^2)$, $e := aba$

... $e^2 = \underbrace{aba \cdot aba}_{} = aba = e$

... $e = aba \;\alpha\; a^2 b a^2 = a^2 \;\alpha\; a$

... $e \leq_{\mathcal{H}} a$ obviously $\square$

Corollary: S inverse, $\underbrace{\alpha \in Con(S)}_{\text{congruence w.r.t. } \circ} \implies S/\alpha$ inverse & $a \alpha b \Rightarrow a' \alpha b'$

Proof: S/α regular? $[a][a'][a] = [aa'a] = [a]$

$[a'][a][a'] = [a'aa'] = [a']$

$\implies [a'] \in V([a])$.

$E_{S/\alpha}$ commutative? take $[a], [b] \in E_{S/\alpha}$

$\overset{\text{Lallement}}{\implies}$ $[a] = [e]$, $[b] = [f]$ for some $e, f \in E_S$

$\implies [a][b] = [e][f] = [ef] = [fe] = [f][e] = [b][a]$

Hence: S/α is inverse

& uniquenes of inverses yields $[a]' = [a']$ $\square$