Representation theory of finite dimensional algebras (NMAG 442)

Notes for the streamed lecture

Jan Šťovíček

April 30, 2020

Department of Algebra, Charles University, Prague

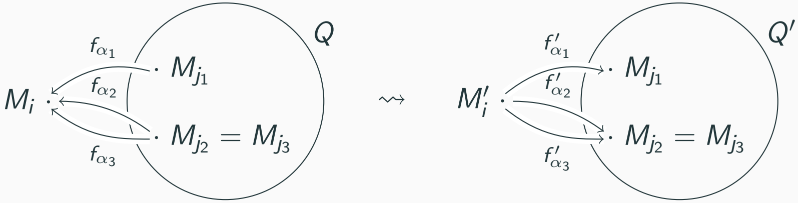
Table of contents

Reflection functors—continued

Reflection functors—reminder [Kra, §3.3]

- Let Q be a quiver with a sink $i \in Q_0$ and $Q' := \sigma_i Q$.
- We have additive functors $S_i^-\colon \operatorname{Rep}_K(Q') \rightleftarrows \operatorname{Rep}_K(Q)\colon S_i^+$.
- If $M = (M_i, f_\alpha) \in \operatorname{Rep}_K Q$, then $S_i^+(M)$ is defined via

$$0 \longrightarrow M_i' \xrightarrow{(f_\alpha')} \bigoplus_{(\alpha\colon j \to i) \in Q_1} M_j \xrightarrow{(f_\alpha)} M_i,$$



Reflections versus reflection functors [Kra, Lemma 3.3.2]

- Recall: We natural split monomorphism

$$\iota_{i,M} \colon S_i^- S_i^+(M) \rightarrowtail M,$$

  where Coker $\iota_{i,M}$ is a direct sum of copies of the simple $S(i)$.
- M has no summand isomorphic to $S(i)$ iff $(f_\alpha) \colon \bigoplus_{\alpha\colon j\to i} M_j \to M_i$ is surjective.
- In that case, $\dim S_i^+(M)_i = \sum_{\alpha\colon j\to i} M_j - M_i$.
- On the other hand, $\sigma_i(\underline{\dim} M) = \underline{\dim} M - (\underline{\dim} M, e_i)e_i$, so

$$\begin{aligned}\sigma_i(\underline{\dim} M)_i &= \dim M_i - (\underline{\dim} M, e_i)\\ &= \dim M_i - (2\dim M_i - \sum_{\alpha\colon j\to i} \dim M_j)\\ &= \sum_{\alpha\colon j\to i} \dim M_j - \dim M_i.\end{aligned}$$

- Thus, if $M \in \mathrm{rep}_K(Q)$ and M has no summands isomorphic to $S(i)$, then $\underline{\dim} S_i^+(M) = \sigma_i(\underline{\dim} M)$. Dually for S_i^-.

Bijections between indecomposable representations [Kra, §3.3]

Theorem ([Kra, Theorem 3.3.5])
Let Q be a quiver with sink $i \in Q_0$ and $Q' = \sigma_i Q$. Then the functors S_i^+ and S_i^- induce mutually inverse bijections between

1. the isomorphism classes of indecomposable representations of Q and
2. the isomorphism classes of indecomposable representations of Q',

with the exception of the simple representation $S(i)$ (both over Q and Q'), which is annihilated by these functors.

Moreover, $\underline{\dim} S^{\pm} M = \sigma_i(\underline{\dim} M)$ for every indecomposable representation M of the corresponding quiver which is not isomorphic to $S(i)$.

Coxeter functors

Coxeter functors [Kra, §3.4]

- Let Q be a quiver with admissibly ordered vertices $Q_0 = \{1, 2, \ldots, n\}$.
- Recall, $(\exists \alpha \colon i \to j) \implies (i > j)$, or equivalently: i is a sink of $\sigma_{i-1} \cdots \sigma_1 Q$ for each $i \in Q_0$
- The Coxeter functors $C^- \colon \operatorname{Rep}_K(Q) \rightleftarrows \operatorname{Rep}_K(Q) \colon C^+$ are defined as the compositions

$$C^- \colon \operatorname{Rep}_K(Q) \underset{S_n^+}{\overset{S_n^-}{\rightleftarrows}} \operatorname{Rep}_K(\sigma_{n-1} \cdots \sigma_1 Q) \underset{S_{n-1}^+}{\overset{S_{n-1}^-}{\rightleftarrows}} \cdots$$

$$\cdots \underset{S_3^+}{\overset{S_3^-}{\rightleftarrows}} \operatorname{Rep}_K(\sigma_2 \sigma_1 Q) \underset{S_2^+}{\overset{S_2^-}{\rightleftarrows}} \operatorname{Rep}_K(\sigma_1 Q) \underset{S_1^+}{\overset{S_1^-}{\rightleftarrows}} \operatorname{Rep}_K(Q) \colon C^+$$

Independence on the admissible ordering

Lemma ([Kra, Lemma 3.4.1])
The functors $C^{\pm}\colon \mathrm{Rep}_K(Q) \to \mathrm{Rep}_K(Q)$ do not depend on the choice of the admissible ordering of vertices of Q.

Proof.

- Key observation: If $i \neq j$ are two sinks of Q, then $S_i^+ S_j^+ = S_j^+ S_i^+$.
- Suppose that $Q_0 = \{1, 2, \dots, n\}$ is admissibly ordered and $Q_0 = \{i_1, i_2, \dots, i_n\}$ is another admissible ordering.
- Then i_1 is a sink. By the above,
$$S_{i_1}^+ S_{i_1-1}^+ \cdots S_1^+ = S_{i-1}^+ \cdots S_1^+ S_{i_1}^+,$$
so
$$S_n^+ \cdots S_1^+ = S_n^+ \cdots \widehat{S_{i_1}^+} \cdots S_1^+ S_{i_1}^+.$$
- Similarly $S_n^+ \cdots \widehat{S_{i_1}^+} \cdots S_1^+ S_{i_1}^+ = S_n^+ \cdots \widehat{S_{i_1}^+} \cdots \widehat{S_{i_2}^+} \cdots S_1^+ S_{i_2}^+ S_{i_1}^+$, and so on. □

Projectives as reflections of simples

Lemma ([Kra, Lemma 3.4.2(1)])
Let Q be a quiver with admissibly ordered vertices $Q_0 = \{1, 2, \ldots, n\}$. Given $i \in Q_0$, then $\underline{\dim} P(i) = \sigma_1 \cdots \sigma_{i-1}(e_i)$ and $\underline{\dim} I(i) = \sigma_n \cdots \sigma_{i+1}(e_i)$.

Proof.

- This is a direct computation:

$$\begin{aligned}
\sigma_{i-1}(e_i) &= e_i - (e_i, e_{i-1})e_{i-1} = e_i + \left|\{\alpha : i \to i-1\}\right| \cdot e_{i-1}, \\
\sigma_{i-2}\sigma_{i-1}(e_i) &= \sigma_i(e_i) - (\sigma_i(e_{i-1}), e_{i-2})e_{i-2} \\
&= e_i - (e_i, e_{i-1})e_{i-1} \\
&\quad - (e_i, e_{i-2})e_{i-2} + (e_i, e_{i-1})(e_{i-1}, e_{i-2})e_{i-2} \\
&= e_i + \left|\{\alpha : i \to i-1\}\right| \cdot e_{i-1} + \left|\{\alpha : i \rightsquigarrow i-2\}\right| \cdot e_{i-2}.
\end{aligned}$$

- In general, induction shows for each $0 \leq \ell \leq i-1$:

$$\sigma_{i-\ell} \cdots \sigma_{i-1}(e_i) = \sum\nolimits_{j=0}^{\ell} \left|\{\alpha : i \rightsquigarrow i-\ell\}\right| \cdot e_{i-\ell}. \qquad \square$$

Projectives as reflections of simples—continued

Lemma ([Kra, Lemma 3.4.2(2)])
Let Q be a quiver with admissibly ordered vertices $Q_0 = \{1, 2, \ldots, n\}$. Given $i \in Q_0$, we have

1. $P(i) \cong S_1^- \cdots S_{i-1}^-\big(S(i)\big)$ (here $S(i) \in \mathrm{rep}_{\mathsf{K}}(\sigma_{i-1} \cdots \sigma_1 Q)$),
2. $I(i) \cong S_n^+ \cdots S_{i+1}^+\big(S(i)\big)$ (here $S(i) \in \mathrm{rep}_{\mathsf{K}}(\sigma_{i+1} \cdots \sigma_n Q)$).

Proof.

- We know that $\underline{\dim} P(i) = \sigma_1 \cdots \sigma_{i-1}(e_i)$.
- Therefore, for each $0 \le \ell \le i-1$:

$$\underline{\dim} S_\ell^+ \cdots S_1^+\big(P(i)\big) = \sigma_{\ell+1} \cdots \sigma_{i-1}(e_i).$$

- In particular (for $\ell = i-1$), $\underline{\dim} S_{i-1}^+ \cdots S_1^+\big(P(i)\big) = e_i$, so

$$S_{i-1}^+ \cdots S_1^+\big(P(i)\big) \cong S(i).$$

- It follows that $P(i) \cong S_1^- \cdots S_{i-1}^-\big(S(i)\big)$. $\square$

Indecomposables annihilated by Coxeter functors

Proposition ([Kra, Proposition 3.4.3])
Let Q be a finite acyclic quiver, K a field and $M \in \mathit{ind}\text{-}KQ$.

1. $C^+(M) = 0$ iff M is projective. Otherwise, $C^-C^+(M) \cong M$ and $\underline{\dim} C^+(M) = c(\underline{\dim} M)$.
2. $C^-(M) = 0$ iff M is injective. Otherwise, $C^+C^-(M) \cong M$ and $\underline{\dim} C^-(M) = c^{-1}(\underline{\dim} M)$.

Proof.

- Let $Q_0 = \{1, 2, \ldots, n\}$ be an admissible ordering.
- Suppose that $C^+(M) = 0$ and let $1 \leq i \leq n$ be smallest possible such that $S_i^+ \cdots S_1^+(M) = 0$.
- Then $S_{i-1}^+ \cdots S_1^+(M) \cong S(i)$, so
$$M \cong S_1^- \cdots S_{i-1}^-\big(S(i)\big) \cong P(i).$$
- Otherwise $C^-C^+(M) \cong M$ and $\underline{\dim} C^+(M) = c(\underline{\dim} M)$ by the theorem about reflection functors. □

Action of the Coxeter functors on indecomposables

Corollary
Let Q be a finite acyclic quiver. Then the functors C^+ and C^- induce mutually inverse bijections between

1. the isomorphism classes of non-projective indecomposable representations of Q and
2. the isomorphism classes of non-injective indecomposable representations of Q.

Moreover, $\underline{\dim} C^{\pm}(M) = c^{\pm 1}(\underline{\dim} M)$ for every indecomposable representation M which is non-projective (for C^+) or non-injective (for C^-), respectively.

Corollary
The Coxeter transformation $c = \sigma_n \cdots \sigma_2 \sigma_1 \colon \mathbb{Z}^n \to \mathbb{Z}^n$ does not depend on the choice of the admissible ordering of vertices.

More on the Coxeter transformation

Lemma ([Kra, Lemma 4.4.1])
Let Q be a finite acyclic quiver ($n = |Q_0|$) and K a field.

- $c(\underline{\dim} P(i)) = -\underline{\dim} I(i)$ for each $i \in Q_0$.
- $\{\underline{\dim} P(i) \mid i \in Q_0\}$ and $\{\underline{\dim} I(i) \mid i \in Q_0\}$ are bases of $\mathbb{Z}^n$.

Proof.

1. $c(\underline{\dim} P(i)) = c\sigma_1 \cdots \sigma_{i-1}(e_i) = \sigma_n \cdots \sigma_{i+1}\sigma_i(e_i) = -\sigma_n \cdots \sigma_{i+1}(e_i) = -\underline{\dim} I(i)$.
2. We have for each $i \in Q_0$:

$$
\begin{aligned}
e_i &= \underline{\dim} P(i) - \sum_{\alpha\colon i \to j} \underline{\dim} P(j) && \text{since } \operatorname{rad} P(i) \cong \bigoplus_{\alpha\colon i \to j} P(j), \\
&= \underline{\dim} I(i) - \sum_{\alpha\colon j \to i} \underline{\dim} I(j) && \text{since } I(i)/\operatorname{soc} I(i) \cong \bigoplus_{\alpha\colon j \to i} I(j).
\end{aligned}
$$

□

Preprojective and preinjective representations

Preprojective and preinjective representations [Kra, §3.5]

Notation
Let K be a field, Q a finite acyclic quiver and $r \in \mathbb{Z}$. Then

$$C^r = \begin{cases} (C^+)^r & \text{if } r > 0, \\ 1_{\mathrm{Rep}_K(Q)} & \text{if } r = 0, \\ (C^-)^{|r|} & \text{if } r < 0. \end{cases}$$

Definition
Let M be an indecomposable representation of Q. Then

1. M is preprojective if $M \cong C^r P(i)$ for some $i \in Q_0$ and $r \leq 0$.
2. M is preinjective if $M \cong C^r I(i)$ for some $i \in Q_0$ and $r \geq 0$.
3. M is regular otherwise (equivalently: $C^r(M) \neq 0 \; \forall r \in \mathbb{Z}$).

Determination by dimension vectors

Proposition ([Kra, Proposition 3.5.2])
Let K be a field and Q a finite acyclic quiver. If M, N are two indecomposable representations and M is preprojective or preinjective, then

$$M \cong N \quad \Longleftrightarrow \quad \underline{\dim} M = \underline{\dim} N.$$

Proof.

- If M is preprojective, then $M \cong C^r P(i)$ for $i \in Q_0$ and $r \leq 0$.
- If $\underline{\dim} M = \underline{\dim} N$, then

$$\underline{\dim} N = (\sigma_1 \cdots \sigma_n)^r \sigma_1 \cdots \sigma_{i-1}(e_i).$$

- In particular $S_{i-1}^+ \cdots S_1^+ C^{-r}(M) \cong S(i)$ and, thus

$$N \cong C^r S_1^- \cdots S_{i-1}^-\big(S(i)\big) \cong C^r P(i) \cong M. \qquad \square$$

A unique form of preprojectives and preinjectives

Proposition ([Kra, Proposition 3.5.2])
Let K be a field and Q a finite acyclic quiver.

1. $C^r P(i) = C^s P(j) \neq 0$ implies $i = j$ and $r = s$.
2. $C^r I(i) = C^s I(j) \neq 0$ implies $i = j$ and $r = s$.

Proof.

- If $C^r P(i) \cong C^s P(j) \neq 0$, then $P(i) \cong C^{s-r} P(j)$, so $s - r \leq 0$.
- By symmetry also $r - s \leq 0$, so $r = s$.
- Now $P(i) \cong P(j)$, which implies $i = j$ (e.g. look at the quotients modulo the radical). □