I. Goal: Describe categories like \( \text{mod} A, \text{D}^b(A), \text{K}^b(A) \), \( \text{mod}_R \), \( \text{Coh} X \), \( \text{D}^b(\text{Coh} X) \)

in terms of generators and relation.

\[ \Rightarrow \text{Pictures of the categories} \ldots \]

- What does it mean: Generators & Relations

- Let \( \Gamma \) be a quiver (not necessarily finite), e.g.

- Let \( \mathbb{K} \) be a field (or a commutative ring)

\[ \Rightarrow \text{Path category } \mathbb{K}\Gamma : \]

  - Objects: Vertices of \( \Gamma \)
  - Morphisms: \( \text{Hom}(x,y) = \text{vector space whose basis is set of all paths } x \to y \) in \( \Gamma \)
  - Composition: Induced by concatenation of paths
  - Units: Trivial paths

\[ \Rightarrow \mathbb{K}\Gamma \text{ is a } \mathbb{K}\text{-linear category} \]

- If \( \mathfrak{a} \subseteq \mathbb{K}\Gamma \) is a 2-sided ideal (i.e. \( \forall x, y \in \mathbb{K}\Gamma : \mathfrak{a}(x, y) \subseteq \text{Hom}(x, y) \) and closed under composition with any map)

\[ \Rightarrow \text{Quotient } \mathbb{K}\text{-linear category } \mathbb{K}\Gamma / \mathfrak{a} = \frac{\mathbb{K}\Gamma}{\mathfrak{a}} \ldots \text{Objects: Same as in } \mathbb{K}\Gamma \]

  Morphisms: \( \text{Hom}_{\mathbb{K}\Gamma/\mathfrak{a}}(x, y) = \frac{\text{Hom}_{\mathbb{K}\Gamma}(x, y)}{\mathfrak{a}(x, y)} \)
II. Examples:
- $Q = (\{1 \rightarrow 2 \rightarrow 3\})$, can we present $\text{End}_Q$ by generators & relations in this way?
- We better look at $\text{End}_Q = \text{A skeleton of the full subcategory of } \text{mod}_Q \text{ spanned by indecomposables}$. 

\[ \begin{array}{c}
P_0 = \mathbb{I}_2 \\
\begin{array}{c}
\vdots \\
1 \\
2 \\
3 \\
\end{array} \\
\begin{array}{c}
0 \rightarrow 3 \rightarrow 2 \\
0 \rightarrow 1 \rightarrow 0 \\
2 \rightarrow 0 \rightarrow 0 \\
\end{array} \\
S_3 \\
S_2 \\
S_7
\end{array} \]

$\Rightarrow \text{End}_Q \cong \mathcal{O}_{P,0} \left| \Gamma = \begin{array}{c}
1 \\
\end{array} \right| \Delta = (\text{mesh relations})$
We can do a similar trick for $D^b(A)$ [Chapell, 1987].

We look at $\text{mod} \ D^b(A)$.

Fact: $\text{mod} \ D^b(A) \cong \mathcal{C}_\Gamma$ (mesh rels).

More in detail:

- Comments on what we use here:

  1. $A$ is hereditary, so

     (a) Every complex in $D^b(A)$ is iso in $D^b(A)$ to a bounded cpx of f.g. projectives ($D^b(A) \cong \mathcal{C}(\text{mod} \ A)$).

     (b) Every $x \in D^b(A)$ is iso to its homology, i.e. $\mathcal{X} \cong \bigoplus_{n \in \mathbb{Z}} H_n(x)[n]$

- Complex of projectives:

  $\cdots \to P_{n+2} \to P_{n+1} \xrightarrow{d} P_n \xrightarrow{d} P_{n-1} \to \cdots$

  $\xrightarrow{\Theta} \bigoplus_{n \in \mathbb{Z}} \xrightarrow{\mathbb{Z}}$

  $\text{Complex of projectives} \xrightarrow{\text{mod}} \cdots \to 0 \to B_1(x) \xrightarrow{d} B_0(x) \xrightarrow{d} 0 \to \cdots$

  $\downarrow \downarrow \downarrow \downarrow$

  $H_1(x) \cdots \to 0 \to 0 \to H_0(x) \to 0 \to \cdots$
1. \[ \text{every } x \in \text{Mod}_R \text{ is a finite } \bigoplus \text{ of objects with local endo. rings} \]

2. **Krull-Schmidt Theorem (a part of it)**

- Suppose \( C \) is an additive category and \( x \in C \) decomposes to a finite \( \bigoplus \) of objects with local endo. rings. Then every other such decomposition differs only up to the order and isomorphism of sums.

**Sketch**:
\[ \text{Hom}(x, -) : \text{add } x \xrightarrow{\cong} \text{proj } \text{End}(x) \]

*Semiperfect*
III. THE RADICAL OF A PREADDITIONAL ABELIAN CATEGORY

Recall: A ring $R$ in $\text{rad}(R) = \{ n \in R \mid \exists a \in R : 1_{R^n} - na \text{ is invertible}\}

(Jacobson radical) = \{ n \in R \mid \exists a \in R -\text{no} -\text{side ideal} \}

DEF: Let $\mathcal{C}$ be a preadditive category. The radical $\text{rad}_{\mathcal{C}}$ is the 2-sided ideal (needs to be proved!) given as follows:

$\forall x, y \in \mathcal{C} : \text{rad}_{\mathcal{C}}(x, y) = \{ f \in \text{Hom}_{\mathcal{C}}(x, y) \mid \exists g : g \cdot f \cdot x = 1_{x-gy} \text{ is invertible} \}

Powers of the radical:

$\text{rad}_{\mathcal{C}}^m(x, y) = \{ \sum_{\text{finite}} x \rightarrow r_1 \rightarrow r_2 \rightarrow \ldots \rightarrow r_m \rightarrow y \} \in \text{rad}_{\mathcal{C}}

\cdot \text{rad}_{\mathcal{C}}(x, y) = \bigcap_{m=1}^{\infty} \text{rad}_{\mathcal{C}}^m(x, y)

\Rightarrow \text{Moreover, } \text{rad}_{\mathcal{C}}^\infty \left( \frac{C}{\text{rad}_{\mathcal{C}}} \right) = \{0\}

Also 2-sided ideals
PROPERTIES OF \( \text{rad}_C \):

1. IF \( f = (f_{ij}) : \bigoplus_{i=1}^n X_i \rightarrow \bigoplus_{j=1}^n Y_j \) THEN \( \text{rad}_C \) = \( \bigoplus_{ij} f_{ij} \cdot \text{rad}_C \)

   SAME FOR \( \text{rad}_m \), \( m \in \mathbb{N} \cup \{0\} \), OR MORE GERNERALLY,
   FOR ANY \( 2 \)-SIDED IDEAL \( I \) OF \( C \).

2. \( \text{rad}_C (r, z) = \text{rad} (\text{End}(C)) \)
   \( r \in C \)

   **BEWARE! DOES NOT WORK FOR \( \text{rad}_m \), \( m \geq 2 \)!!!**

   \( \text{rad}_C (r, z) \leq \text{rad}_m (\text{End}(C)) \)

3. SUPPOSE \( x, y \in C \) HAVE LOCAL ENDO. RINGS. THEN

   \( \text{rad}_C (x, y) = \frac{1}{2} \text{ ker } : x \rightarrow y \) AND \( \text{ NON-ISO } \)

   IN PARTICULAR IF \( x \neq y \) THEN \( \text{rad}_C (x, y) = \text{Hom}_C (x, y) \)
Suppose \( \mathcal{C} \) is skeletal (different objects are non-iso), \( \mathbb{K} \)-linear (\( \mathbb{K} \) is a field). Hom-finite, all objects have local endomorphism rings.

\[
\text{Quiver of } \mathcal{C} \quad \text{... vertices: objects of } \mathcal{C} \\
\text{arrows: } \# \text{ arrows } x \to y = \dim_{\mathbb{K}} \text{End}_{\mathcal{C}} (x, y)
\]

As in Gabriel's theorem, we have a full \( \mathbb{K} \)-linear functor!

\[
\mathcal{G}_\mathcal{C} \quad \longrightarrow \quad \mathcal{G} \quad \text{on objects}
\]

\[
\mathcal{G} \cong \text{End}_{\mathcal{C}}
\]

**Def.** If \( A \) is a finite-dimensional algebra and \( \mathcal{O} = \text{mod} A \), then \( \Gamma \mathcal{O} = \Gamma_A \) is called the Auslander-Reiten quiver of \( A \).
IV. **Auslander–Reiten Theory** (Why! Tools to Compute \( \text{rad}_A \), \( \text{rad}^2_A \))

- A FIN. DIM. ALG. / \( A \)

\[ \xrightarrow{\text{Auslander–Reiten Translation}} \text{Certain Bijection Between} \]

\[ \{ \text{Non-Projective Indecomposables} \} \xrightarrow{\sim} \{ \text{Non-Ind. Indecomposables} \} \]

- \( T \) (ASS [THM IV.2.13]): \( \text{If } n \in \text{mod} A \cdot \text{ We have Functorial Iso's} \)

\[ \text{Ext}^0_A(M, n) \cong \text{DHom}_A(M, \Omega n) \cong \text{DHom}_A(\Omega^2 n, n) \]

- **Functorial**

\[ \text{IF } \begin{array}{c} n \in \text{Indecomposable Non-Projective} \end{array} \]

\[ \text{Ext}^1_A(M, n) \cong \text{DHom}_A(M, n) = \text{D}(\text{End}_A(n) \cong \text{Hom}_A(n, n)) \]

End\( (A) \)-BIMODULE WITH

SImple SOLE FROM BOTH LEFT AND RIGHT

THE SOLE IS 1-DIM, REPRESENTED BY

\[ 0 \rightarrow \Omega n \rightarrow E \rightarrow n \rightarrow 0 \text{ SEQ.} \]

\[ \text{IF } n \in \text{rad}_A A \]

- **Generators**

FOR \( \text{rad}_A A \) AS A RIGHT IDEAL