Some consequences: (in an arbitrary semigroup)

- $\mathcal{H}$-classes contain ≤ 1 idempotent
- maximal subgroups are precisely $\mathcal{H}$-classes w/ idempotent
- max. subgroups are disjoint
- S idempotent $\Rightarrow$ $\mathcal{H}$-classes have 1 elt., i.e., $\mathcal{H} = id$

  S group $\Rightarrow$ $\mathcal{H} = G \times G$

Regularity

a regular $\equiv$ the following equivalent conditions hold:

(1) $\exists b \quad aba = a$

(2) $\exists a' \quad aa'a = a \;\&\; a'aa' = a'$

(3) $L_a \cap E_S \neq \emptyset$

(4) $R_a \cap E_S \neq \emptyset$

Proof: (1) $\Rightarrow$ (2) $\quad a' := bab$

(2) $\Rightarrow$ (1) $\quad$ trivial

(1) $\Rightarrow$ (3) $\quad e := ba$

(3) $\Rightarrow$ (1) $\quad e \in L_a \cap E_S \Rightarrow \exists b \; ba = e \Rightarrow aba = ae = a$ (Lemma 11)

(4) dually

$Reg(S) := \{a \in S : a \text{ regular}\}$

S regular $\equiv$ $Reg(S) = S$

Ex.:
- idempotent semigroups
- groups
- T_X (see also eggbox for T_3)

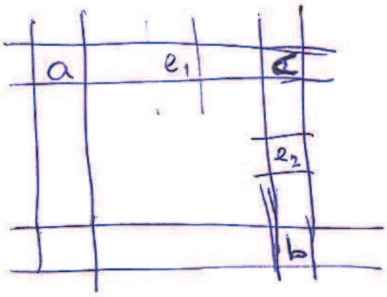
(*) NOTE: all (1)–(4) hold for a nice reason

~~a regular $\Leftrightarrow$ $A_S(a) = \{b : aba = a\}$ associates of a~~

~~$V(a) = \{a' : aa'a = a \;\&\; a'aa' = a'\}$ inverses of a~~

Prop.: $Reg(S)$ is union of $\mathcal{D}$-classes, i.e., $\left.\begin{array}{l} a \in Reg(S) \\ a \mathcal{D} b \end{array}\right\} \Rightarrow b \in Reg(S)$

Proof:



$a \mathcal{D} b \overset{\text{Prop.}}{\Longleftrightarrow} \exists c \quad a \mathcal{R} c \mathcal{L} b$

a regular $\Rightarrow \exists e_1 \in R_a \cap E_S = R_c \cap E_S$

$\Rightarrow$ c regular

$\Rightarrow \exists e_2 \in L_c \cap E_S = L_b \cap E_S$

$\Rightarrow$ b regular □

(*)
- ◉ S regular $\Rightarrow S/\alpha$ regular
- ◉ S_i regular $\Rightarrow \prod S_i$ regular
- ◉ S regular, $T \leq S \not\Rightarrow$ T regular
  ... $(\mathbb{N},+) \leq (\mathbb{Z},+)$

$As(a) := \{ b : aba = a \}$ ... associates of a

$V(a) := \{ a' : aa'a = a,\ a'aa' = a' \}$ ... inverses of a

◉ $V(a) \subseteq As(a)$, $\quad As(a) \to V(a)$
$\qquad b \mapsto bab$

Lemma:
◉ $As(a) \to E_S \cap L_a$ is onto
$\qquad b \mapsto ba$

Proof: $e \in E_S \cap L_a \Rightarrow$ $ae = a$, $\exists x\ xa = e$, $b := xax$

$ba = xaxa = ee = e$ ✓

? $b \in V(a)$?

$aba = axaxa = ae^2 = ae = a$

$bab = xaxaxax = e^3x = ex = xax = b$ □

~~◉ V(a) → $E_S \cap L_a$ is onto~~

~~b ↦ ba~~

HENCE: $|E_S \cap L_a| \leq |V(a)| \leq |As(a)|$ (a regular = all ≠ 0)
(R_a)

Prop.: a regular $\Rightarrow \left[\; |V(a)| = 1 \iff \begin{array}{l} |E_S \cap L_a| = 1 \\ \& \; |E_S \cap R_a| = 1 \end{array} \right]$

Proof: $(\Rightarrow)$ from ⓘⓘⓘ above

$(\Leftarrow)$ let $\grave{a}, a^* \in V(a)$, $E_S \cap L_a = \{e\}$, $E_S \cap R_a = \{f\}$ $\Big\}$ $\grave{a}a = a^*a = e$, $a\grave{a} = aa^* = f$ (by $V(a) \Rightarrow \dots$ $b \mapsto ba$)

$$\Downarrow$$

$$\grave{a} = \grave{a}a\grave{a} = a^*a\grave{a} = a^*aa^* = a^* \qquad \square$$

Prop.: a, b regular $\Rightarrow$

$a \mathcal{L} b \overset{(1)}{\iff} \forall \grave{a} \in V(a)\ \exists \grave{b} \in V(b) \quad \grave{a}a = \grave{b}b$

$\overset{(2)}{\iff} \exists \grave{a} \in V(a)\ \exists \grave{b} \in V(b) \quad \grave{a}a = \grave{b}b$

Proof:

$a \mathcal{L} b \Rightarrow (1)$: $a = xb$, $b = ya$ $\Big\}$ $\grave{a}a = \grave{a}xb$

$\grave{b} := \grave{a}x \qquad ?\ \grave{b} \in V(b)\ ?$

$$b\grave{b}b = b\grave{a}xb = ya\grave{a}a = ya = b$$

$$\grave{b}b\grave{b} = \grave{a}xb\grave{a}x = \grave{a}a\grave{a}x = \grave{a}x = \grave{b}$$

$(1) \Rightarrow (2)$ trivial

$(2) \Rightarrow a\mathcal{L}b$: $\grave{a}a = \grave{b}b \Rightarrow a = a\grave{a}a = a\grave{b}b$

$\Rightarrow b\grave{a}a = b\grave{b}b = b$ $\Big\}$ $a \mathcal{L} b \qquad \square$

Inverse semigroups

$\equiv \; \forall a \; \exists! \, \grave{a} \quad a\grave{a}a = a \;\&\; \grave{a}a\grave{a} = \grave{a}$

i.e., $|V(a)| = 1 \quad \forall a$

Ex.: • groups, semilattices
• I_X = partial bijections $= \{ f \in PT_X : f \text{ 1-1} \} \leq PT_X$

... (not easy to prove !! (later)

Note: $(a')' = a$, $(ab)' = b'a'$ ... —— " —— either !! (later)

Theorem: TFAE: (1) inverse

(2) every $\mathcal{L}$-class contains exactly one idempotent & every $\mathcal{R}$-class —— " ——

(3) regular & all idempotents commute

(hence $E_S \leq S$
... $ef \cdot ef \overset{\text{commut.}}{=} ee \cdot ff = ef$)

Proof: (1) $\Leftrightarrow$ (2) see Prop. on (10)

(3) $\Rightarrow$ (2) at least 1 ... from regularity

at most 1 ... $e \mathcal{L} f \Rightarrow e = ef = fe = f$ (Lemma U)

(2) $\Rightarrow$ (3) $e, f \in E_S \quad z := (ef)'$ (the unique inverse)

Claim 1: $fze = z$

... $(ef)(fze)(ef) = efzef = ef$
$(fze)(ef)(fze) = fzefze = fze$
} fze is inverse to ef $\Rightarrow fze = z$

Claim 2: $ef \in E_S$

... $zz = fzefze = fze = z$

$\Rightarrow z$ is inverse for z
ef —— " —— z
} $ef = z \in E_S$

☉ ... $u \in E_S \Rightarrow u' = u$!

Claim 3: $fe \in E_S$

... dually

$$\Rightarrow \begin{rcases} (ef)(fe)(ef) = efef = ef \\ (fe)(ef)(fe) = fefe = fe \end{rcases} ef = fe$$

□

Basic properties of inverse semigroups:

(1) $(ab)' = b'a'$

(2) $(a')' = a$

(3) $e \in E_S \Rightarrow aea' \in E_S$ & $E_S \leq S$

(4) $a \mathcal{L} b \iff a'a = b'b$

$a \mathcal{R} b \iff aa' = bb'$

(5) $e, f \in E_S,\ e \mathcal{D} f \Rightarrow \exists a\ \ a'a = e$ & $aa' = f$

(in particular, for $e = f \in E_S$: $\exists a\ \ e = a'a = aa'$)

Proof: (1) $\underbrace{(ab)(b'}_{\in E_S}\underbrace{a')(ab)}_{\in E_S} \overset{\text{comm.}}{=} aa'abb'b = ab$

$(b'a')(ab)(b'a') \overset{\text{com}}{=} b'a'$

$\Rightarrow$ (unique) $(ab)' = b'a'$

(2) a is inverse for a' $\Rightarrow (a')' = a$

(3) $e, f \in E_S \Rightarrow ef \cdot ef \overset{\text{com}}{=} eeff = ef$

$\Rightarrow ae\underbrace{a'a}_{\in E_S}ea' = \underline{aa'a}\,\underline{ee}a' = aea'$

(4) see Prop. on ⑩

(5) $\exists a\ \ e \mathcal{L} a \mathcal{R} f$

$\Downarrow(4)$ $\Downarrow(4)$

$a'a = ee = e \qquad aa' = ff = f$

□

→ EQUATIONAL BASE

Coro: $(S, \cdot, ')$ is an inverse semigroup $\iff$
- $x(yz) = (xy)z$
- $xx'x = x,\ x'xx' = x'$
- $(xx')(yy') = (yy')(xx')$

$\forall x, y, z$

Proof: ($\Rightarrow$) use Thm (3) & property (5) ($\Leftarrow$) HW ⊛

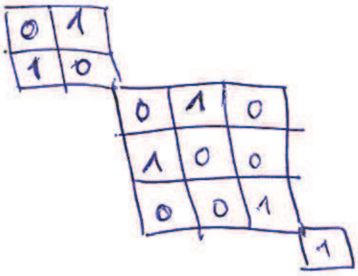
⊛ Hint: prove that if $e \in E_S$ then $e = ee'$. Hence idempotents commute.

Eggbox for inverse semigroups: ... every row / column exactly one idempotent

$\rightsquigarrow$ ~~put~~ $<^{0}_{1}$ # of idp. in an $\mathcal{H}$-class

$\rightsquigarrow$ permutation matrices $\hat{=}$ $\mathcal{D}$-classes

Ex.:



Prop.: I_X is an inverse semigroup

Proof: $I_X = \{\alpha : A \to B \text{ bijective} : A, B \subseteq X\}$

$$\alpha' := \alpha^{-1} : B \to A$$

we have $\left.\begin{array}{l} \alpha\alpha' = id_B \\ \alpha'\alpha = id_A \end{array}\right\} \Rightarrow \begin{array}{l} \alpha'\alpha\alpha' = \alpha' \\ \alpha\alpha'\alpha = \alpha \end{array} \Rightarrow$ regular

... ? unique ? ... difficult

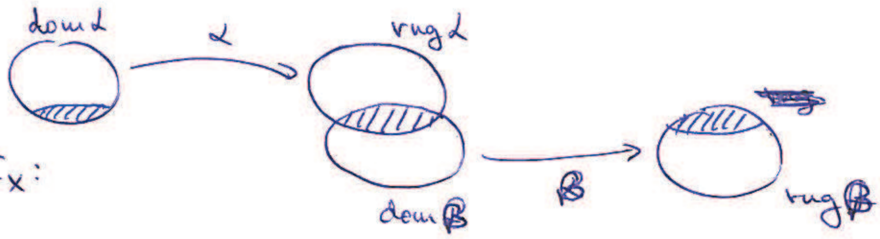
... ? idempotents commute ?

Claim: $\alpha\alpha = \alpha \iff \exists A \subseteq X \quad \alpha = id_A$

$\hookrightarrow$ Then it is trivial: $id_A \, id_B = id_{A \cap B} = id_B \, id_A$ $\square$

Pf. of Claim:

Consider $\alpha, \beta \in I_X$:



$\rightsquigarrow$ (o) $\quad \operatorname{dom} \beta\alpha = \alpha^{-1}(\operatorname{rng}\alpha \cap \operatorname{dom}\beta)$

$\quad \operatorname{rng} \beta\alpha = \beta(\operatorname{rng}\alpha \cap \operatorname{dom}\beta)$

($\Rightarrow$) We ~~want~~ assume $\alpha = \beta$ & $\alpha\alpha = \alpha$, hence

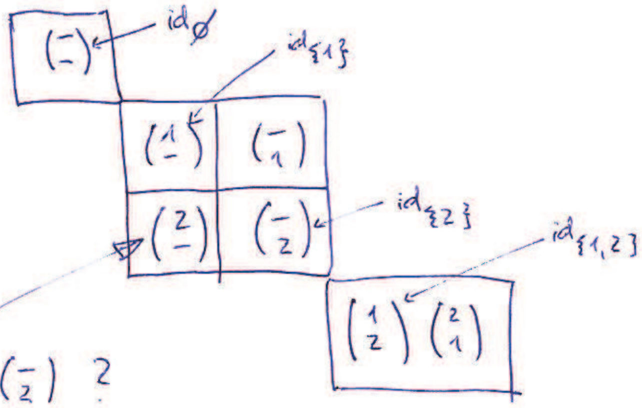
$\alpha^{-1}(\operatorname{rng}\alpha) = \operatorname{dom}\alpha = \operatorname{dom}\alpha\alpha = \alpha^{-1}(\operatorname{rng}\alpha \cap \operatorname{dom}\alpha)$

$\alpha(\operatorname{dom}\alpha) = \operatorname{rng}\alpha = \operatorname{rng}\alpha\alpha = \alpha(\ \text{—} \cap \text{—}\ )$

$\overset{\alpha \ 1-1}{\Longrightarrow} \quad \operatorname{rng}\alpha = \operatorname{rng}\alpha \cap \operatorname{dom}\alpha = \operatorname{dom}\alpha$

$\Rightarrow \alpha$ is permutation on $A \overset{\alpha\alpha = \alpha}{\Longrightarrow} \alpha = id_A$ $\square$

Ex: $I_2 =$



why $\binom{1}{-} \mathcal{L} \binom{2}{-} \mathcal{R} \binom{-}{2}$?

$\mathcal{L}: \quad \binom{2}{-}'\binom{2}{-} = \binom{-}{1}\binom{2}{-} = \binom{1}{-}$

$\mathcal{R}: \quad \binom{2}{-}\binom{2}{-}' = \binom{2}{-}\binom{-}{1} = \binom{-}{2}$

Vagner - Preston theorem: S inverse smg. $\Rightarrow$ $S \hookrightarrow I_S$

1952 USSR, 1954 USA

$$a \mapsto \lambda_a \restriction_{a'S}$$

Ex.: groups ... $a'S = S$, hence $S \hookrightarrow$ symetric group over S, $a \mapsto \lambda_a$ (Cayley rep.)

Ex.: $\wedge$-semilattices ...
- $a'S = aS = \downarrow a = \{x : x \leq a\}$
- a idempotent $\Rightarrow$ $\varphi(a)$ idempotent for any φ hom.

  $\Rightarrow \lambda_a \restriction_{\downarrow a} = id_{\downarrow a}$

  (indeed, $\lambda_a(x) = a \wedge x = x \quad \forall x \leq a$)
- $id_A \circ id_B = id_{A \cap B}$

Hence, in fact, $S \hookrightarrow (\mathcal{P}(S), \cap)$ ... semilattice of subsets

$$a \mapsto \downarrow a$$

Proof: $\sigma_a := \lambda_a \restriction_{a'S}$, denote φ the mapping $a \mapsto \sigma_a$

1) $\sigma_a : a'S \to aS$ onto ... $\lambda_a(a's) = aa's \in aS$

$aS \ni as = aa'as = \lambda_a(\underbrace{a'as}_{\in a'S})$

2) $\sigma_a \in I_S$ (i.e., it is 1-1) ... $\sigma_a(x) = \sigma_a(y)$ for some $\underbrace{x, y \in a'S}_{x = a's,\ y = a't}$

$$\Rightarrow \quad a a' s = a a' t$$

$$\Rightarrow \quad a' a a' s = a' a a' t$$

$$x = a's \qquad\qquad a't = y$$

3) <u>φ is 1-1</u>: $\sigma_a = \sigma_b \Rightarrow \operatorname{dom}\sigma_a = \operatorname{dom}\sigma_b$, where $\operatorname{dom}\sigma_a = a'S$ and $\operatorname{dom}\sigma_b = b'S$

$$\Rightarrow a' \mathcal{R} b'$$

$$\overset{\text{Prop. (4)}}{\Longleftrightarrow} \quad a'a'' = b'b'', \quad a'a'' = a'a, \quad b'b'' = b'b \quad \in a'S = b'S$$

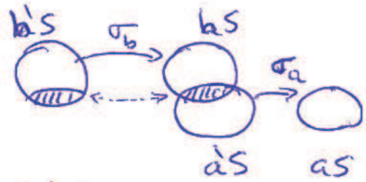
$$\Rightarrow \sigma_a(a'a) = \sigma_b(b'b), \qquad \sigma_a(a'a) = aa'a = a, \qquad \sigma_b(b'b) = bb'b = b$$

4) <u>φ is hom.</u>: ? $\sigma_a \circ \sigma_b = \sigma_{a\cdot b}$?

4a) Indeed $\lambda_a \circ \lambda_b = \lambda_{a\cdot b}$, since $\forall x \in S$: $\lambda_a(\lambda_b(x)) = a(bx)$, $\lambda_{a\cdot b}(x) = (ab)x$ ✓

4b) $\operatorname{dom} \sigma_a \circ \sigma_b \overset{?}{=} \operatorname{dom}\sigma_{a\cdot b} = (ab)'S$?

$$\operatorname{dom}\sigma_a\sigma_b = \sigma_b^{-1}(bS \cap a'S) \quad \dots$$



$$= \sigma_b^{-1}(bb'S \cap a'aS)$$

$$\overset{(*)}{=} \sigma_b^{-1}((bb')(a'a)S)$$

$$\overset{(\Delta)}{=} \sigma_{b'}((bb')(a'a)S)$$

$$= b'bb'a'aS = b'a'aS = b'a'S = (ab)'S \quad ✓$$

$bS = bb'S$ because $b = bb'b$

$a'S = a'aS$ because $a' = a'aa'$

(*) $eS \cap fS = efS \quad \forall e,f \in E_S$:

($\supseteq$) $efS \subseteq eS$, $feS \subseteq fS$

($\subseteq$) $z \in eS \cap fS$

$\Rightarrow z = ex = fy$

$\Rightarrow efz = effy = efy = ez = eex = ex = z$

$\Rightarrow z \in efS$

(Δ) ? $\sigma_b\sigma_{b'} = id_{bS}$, $\sigma_{b'}\sigma_b = id_{b'S}$?

$\sigma_{b'}: b''S = bS \to b'S$

$\Rightarrow$ dom, rng OK

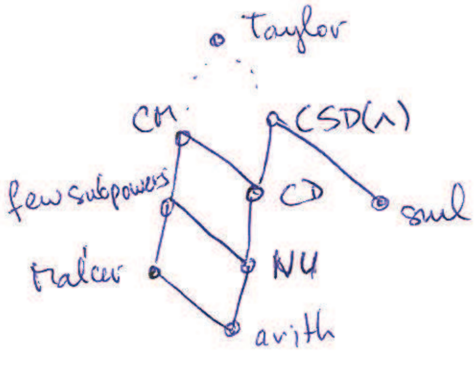
$\sigma_b\sigma_{b'}(bs) = bb'bs = bs$ ✓

$\sigma_{b'}\sigma_b(b's) = b'bb's = b's$ ✓

□

Inverse semigroups & Mal'tsev conditions

Ex: groups — are ~~[illegible]~~ Mal'cev, no sml. term, not CSD(∧) [HW]
semilattices — sml. term, not CM [HW]



Taylor term: idemp. & $t\begin{pmatrix} x & & \\ & \ddots & \\ & & x \end{pmatrix} = t\begin{pmatrix} y & & \\ & \ddots & \\ & & y \end{pmatrix}$

$$t(xyzuv) := xy^{-1}zu^{-1}v$$

... idemp.: $xx^{-1}xx^{-1}x = xx^{-1}x = x$

.. $t(xxyyx) = t(yyxxx)$ ... ~~[illegible]~~ for coord. 1..4

$t(xxxyy) = t(xyyxx)$ ... coord. 3..5

Congruences of inverse semigroups $\longleftrightarrow$ (normal subsemigroup, normal congruence on E_S)

α congruence on S (inverse) $\leadsto$ <u>trace</u> $tr\,\alpha = \alpha \cap E_S^2$

<u>kernel</u> $Ker\,\alpha = \bigcup_{e \in E_S} [e]_\alpha$

$\Rightarrow$ <u>Ex.:</u> groups: $tr\,\alpha$ triv., $Ker\,\alpha = [1]$ // semilattices: $tr\,\alpha = \alpha$, $Ker\,\alpha = E_S$

<u>def.:</u> $T \leq S$ is called <u>full</u> $\equiv$ $E_T = E_S$

<u>self-conjugate</u> $\equiv$ $\forall t \in T\ \forall a \in S\ \ ata' \in T$

<u>normal</u> $\equiv$ full, self-conjugate, inverse

<u>Lemma:</u> $Ker\,\alpha$ is normal

<u>Pf:</u> full ✓

$Ker\,\alpha \leq S$? $\quad a \in [e], b \in [f] \to ab \in [ef]$
$\quad a\alpha e \quad b\alpha f \quad ab\,\alpha\, \underbrace{ef}_{\in E_S}$

self-conjugate? $\quad a \in [e], b \in S \Rightarrow bab' \in [beb']$
$\quad a\alpha e \quad bab'\,\alpha\, \underbrace{beb'}_{\in E_S}$

inverse? $\quad a \in [e] \Rightarrow a' \in [e]$
$\quad a\alpha e \quad a'\alpha e' = e$ □

<u>def.:</u> $T \leq S$ normal, $\sigma \in Con\,T$ is called <u>normal</u> $\equiv$ $\forall a,b \in T\ \forall c \in S$

$a\sigma b \Rightarrow c a c' \,\sigma\, c b c'$

<u>Lemma:</u> $tr\,\alpha$ is normal (on $E_S \leq S$ normal)

<u>Pf:</u> $a(tr\,\alpha)b \Rightarrow a\alpha b\ \&\ a,b \in E_S \Rightarrow cac'\,\alpha\,cbc'\ \&\ cac', cbc' \in E_S \Rightarrow cac'\,(tr\,\alpha)\,cbc'$ □

<u>Ex.:</u> S/α is a group iff $tr\,\alpha = E_S^2$

<u>Proof:</u>

<u>Note:</u> S inverse $\Rightarrow$ [S group $\Leftrightarrow |E_S| = 1$]
($\Leftarrow$ ... look at eggbox)

($\Rightarrow$) obvious
($\Leftarrow$) from Lallement

□

Theorem: S inverse smg.

(1) $\alpha \in \operatorname{Con} S \Rightarrow (\operatorname{Ker}\alpha, \operatorname{tr}\alpha)$ is a congruence pair

(2) (N, τ) congruence pair $\Rightarrow$

$$\alpha_{(N,\tau)} := \{(a,b) : (a'a, b'b) \in \tau \ \& \ ab' \in N\} \in \operatorname{Con} S$$

(3) $\operatorname{Ker}\alpha_{(N,\tau)} = N, \quad \operatorname{tr}\alpha_{(N,\tau)} = \tau$

$\alpha_{(\operatorname{Ker}\alpha, \operatorname{tr}\alpha)} = \alpha$

def.: (N,τ) congruence pair $\equiv$
- $N \subseteq S$ normal
- $\tau \in \operatorname{Con}(E_S)$ normal

(CP1) $\left.\begin{array}{l} ae \in N \\ (e, a'a) \in \tau \end{array}\right\} \Rightarrow a \in N$

(CP2) $a \in N \Rightarrow (aa', a'a) \in \tau$

Ex.: in groups: $N \trianglelefteq S$, τ trivial on E_S trivial

$\alpha \rightsquigarrow N = [1]$, $N \rightsquigarrow \alpha = \{(a,b) : \underbrace{a^{-1}a = b^{-1}b}_{\text{triv.}} \ \& \ ab^{-1} \in N\}$

in semilattices: $N \subseteq S$ full $\rightsquigarrow N = S$, $\alpha = \{(a,b) : (a,b) \in \tau \ \& \ \underbrace{ab \in N}_{\text{triv.}}\}$

hence $\alpha = \tau$

Proof: (1) $\operatorname{Ker}\alpha \subseteq S$ normal, $\operatorname{tr}\alpha \in \operatorname{Con}(E_S)$ normal ... see Lemmas

(CP1) $\left.\begin{array}{l} ae \in [f] \subseteq \operatorname{Ker}\alpha \\ e \ \tau_\alpha \ a'a \end{array}\right\} \Rightarrow a = a(a'a) \ \alpha \ ae \ \alpha \ f$

$\Rightarrow a \in [f] \subseteq \operatorname{Ker}\alpha$

(CP2) $\begin{array}{l} a \in [f] \subseteq \operatorname{Ker}\alpha \\ a \alpha f \end{array} \Rightarrow \begin{array}{l} aa' \ \alpha \ ff = f \\ a'a \ \alpha \ ff = f \end{array}$

$\Rightarrow aa' \ \alpha \ a'a$

(2) reflex.: $aa' \in N$ because full

sym.: $ab' \in N \Rightarrow (ab')' \in N$ because inverse

$(ab')' = ba'$

trans.: $ab' \in N,\ bc' \in N \Rightarrow a\underbrace{b'b}_{\in E_S}c' \in N \overset{(*)}{\Rightarrow} ac' \in N$

$\circledast \equiv \quad aeb \in N \,, \quad e \,\tau\, a'a \;\Rightarrow\; ab \in N$

$\ldots \; aeb = aebb'b = abb'eb = abf$, $\quad b'eb =: f \in E_s$

$f = b'eb \;\tau\; b'a'ab = (ab)'ab$ (↑ because τ is normal)

$\Big\}\; \overset{(CP1)}{\Longrightarrow} \; ab \in N$

<u>invariant</u>: assume $a \,\alpha\, b$, prove $ca \,\alpha\, cb$, $ac \,\alpha\, bc$

$a'a \,\tau\, b'b$

$ab' \in N$

<u>$ca \,\alpha\, cb$</u>: $(ca)'(ca) = a'c'ca \;\tau\; b'c'cb = (cb)'(cb)$ (Δ)

$(ca)(cb)' =$ ~~[struck]~~ $cab'c' \in N$ because self-conj.

<u>$ac \,\alpha\, bc$</u>: $(ac)'(ac) = c'a'ac \;\tau\; c'b'bc = (bc)'(bc)$ (↑ because τ normal)

$(ac)(bc)' = acc'b' = acc'b'b^*b'$

$= ab'b^*cc'b \in N$ ($ab' \in N$, $b^*cc'b \in E_s \subseteq N$ because full)

$\Delta \equiv \quad a'a \,\tau\, b'b \,,\; ab' \in N \;\Rightarrow\; a'ea \,\tau\, b'eb \quad \forall e \in E_s$

$\ldots \; a'ea = a'ea(a'a)a'ea$ ... idemp.

$\tau\; a'ea(b'b)a'ea$ ... τ normal

$= a'e\,ab'(ab')'ea$

$\tau\; a'e(ab')'ab'\,ea$ ... by (CP2) & τ normal

$= a'eba'ab'ea$

$= a'(ab'e)'(ab'e)a$ NOTE: $ab'e \in N$ since full

$\tau\; a'(ab'e)(ab'e)'a$ ... by ~~(CP2)* & [struck]~~ τ normal

$= a'a\,b'\,ee\,b\,a'a$

$\tau\; b'b\,b'\,e\,b\,b'b = b'eb$ ✓

(3) $\underline{\mathrm{Ker}\,\alpha_{(N,\tau)} = N}$:

$\subseteq$ $a \in \mathrm{Ker}\,\alpha \Rightarrow \begin{array}{c} a \in [e] \\ (a \alpha e) \end{array} \overset{\text{def}}{\Rightarrow} \left. \begin{array}{l} \grave{a} a \;\alpha\; \grave{e} e = e \\ \& \; a\grave{e} = ae \in N \end{array} \right\} \overset{(CP1)}{\Rightarrow} a \in N$

$\supseteq$ $\left. \begin{array}{l} a \in N \\ e := \grave{a} a \end{array} \right\} \Rightarrow \; ?\; a \alpha e \;?$ hence $a \in \mathrm{Ker}\,\alpha$

$\grave{a} a = e = \grave{e} e$, hence $\grave{a} a \;\tau\; \grave{e} e$

& $a\grave{e} = a\grave{a} a = a \in N$

$\left. \begin{array}{l} \underline{\mathrm{tr}\,\alpha_{(N,\tau)} = \tau} : \\ \underline{\alpha_{(\mathrm{Ker}\,\alpha,\, \mathrm{tr}\,\alpha)} = \alpha} : \end{array} \right\}$ HW

$\square$