NMAG442 Representation Theory of Finite-Dimensional Algebras

Excercise session 7—May 28, 2020

This last exercise session is concerned mainly with the characterization of finite representation type using chains of morphisms and powers of the radical of a category. The main reference is [2], section 9.

Let k be a fixed algebraically closed field.

Our goal will be to prove the following theorem:

Theorem 1 (Theorem 8.2.1 in [2])**.** *For a quiver Q, the following are equivalent:*

(i) The number of isomorphism classes of indecomposable representations is finite.

(ii) There is a global bound for the length of every indecomposable representation.

(iii) We have $\bigcap_{n\geq 0} \operatorname{Rad}^n(X,Y) = 0$ for every pair of X, Y of representations.

(iv) Given an infinite family of non-isomorphisms $\varphi_i : X_i \to X_{i+1}, i \geq 1$, between indecomposable representations, there exists $n \geq 1$ such that $\varphi_n \dots \varphi_1 = 0$.

Exercise 1. Reason that the implication $(i) \Rightarrow (ii)$ holds, and prove that the implication $(ii) \Rightarrow (iii)$. (Hint: Reduce the question to the case of two indecomposable, and make use of the explicit length-dependent bound in the Harada-Sai lemma.)

Exercise 2. Given an infinite family of non-isomorphisms $\varphi_i : X_i \to X_{i+1}, i \geq 1$, between indecomposable representations, such that for each $n \geq 1$ such that $\varphi_n \dots \varphi_1 \neq 0$. Prove that:

(i) There is $r \geq 1$ such that $\operatorname{Ker} \varphi_r \dots \varphi_1 = \operatorname{Ker} \varphi_n \dots \varphi_1$ for all $n \geq r$, and $\operatorname{Ker} \varphi_r \dots \varphi_1 \neq X_1$.

(ii) Find a homorphism $X_1 \overset{\psi}{\to} I$ such that for each $n \geq r$ it factors through the map $\varphi_n \dots \varphi_1$.

(iii) Observe that $\varphi_n \dots \varphi_1 \neq 0 \in \operatorname{Rad}^n(X_1, X_n)$ and that such $X_1 \overset{\psi}{\to} I$ lies in $\operatorname{Rad}^n(X_1, I)$ for all $n \geq 1$.

Exercise 3. Show that the implication $(iii) \Rightarrow (iv)$ holds. (Hint: Use the exercise above.)

Exercise 4. Provided that Q is a Euclidean quiver and δ_Q is the generator of the radical of the associated quadratic form, then there is a regular representation X of dimension vector δ_Q such that $\operatorname{Hom}_{kQ}(X, X) \cong k$ and that it has no regular subrepresentations (see section §9 in [1], for instance). Show that:

(i) $\operatorname{Ext}^1_{kQ}(X, X) \cong k$

(ii) If we set $X_1 = X$, then, by induction, for each $i \geq 1$, we have that the dimension of $\operatorname{Hom}_{kQ}(X_1, X_i)$ is one, and thus $\dim_k \operatorname{Ext}^1_{kQ}(X_1, X_i) = 1$, and set X_{i+1} such that the following short exact sequence does not split $0 \to X_i \to X_{i+1} \to X_1 \to 0$. (Hint: Use Lemma 9.1.3 in [2], the fact that X has no regular subrepresentations, and discuss possible maps $X_1 \to X_{i+1} \to X_{i+1}/X_i$ all of which are regular).

(iii) For each $i \geq 1$, X_i is indecomposable. (Hint: $\operatorname{Hom}_{kQ}(X_i, X_i) \cong k[x]/(x^i)$ as the image of each endomorphism of X_i needs to be isomorphic to $X_j, j \leq i$, and $\dim_k \operatorname{Hom}_{kQ}(X_j, X_i) = 1$ by induction, similarly as above.)

(iv) The system of maps $X_i \to X_{i+1}, i \geq 1$ satifies the assumptions in Exercise 2.

Exercise 5. Show that the implication $(iv) \Rightarrow (i)$ holds. (Hint: Combinatorially, reduce non-Dynkin cases to Euclidean cases, as in the proof of Corollary 5.3.3 in [2], and use the results in exercise above.)

References

[1] Crawley-Boevey, W. Lectures on representations of quivers.

[2] Krause, H. Representations of quivers via reflection functors. *arXiv preprint arXiv:0804.1428* (2008).

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