Theorem: comp. regular $\Longleftrightarrow$ semilattice of completely simple

$$\text{completely simple} \;\; \text{III} \;\; \& \begin{cases}\text{comp. regular}\\ \text{ideal-simple}\end{cases}$$

Proof: ($\Leftarrow$) $S = \bigcup$ of $\underbrace{\text{comp. reg. smg's}}_{\forall a \text{ is c.r.} \;\equiv\; \exists b \in As(a)\; ab=ba}$

$\underbrace{\qquad\qquad\qquad\qquad}_{\forall a \text{ is c.r.}}$

($\Rightarrow$) decomposition by $\mathcal{J}$

need to prove:
- $\mathcal{J}$ is a congruence
- $S/\mathcal{J}$ is a semilattice
- blocks of $\mathcal{J}$ are comp. simple $\boxtimes$

Ex. of completely simple semigroups:

- groups
- projection smg's $(\underset{\text{right}}{xy=y}, \; \underset{\text{left}}{xy=x})$
- Rees matrix semigroup $M(\underset{\text{group}}{G}; \underbrace{I, \Lambda}_{\text{index sets } (\neq\emptyset)}, P)$ — $\Lambda \times I$-matrix over G, $P = (p_{\lambda i})$ "sandwich matrix"

elements: $I \times G \times \Lambda$

operation: $(i, a, \lambda) \circ (j, b, \mu) := (i, a p_{\lambda j} b, \mu)$

... why matrix semigroup?

$$(i, a, \lambda) \rightsquigarrow \begin{array}{cc} \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & a & \vdots \\ 0 & \cdots & 0 \end{pmatrix} & \lambda \\ i & \end{array} =: (a)_{\lambda i}$$

operation: $(a)_{\lambda i} \circ (b)_{\mu j} := \underbrace{(a)_{\lambda i} \cdot P \cdot (b)_{\mu j}}_{\text{matrix multiplication}}$