Representation theory of finite dimensional algebras (NMAG 442)

Notes for the streamed lecture

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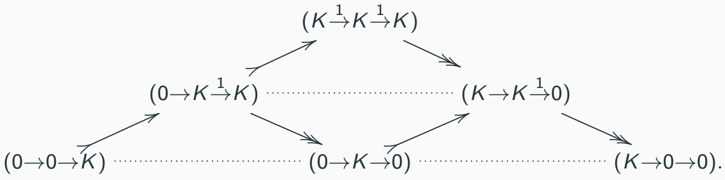
Reminder and aims

Classification results so far

1. If Q is an orientation of a Dynkin diagram of type A, D or E, then *ind*-KQ has finitely many objects and these correspond bijectively to positive roots in $\mathbb{Z}^{|Q_0|}$.
2. If K is algebraically closed and $Q = (\bullet \rightrightarrows \bullet)$, then the indecomposable representations are precisely
    2.1 the preprojectives $P_n \colon K^n \overset{\binom{I}{0}}{\underset{\binom{0}{I}}{\rightrightarrows}} K^{n+1}$, $n \geq 0$,
    2.2 the preinjectives $I_n \colon K^{n+1} \overset{(I\ 0)}{\underset{(0\ I)}{\rightrightarrows}} K^n$, $n \geq 0$,
    2.3 the regular ones $R_{n,\lambda}$, $n \geq 1$, $\lambda \in \mathbb{P}^1_K$, with $\underline{\dim} R_{n,\lambda} = (n, n)$.

The aim

- We wish to understand (parts of) the category *ind*-KQ, where K is a field and Q a finite acyclic quiver, in terms of generating morphisms and relations among them.
- E.g. if K is any field, $Q = (1 \to 2 \to 3)$, then *ind*-KQ is generated by the quiver



- The relations are generated by two zero relations and one commutativity relation (the dotted lines above).

The radical of a category of modules

Definition of the radical [Kra, §2.3]

Definition
Let $\mathcal{C}$ be a small preadditive category and $X, Y \in \operatorname{obj} \mathcal{C}$. Then we define

$$\begin{aligned}\operatorname{Rad}_{\mathcal{C}}(X, Y) &= \{\varphi\colon X \to Y \mid 1_X - \psi\varphi \text{ is invertible } \forall \psi\colon Y \to X\} \\ &= \{\varphi\colon X \to Y \mid \psi\varphi \in \operatorname{rad}\operatorname{End}_{\mathcal{C}}(X)\ \forall \psi\colon Y \to X\}.\end{aligned}$$

Remarks

1. This generalizes the notion of radical of a ring A—take for $\mathcal{C}$ the category with a single object $*$ such that $\operatorname{End}_{\mathcal{C}}(*) = A$.
2. A morphism $\bigoplus_{i=1}^{m} X_i \to \bigoplus_{i=1}^{n} Y_j$ is in the radical iff all the components $X_i \to Y_j$ are in the radical (exercise).
3. $\operatorname{Rad}_{\mathcal{C}}(X, X) = \operatorname{rad}\operatorname{End}(X)$ (exercise).
4. Main interest: $\mathcal{C} = \textit{mod-}A$ or $\mathcal{C} = \textit{ind-}A$, A fin. dim. alg.
5. If $\mathcal{C} = \textit{ind-}A$, then
$$\operatorname{Rad}_{\mathcal{C}}(X, Y) = \{\varphi\colon X \to Y \mid \varphi \text{ is non-isomorphism}\}.$$

Powers of the radical [Kra, §6.1]

Definition
Let $\mathcal{C}$ be a small preadditive category and $X, Y \in \operatorname{obj}\mathcal{C}$. We inductively define

- $\operatorname{Rad}^0_{\mathcal{C}}(X,Y) = \operatorname{Hom}_{\mathcal{C}}(X,Y)$,
- $\operatorname{Rad}^{n+1}_{\mathcal{C}}(X,Y) = \left\{ \sum_{i=1}^n \varphi''_i \varphi'_i \mid \begin{array}{l} \varphi'_i \in \operatorname{Rad}_{\mathcal{C}}(X, Z_i), \\ \varphi''_i \in \operatorname{Rad}^n_{\mathcal{C}}(Z_i, Y) \end{array} \right\}$,
- $\operatorname{Rad}^\infty_{\mathcal{C}}(X,Y) = \bigcap_{n \geq 0} \operatorname{Rad}^n_{\mathcal{C}}(X,Y)$.

Remarks

1. If $\mathcal{C}$ is additive (such as $\mathcal{C} = \textit{mod-}A$), then $\operatorname{Rad}^{n+1}_{\mathcal{C}}(X,Y) = \{\varphi''\varphi' \mid \varphi' \in \operatorname{Rad}_{\mathcal{C}}(X,Z) \text{ and } \varphi'' \in \operatorname{Rad}^n_{\mathcal{C}}(Z,Y)\}$:
$$X \overset{\varphi' = (\varphi'_i)}{\longrightarrow} \bigoplus_{i=1}^n Z_i \overset{\varphi'' = (\varphi''_i)}{\longrightarrow} Y.$$
2. Beware: $\operatorname{Rad}^2_{\mathcal{C}}(X,X) \neq \operatorname{rad}^2 \operatorname{End}_{\mathcal{C}}(X)$ in general!

Irreducible morphisms [Kra, §6.2]

Analogy
We hope to understand generating morphisms for the category *ind-A* via representatives of cosets in $\text{Rad}(X, Y)/\text{Rad}^2(X, Y)$ as we understood A itself (in Gabriel's theorem) via representatives of cosets in $\text{rad}(A)/\text{rad}^2(A)$.

Definition
A morphism $\varphi\colon X \to Y$ in *mod-A* is irreducible if

1. φ is neither a split mono nor a split epi, and
2. whenever we have $\varphi = \varphi''\varphi'$ in *mod-A*, then φ' is a split mono or φ'' is a split epi.

Lemma ([Kra, Lemma 6.2.1])
Any irreducible morphism is a monomorphism or an epimorphism.

Proof.
Consider the factorization $X \overset{\varphi'}{\twoheadrightarrow} \text{Im}\,\varphi \overset{\varphi''}{\rightarrowtail} Y$. □

Irreducible morphisms and the radical [Kra, §6.2]

Lemma ([Kra, Lemma 6.2.2])
Let $\varphi\colon X \to Y$ be a morphism in mod-A.

1. *If X is indec., then $\varphi \in \operatorname{Rad}(X, Y)$ iff φ is not a split mono.*
2. *If Y is indec., then $\varphi \in \operatorname{Rad}(X, Y)$ iff φ is not a split epi.*
3. *If X and Y are both indecomposable, then $\varphi \in \operatorname{Rad}(X, Y) \setminus \operatorname{Rad}^2(X, Y)$ iff φ is irreducible.*

Proof.

1. We have $\varphi = (\varphi_i)\colon X \to \bigoplus_{i=1}^{n} Y_i = Y$. Then $\varphi \in \operatorname{Rad}(X, Y)$ iff all $\varphi_i \in \operatorname{Rad}(X, Y_i)$ iff all φ_i are non-isomorphisms iff φ is not a split mono.
2. is dual to 1.
3. is an immediate consequence of 1. and 2. □

Generating morphisms [Kra, §6.2]

Proposition ([Kra, Proposition 6.2.4])
Let $X, Y \in$ *ind*-A be indecomposable and suppose that $\mathrm{Rad}^n(X, Y) = 0$ for some $n \geq 0$. Then every non-isom. $\varphi\colon X \to Y$ a sum of compositions of irreducible morphisms in *ind*-A.

Proof.

- If φ is irreducible (equivalently $\varphi \notin \mathrm{Rad}^2(X, Y)$), we are done.
- Otherwise $\varphi = \sum_{i=1}^{n} \varphi_i'' \varphi_i'$, where $\varphi_i' \in \mathrm{Rad}(X, Z_i)$ and $\varphi_i'' \in \mathrm{Rad}(Z_i, Y)$ and the Z_i are all indecomposable.
- We repeat the procedure for each φ_i' and φ_i''—either they are irreducible or they have a similar expression, and so on.
- After n steps, we obtain an expression $\varphi = \sum_j \varphi_{jn_j} \cdots \varphi_{j2}\varphi_{j1}$, where φ_{jk} are radical maps in *ind*-A and irreducible if $n_j < n$.
- However, all the terms $\varphi_{jn_j} \cdots \varphi_{j2}\varphi_{j1}$ with $n_j = n$ vanish as we assume $\mathrm{Rad}^n(X, Y) = 0$. □

The Harada-Sai lemma and consequences

The Harada-Sai lemma [Kra, §6.3]

Lemma (Harada-Sai, [Kra, Lemma 6.3.1])
Let $n \geq 1$ and suppose we have in ind-A a chain of non-isomorphisms $X_1 \xrightarrow{\varphi_1} X_2 \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_{2^n-2}} X_{2^n-1} \xrightarrow{\varphi_{2^n-1}} X_{2^n}$ between modules of dimension $\leq n$. Then $\varphi_{2^n-1} \cdots \varphi_1 = 0$.

Proof.

- We prove that $\dim \operatorname{Im}(\varphi_{2^m-1} \cdots \varphi_1) \leq n - m$ by induction on $1 \leq m \leq n$. Clear for $m = 1$ as $\operatorname{Im}(\varphi_1) \lneq X_2$.
- If $m > 1$, consider $X_1 \xrightarrow{\varphi'} X_{2^{m-1}} \xrightarrow{\varphi_{2^{m-1}}} X_{2^{m-1}+1} \xrightarrow{\varphi''} X_{2^m}$.
- If $\dim \operatorname{Im}(\varphi'' \varphi_{2^{m-1}} \varphi') = n - m + 1$, the same holds for $\operatorname{Im}(\varphi''), \operatorname{Im}(\varphi'' \varphi_{2^{m-1}}), \operatorname{Im}(\varphi_{2^{m-1}} \varphi'), \operatorname{Im}(\varphi')$ by induction.
- So $\operatorname{Ker}(\varphi'' \varphi_{2^{m-1}}) \cap \operatorname{Im}(\varphi') = 0$, $\dim X_{2^{m-1}} = \dim \operatorname{Ker}(\varphi'' \varphi_{2^{m-1}})$ $+ \dim \operatorname{Im}(\varphi'' \varphi_{2^{m-1}}) = \dim \operatorname{Ker}(\varphi'' \varphi_{2^{m-1}}) + \dim \operatorname{Im}(\varphi')$.
- Hence $X_{2^{m-1}} = \operatorname{Ker}(\varphi'' \varphi_{2^{m-1}}) \oplus \operatorname{Im}(\varphi')$ and, as $X_{2^{m-1}} \in$ *ind-A*, $\varphi_{2^{m-1}}$ is monic. Dually, $\varphi_{2^{m-1}}$ is epic, so an isomorphism $\lightning$. □

Finite representation type

- Suppose that A is a finite dimensional algebra which is of finite representation type (i.e. *ind*-A has finitely many objects).
- Then $\exists N > 0$ such that $\mathrm{Rad}^N(X, Y) = 0$ for all $X, Y \in$ *ind*-A by Harada-Sai.
- In particular, each non-isomorphism in *ind*-A is a sum of compositions of irreducible morphisms by [Kra, Proposition 6.2.4].