Completely regular semigroups

Recall: inverse smg ≡ every $\mathcal{L}$-class, $\mathcal{R}$-class exactly 1 idempotent

i.e., $|L_a \cap E_S| = |R_a \cap E_S| = 1 \quad \forall a$

i.e., eggbox $\leftrightarrow$ permutation matrices

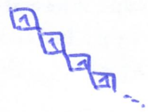
Now: completely regular smg ≡ every $\mathcal{H}$-class exactly 1 idempotent

i.e., $|H_a \cap E_S| = 1 \quad \forall a$

i.e., eggbox $\leftrightarrow$ matrices of 1's

i.e., union of groups

(inverse & completely regular ≡ Clifford smg

≡ eggbox )

Ex.: • groups, semilattices (are Clifford)
• idempotent semigroups

Aim: to show that any c.reg. can be "glued" from groups and projections over a semilattice

Prop.: a regular $\Rightarrow$ TFAE:

(1) $\exists b \in As(a) \quad ab = ba$

(2) $\exists!\, a' \in V(a) \quad aa' = a'a$

(3) $H_a \cap E_S \neq \emptyset$

(4) H_a is a group

def.: completely regular element

Proof: (1) $\Rightarrow$ (2) $\quad a' := bab \underbrace{\in V(a)}_{\text{see earlier}}$

$$\Rightarrow \quad aa' = abab \overset{(1)}{=} baba = a'a$$

if $aa^* = a^*a$, $a^* \in V(a)$, then:

$$a^* = a^*aa^* = a(a^*)^2 = aa'aa'a\,(a^*)^2$$
$$= (a')^2\, a^3\, (a^*)^2$$

$$a' = a'aa' = (a')^2 a = (a')^2 aa^*aa^*a$$