Representation theory of finite dimensional algebras (NMAG 442)

Notes for the streamed lecture

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Roots of Dynkin and Eucledian diagrams—continued

Reminder [Kra, §4.3]

- Let Γ be a Dynkin or a Euclidean diagram and

$$q(x) = \sum_{i \in \Gamma_0} x_i^2 - \sum_{i \leq j} d_{ij} x_i x_j$$

- Then $(x, y) = q(x + y) - q(x) - q(y)$ is positive semidefinite and $q(x) = \frac{1}{2}(x, x)$.
- A root is a non-zero element of $\Delta = \{x \in \mathbb{Z}^n \mid q(x) \leq 1\}$.
- Facts about roots ([Kra, Prop. 4.3.1]):
    1. The basis vector e_i is a root for each $i \in \Gamma_0$.
    2. x is a root iff $-x$ is a root.
    3. Each root x is positive $(x > 0)$ or negative $(x < 0)$.

Finiteness for roots—the Euclidean case [Kra, §4.3]

Proposition (Proposition 4.3.1(2) and (4))
Let Γ be Euclidean. Then:

1. If $x \in \Delta$ and $y \in \operatorname{rad} q$, then $x + y \in \Delta$.
2. $\Delta / \operatorname{rad} q$ is finite.

Proof.

- $q(x+y) = q(x) + (x, y) + q(y) = q(x)$. This proves 1.
- Let $\delta \in \mathbb{Z}^n$ be the smallest positive radical vector and $i \in \Gamma_0$ such that $\delta_i = 1$.
- If $x \in \Delta$, then $y := x - x_i\delta \in \Delta$ defines the same coset in $\Delta / \operatorname{rad} q$ and $y_i = 0$.
- Moreover, both $\delta + y$ and $\delta - y$ are positive roots (look at the i-th coordinate!)
- Hence $-\delta < y < \delta$. □

Finiteness for roots—the Dynkin case [Kra, §4.3]

Corollary (Proposition 4.3.1(5))
If Γ be Dynkin, then Δ is finite.

Proof.

- There is a Euclidean diagram $\tilde{\Gamma}$ such that Γ is obtained by deleting a vertex i from $\tilde{\Gamma}$.
- A root of Γ can be viewed as a root of $\tilde{\Gamma}$ whose i-th coordinate is 0. $\square$

Reflections

Simple reflections [Kra, §3.2]

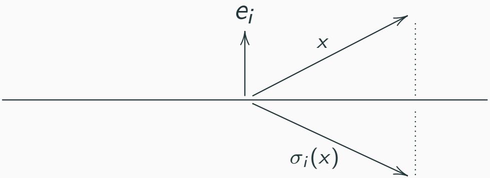
- Let Q be a finite quiver, $q(x) = \sum_{i \in Q_0} x_i^2 - \sum_{\alpha\colon i \to j} x_i x_j$ and $(x, y) = q(x + y) - q(x) - q(y)$, as before.
- Assume Q has no loops, i.e. no . Then $(e_i, e_i) = 2\langle e_i, e_i \rangle = 2$.
- In that case, we can always define the reflection with respect to vertex i:

$$\begin{aligned} \sigma_i \colon \mathbb{Z}^n &\to \mathbb{Z}^n \\ x &\mapsto x - 2\frac{(x, e_i)}{(e_i, e_i)} e_i = x - (x, e_i) e_i. \end{aligned}$$

- Observation: $\sigma_i^2 = 1_{\mathbb{Z}^n}$.
- Observation: $\big(\sigma_i(x), \sigma_i(y)\big) = (x, y) \quad (\forall x, y \in \mathbb{Z}^n)$.
- Observation: If the underlying graph of Q is Dynkin or Euclidean, then σ_i permutes roots (as $q(x) = q\big(\sigma_i(x)\big)$).

Why reflections?

- If q is positive def. ($= Q$ Dynkin), then $(-,-)\colon \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ extends to a scalar product $(-,-)\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$.
- Then we can also extend σ_i to $\mathbb{R}^n \to \mathbb{R}^n$ and we really get a reflection with respect to the hyperplane orthogonal to e_i:



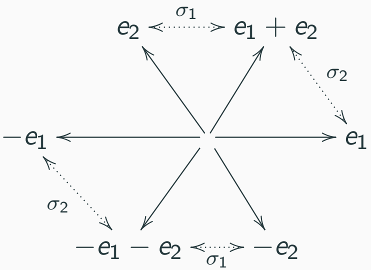
Reflections and roots [Kra, §4.3]

Lemma ([Kra, Lemma 4.3.2])
Let Q be a quiver whose underlying graph is Dynkin or Euclidean, and let $i \in \Gamma_0$. If x is a positive root and $\sigma_i(x)$ is not positive, then $x = e_i$.

Proof.

- If $\sigma_i(x)$ is not positive, then $\sigma_i(x) < 0$.
- But $\sigma_i(x) = x - (x, e_i)e_i$, so $\sigma_i(x)_j = x_j$ for each $j \neq i$.
- It follows that $x_j = 0$ for all $j \neq i$, so $x = e_i$. $\square$

Example



Coxeter transformation

Change or orientation and admissible orderings [Kra, §3.1]

Definition
Let Q be a finite quiver. An ordering of vertices $Q_0 = \{1, 2, \dots, n\}$ is admissible, if $\quad (\exists \alpha \colon i \to j) \implies (i > j)$.

Examples
$Q = (3 \to 2 \to 1)$.

Definition
If Q is a quiver and $i \in Q_0$, we denote $\sigma_i Q$ the quiver obtained from Q by changing orientation of the arrows incident at i.

Lemma
An ordering $Q_0 = \{1, 2, \dots, n\}$ is admissible iff i is a sink of $\sigma_{i-1} \cdots \sigma_1 Q$ for each $i \in Q_0$.

Examples
$Q = (3 \to 2 \to 1) \rightsquigarrow \sigma_1 Q = (3 \to 2 \leftarrow 1) \rightsquigarrow$
$\sigma_2 \sigma_1 Q = (3 \leftarrow 2 \to 1) \rightsquigarrow \sigma_3 \sigma_2 \sigma_1 Q = Q$.

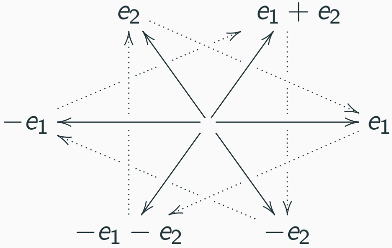
Definition
Let Q be a finite quiver with an admissible ordering of vertices, $Q_0 = \{1, 2, \ldots, n\}$. The automorphism

$$\begin{aligned} c\colon \mathbb{Z}^n &\to \mathbb{Z}^n, \\ x &\mapsto \sigma_n \cdots \sigma_2 \sigma_1(x) \end{aligned}$$

is called the Coxeter transformation.

Example
If $Q = (3 \to 2 \to 1)$, then

$$\begin{aligned} c\colon e_2 &\mapsto e_1, \\ e_1 &\mapsto -e_1 - e_2. \end{aligned}$$



Fixed points of the Coxeter transformation [Kra, §4.4]

Lemma ([Kra, Lemma 4.4.3])
Let $x \in \mathbb{Z}^n$. Then $c(x) = x$ iff $x \in \operatorname{rad} q$.

Proof.
The following statements are equivalent for $x \in \mathbb{Z}^n$:

- $c(x) = x$,
- $x_i = \sigma_i(x)_i \ \big(= x_i - (x, e_i)\big)$ for each i,
- $(x, e_i) = 0$ for each i. □

Coxeter and positivity in the Dynkin case [Kra, §4.4]

- If Q is of Dynkin type, then $c\colon \mathbb{Z}^n \to \mathbb{Z}^n$ permutes the finite set Δ.
- In particular, for each i there is $r_i > 0$ such that $c^{h_i}(e_i) = e_i$.
- It follows that $c^h = 1_{\mathbb{Z}^n}$ for some $h > 0$. The smallest such h is called the Coxeter number.

Lemma ([Kra, Lemma 4.4.4])
Let Q be of Dynkin type and $x \in \mathbb{Z}^n$. Then $\exists r \geq 0$ such that $c^r(x)$ is not positive.

Proof.

- Put $y = \sum_{r=0}^{h-1} c^r(x)$.
- Then $c(y) = y$, so $y \in \operatorname{rad} q = \{0\}$.
- Consequently, $c^r(x)$ is not positive for some $0 \leq r < h$. □

Enumerating roots in the Dynkin case

- Let Q be of Dynkin type with admissibly ordered vertices $Q_0 = \{1, 2, \ldots, n\}$ and x a positive root.
- Let $r \geq 0$ and $1 \leq s \leq n$ be smallest possible such that

$$\sigma_s \sigma_{s-1} \cdots \sigma_1 (\sigma_n \cdots \sigma_2 \sigma_1)^r (x) < 0.$$

- Then $\sigma_{s-1} \cdots \sigma_1 (\sigma_n \cdots \sigma_2 \sigma_1)^r (x) = e_s$ (recall Lemma).
- Thus, each positive root has an expression of the form

$$(\sigma_1 \sigma_2 \cdots \sigma_n)^r \sigma_1 \cdots \sigma_{s-1} (e_s),$$

where all the intermediate roots

$$\sigma_t \cdots \sigma_n (\sigma_1 \sigma_2 \cdots \sigma_n)^{r'} \sigma_1 \cdots \sigma_{s-1} (e_s)$$

for all shorter expressions are also positive!

Reflection functors

Reflection functors [Kra, §3.3]

- Let Q be a quiver with a sink $i \in Q_0$. So i is a source in $Q' := \sigma_i Q$.
- We define additive functors $S_i^- \colon \operatorname{Rep}_K(Q') \rightleftarrows \operatorname{Rep}_K(Q) \colon S_i^+$.
- Consider $M = (M_i, f_\alpha) \in \operatorname{Rep}_K Q$ and the exact sequence

$$0 \longrightarrow M_i' \xrightarrow{(f_\alpha')} \bigoplus_{(\alpha\colon j \to i) \in Q_1} M_j \xrightarrow{(f_\alpha)} M_i$$

- We define $S_i^+(M) = (M_i', f_\alpha')$ as follows
  1. M_i' is as above and $M_j' = M_j$ if $j \neq i$.
  2. If $(\alpha\colon i \to k) \in Q_1'$, then f_α' is as above, and if $(\alpha\colon j \to k) \in Q_1'$ has $j \neq i$, then $f_\alpha' = f_\alpha$.
- If $N = (N_i, g_\alpha) \in \operatorname{Rep}_K(Q')$, then $S_i^-(N)$ is defined dually using

$$N_i \xrightarrow{(g_\alpha)} \bigoplus_{(\alpha\colon i \to j) \in Q_1'} N_j \xrightarrow{(g_\alpha')} N_i' \longrightarrow 0$$

Reflections versus reflection functors [Kra, §3.3]

- Consider Q with a sink $i \in Q_0$, $Q' := \sigma_i Q$, and

$$S_i^-\colon \operatorname{Rep}_K(Q') \rightleftarrows \operatorname{Rep}_K(Q)\colon S_i^+.$$

- Then we have natural morphisms

$$\iota_i\colon S_i^- S_i^+(M) \rightarrowtail M,$$
$$\pi_i\colon N \twoheadrightarrow S_i^+ S_i^-(N).$$

Lemma ([Kra, Lemma 3.3.2])

1. $M \cong (S_i^- S_i^+(M)) \oplus \operatorname{Coker} \iota_i$ and
   $\operatorname{Coker} \iota_i$ is a direct sum of copies of the simple $S(i)$.
2. $N \cong (S_i^+ S_i^-(N)) \oplus \operatorname{Ker} \pi_i$ and
   $\operatorname{Ker} \pi_i$ is a direct sum of copies of the simple $S(i)$.
3. If $M \in \operatorname{rep}_K(Q)$ and M has no summand isomorphic to $S(i)$, then
   $\underline{\dim} S_i^+(M) = \sigma_i(\underline{\dim} M)$.
4. If $N \in \operatorname{rep}_K(Q')$ and N has no summand isomorphic to $S(i)$, then
   $\underline{\dim} S_i^-(N) = \sigma_i(\underline{\dim} N)$.

Lemma ([Kra, Lemma 3.3.3])
Let Q be a quiver, $i \in Q_0$ a sink and $M = (M_j, f_\alpha) \in \mathrm{rep}_K(Q)$ indecomposable. TFAE:

1. $M \ncong S(i)$.
2. $S_i^+(M) \neq 0$.
3. $S_i^+(M)$ is indecomposable.
4. $S_i^- S_i^+(M) \cong M$.
5. The map $(f_\alpha)\colon \bigoplus_{\alpha\colon j \to i} M_j \to M_i$ is surjective.
6. $\sigma_i(\underline{\dim} M) > 0$.
7. $\sigma_i(\underline{\dim} M) = \underline{\dim} S_i^+(M)$.

Bijections between indecomposable representations [Kra, §3.3]

Theorem ([Kra, Theorem 3.3.5])
Let Q be a quiver with sink $i \in Q_0$ and $Q' = \sigma_i Q$. Then the functors S_i^+ and S_i^- induce mutually inverse bijections between

1. the isomorphism classes of indecomposable representations of Q and
2. the isomorphism classes of indecomposable representations of Q',

with the exception of the simple representation $S(i)$ (both over Q and Q'), which is annihilated by these functors.

Moreover, $\underline{\dim} S^{\pm} M = \sigma_i(\underline{\dim} M)$ for every indecomposable representation M of the corresponding quiver which is not isomorphic to $S(i)$.