$\alpha = \sqrt{2} + \sqrt{3}$, MIN. POLYNOM $\alpha$ NAD $\mathbb{Q}$ ?

$\sum_{i=0}^{n} c_i \cdot \alpha^i = 0$, $c_i \in \mathbb{Q}$, $n$ NEJMENŠÍ MOŽNÉ

$\alpha^2 = (\sqrt{2}+\sqrt{3})^2 = 2 + 2\sqrt{6} + 3 = 5 + 2\sqrt{6}$

$\alpha^3 = (\sqrt{2}+\sqrt{3})^3 = 2\sqrt{2} + 3\cdot 2\cdot\sqrt{3} + 3\cdot\sqrt{2}\cdot 3 + 3\sqrt{3} = 11\sqrt{2} + 9\sqrt{3}$

$\alpha^4 = (\sqrt{2}+\sqrt{3})^4 = \dots = 49 + 20\sqrt{6}$

$\alpha^i = a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6}$, $a, b, c, d \in \mathbb{Q}$

$$\begin{aligned}
\alpha^0 &= 1 \\
\alpha^1 &= \sqrt{2} + \sqrt{3} \\
\alpha^2 &= 5 + 2\sqrt{6} \\
\alpha^3 &= 11\sqrt{2} + 9\sqrt{3} \\
\alpha^4 &= 49 + 20\sqrt{6}
\end{aligned}$$

CHCEME $\sum_{i=0}^{4} c_i \alpha^i = 0$

$$\begin{pmatrix} 1 & 0 & 5 & 0 & 49 \\ 0 & 1 & 0 & 11 & 0 \\ 0 & 1 & 0 & 9 & 0 \\ 0 & 0 & 2 & 0 & 20 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = 0$$

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$$\begin{pmatrix} 1 & 0 & 5 & 0 & 49 \\ 0 & 1 & 0 & 11 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 10 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 5 & 0 & 49 \\ 0 & 1 & 0 & 11 & 0 \\ 0 & 0 & 1 & 0 & 10 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \rightsquigarrow \alpha^4 - 10\alpha^2 + 1 = 0$$

$(1 \quad 0 \quad -10 \quad 0 \quad 1)$

$m_{\sqrt{2}+\sqrt{3},\mathbb{Q}} = ?$

$\alpha := \sqrt{2}+\sqrt{3}$

$\rightsquigarrow \alpha^4 - 10\alpha^2 + 1 = 0$

- OT: 1) JE $\underbrace{x^4-10x^2+1}_{f}$ MIN. STUPNĚ TAK, ABY $f(\alpha)=0$ ?

2) EKVIVALENTNĚ, JE $f \in \mathbb{Q}[x]$ IREDUCIBILNÍ ?

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- POUŽIJEME $1, \sqrt{2}, \sqrt{3}$ a $\sqrt{6}$ JSOU LN NAD $\mathbb{Q}$.
- VÍME $1, \sqrt{2}$ JSOU LN NAD $\mathbb{Q}$.
- POUŽIJEME DK. VĚTY O NÁSOBNÉM ROZŠÍŘENÍ $\mathbb{Q} \subseteq \mathbb{Q}(\sqrt{2}) \subseteq \mathbb{Q}(\sqrt{2},\sqrt{3})$

$[\mathbb{Q}(\sqrt{2}):\mathbb{Q}]=2$ $\quad [\mathbb{Q}(\sqrt{2},\sqrt{3}):\mathbb{Q}(\sqrt{2})] = \deg m_{\sqrt{3},\mathbb{Q}(\sqrt{2})}$

BÁZE: $1, \sqrt{2}$ $\quad$ BÁZE: $1, \sqrt{3}$

$= \deg x^2 - 3$

(STAČÍ $\sqrt{3} \notin \mathbb{Q}(\sqrt{2})$, TJ $(a+b\sqrt{2})^2 = (a^2+2b^2) + 2ab\sqrt{2} \neq 3$)

$\rightsquigarrow$ BÁZE $\mathbb{Q}(\sqrt{2},\sqrt{3})$ NAD $\mathbb{Q}$ JE $1, \sqrt{2}, \sqrt{3}, \sqrt{6}$

- PŘ: SPOČÍTEJTE MIN. POLYNOM $3+\sqrt[4]{3}$ NAD $\mathbb{Q}$

$\rightsquigarrow$ $\alpha := 3+\sqrt[4]{3}$ $\quad \left(\alpha^i = a + b\cdot 3^{\frac{1}{4}} + c\cdot 3^{\frac{1}{2}} + d\cdot 3^{\frac{3}{4}},\ a,b,c,d \in \mathbb{Q}\right)$

$$\begin{aligned}
\alpha^0 &= 1\\
\alpha^1 &= 3 + 3^{\frac{1}{4}}\\
\alpha^2 &= 9 + 6\cdot 3^{\frac{1}{4}} + 3^{\frac{1}{2}}\\
\alpha^3 &= 27 + 27\cdot 3^{\frac{1}{4}} + 9\cdot 3^{\frac{1}{2}} + 3^{\frac{3}{4}}\\
\alpha^4 &= 84 + 108\cdot 3^{\frac{1}{4}} + 54\cdot 3^{\frac{1}{2}} + 12\cdot 3^{\frac{3}{4}}
\end{aligned}$$

$$\rightsquigarrow \begin{pmatrix} 1 & 3 & 9 & 27 & 84\\ 0 & 1 & 6 & 27 & 108\\ 0 & 0 & 1 & 9 & 54\\ 0 & 0 & 0 & 1 & 12 \end{pmatrix}\begin{pmatrix} c_0\\ c_1\\ c_2\\ c_3\\ c_4\end{pmatrix} = 0$$

$$(78\ \ -108\ \ 54\ \ -12\ \ 1)$$

$\rightsquigarrow$ $\sum_{i=0}^{4} c_i \alpha^i = 0$

$\rightsquigarrow$ $\alpha$ JE KOŘENEM $\overbrace{x^4 - 12x^3 + 54x^2 - 108x + 78}^{f} \in \mathbb{Q}[x]$

- PROČ JE $f$ MIN. POLYNOM $\alpha$?
- VARIANTA 1: $f$ IREDUCIBILNÍ V $\mathbb{Q}[x]$, EISENSTEIN PRO $p=3$
- VARIANTA 2: $[\mathbb{Q}(\sqrt[4]{3}):\mathbb{Q}] = [\mathbb{Q}(3+\sqrt[4]{3}):\mathbb{Q}] = \deg m_{\sqrt[4]{3},\mathbb{Q}} = \deg x^4-3 = 4$

$$= \deg m_{\sqrt[4]{3}+3,\mathbb{Q}}$$

- VARIANTA 3: $1, 3^{\frac{1}{4}}, 3^{\frac{1}{2}}, 3^{\frac{3}{4}}$ JSOU LN NAD $\mathbb{Q}$, VIZTE VARIANTA 2