NMAG442 Representation Theory of Finite-Dimensional Algebras

Excercise session 2—March 26, 2021

Our goal today is to finish reviewing the material in Section I.3 in [1]. There are two main topics which we cover in this respect: Wedderburn-Artin theorem and (Jacobson) radical of a module over a finite-dimensional algebra. Time permitting, we focus also on some exercises about idempotents, local algebras, and projective covers.

For the sake of completeness and to assist recall, exercises finished during the last session are left in the text below. They are marked with ✓.

Wedderburn-Artin theorem

Definition 1 (Semisimple modules and rings)**.** *A module M in* $\mathrm{Mod}-R$ *is called simple if it has no proper submodules (other than zero submodule and itself). It is called semisimple (or completely reducible) if it is a direct sum of simple $R-$modules. Finally, a ring S is semisimple if it is semisimple as a module over itself.*

Definition 2 (Socle of a module)**.** *Let M in* $\mathrm{Mod}-R$ *be a module. Then,* $\mathrm{soc}(M)$ *is the submodule of M generated by all simple submodules of M. It is referred to as the socle of M.*

Exercise 1. Prove that, for M and N right modules over R:

- (i) M is semisimple if and only if $\mathrm{soc}(M) = M$. (Hint: Use Zorn lemma.) ✓
- (ii) Let $f : M \to N$ be a module homomorphism. Then, $f(\mathrm{soc}(M)) \subseteq \mathrm{soc}(N)$. ✓
- (iii) Epimorphic image of a semisimple module is semisimple. ✓
- (iv) R is semisimple if and only if all right modules over R are semisimple. ✓

Exercise 2 (Schur lemma; 3.1 in chapter I in [1]). Let $S_1, S_2 \in \mathrm{Mod} - R$, and $f : S_1 \to S_2$ be a non-zero homomorphism between them. Then, prove the following:

- (i) If S_1 is simple, f is a monomorphism. ✓
- (ii) If S_2 is simple, f is an epimorphims. ✓
- (iii) If both are simple, f is an isomorphism. ✓

Exercise 3. Find a simple example of a ring R (preferrably a finite-dimensional algebra over a field k) and an $R-$module M such that M is not simple, yet $\mathrm{End}_R(M)$ is a division ring.

Exercise 4 (Corollary 3.2 in chapter I in [1]). Let R be a finite-dimensional algebra over an *algebraically closed* field k. Then, for every S, a simple module over R, prove that $\mathrm{End}_R(S) \cong k$. ✓

Exercise 5 (Wedderburn-Artin theorem; 3.4 in chapter I in [1]). Let R be a ring. Then, prove that the following propositions are equivalent:

(i) R is semisimple.

(ii) R is isomorphic to $M_{m_1}(D_1) \times \cdots \times M_{m_n}(D_n)$ for $m_1, \ldots, m_n \in \mathbb{N}$ and division rings $D_1, \ldots, D_n$.

Exercise 6 (Wedderburn-Artin theorem continued; 3.4 in chapter I in [1]). Let A be a finite-dimensional algebra. Then, prove that the following propositions are equivalent:

(i) A is semisimple.

(ii) $\operatorname{rad} A = 0$.

(Hint: Show that every right ideal in A splits.)

Radical of a module over a finite-dimensional algebra

Definition 3 (Radical of a module; 3.6 in chapter I in [1]). *Let $M \in \operatorname{Mod} - R$. The (Jacobson) radical of M, $\operatorname{rad} M$ is the intersection of all maximal submodules of M.*

Definition 4 (Superflous submodule; 5.5(a) in chapter I in [1]). *Let $M \in \operatorname{Mod} - R$, and let $L \leq M$ be a submodule. Say that L is superflous if for any $N \leq M$ such that $L + N = M$, N needs to equal M.*

Exercise 7 (3.7 in chapter I in [1]). Prove that, for M and N right modules over R:

(i) An element $m \in M$ is in the radical of M if it is in the kernel for every module homomorphism $f : M \to S$ where S is simple.

(ii) $\operatorname{rad}(M \oplus N) = \operatorname{rad} M \oplus \operatorname{rad} N$.

(iii) Let $f : M \to N$ be an $R-$module homomorphism. Then, $f(\operatorname{rad}(M)) \subseteq \operatorname{rad}(N)$.

(iv) If R is a finite dimensional algebra, then $M \cdot \operatorname{rad} R = \operatorname{rad} M$. (Hint: Key is that $R/\operatorname{rad} R$ is semisimple.)

(v) $\operatorname{rad} M$ is a superflous submodule of M if there is maximal submodule over every submodule of M.

Definition 5 (Top of a module). *Let A be a finite-dimensional algebra, and $M \in \operatorname{Mod} - A$. Its top is defined as* $\operatorname{top} M = M/\operatorname{rad} M$.

Exercise 8 (Corollary 3.9(a) in chapter I in [1]). Let A be a finite-dimensional algebra, and $M, N \in \operatorname{Mod} - A$. Then $f : M \to N$, a module homomorphism, is surjective if and only if the induced map $\operatorname{top} f : \operatorname{top} M \to \operatorname{top} N$ is surjective.
(Hint: Determine what $\operatorname{top} f$ is, and use properties of the module radical.)

Idempotents, local algebras, and projective covers

Exercise 9 (4.9, I.4 in [1]). Let us have an algebra over k:

$$B = \left\{ \begin{pmatrix} \lambda & 0 & 0 \\ \alpha_{21} & \lambda & 0 \\ \alpha_{31} & \alpha_{32} & \lambda \end{pmatrix}; \lambda, \alpha_{21}, \alpha_{31}, \alpha_{32} \in k \right\}$$

Show that:

(i) B is indeed a well-defined algebra.

(ii) B is local.

Exercise 10. Given a commutative finite-dimensional algebra over k, find a decomposition thereof.

Exercise 11. Exhibit a k-algebra with no non-trivivial idempotents that is not local.

Exercise 12 (7 in I.6 in [1]). Show that $k[t]/(t^3)$ as a module over $k[t]$ (which is a path algebra of a quiver with a single vertex and a loop) has no projective cover.

References

[1] Assem, I., Skowronski, A., and Simson, D. *Elements of the Representation Theory of Associative Algebras: Volume 1: Techniques of Representation Theory*, vol. 65. Cambridge University Press, 2006.

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