NMAG442 Representation Theory of Finite-Dimensional Algebras

Excercise session 6—May 5, 2021

Our goal today is to calculate global dimension of an algebra using its Cartan matrix and to discuss examples of how reflection functors work and what they do with dimension vectors of representations.

We work over an algebraically closed k and with finite-dimensional modules.

Cartan matrices

Exercise 1. Calculate the Cartan matrix for the path algebra of the following quiver:

$$\begin{tikzcd}
& 1 \arrow[dl, "\alpha"'] & \\
2 \arrow[rr, "\beta"'] & & 3 \arrow[ul, "\gamma"']
\end{tikzcd}$$

bound by relations $\alpha\beta, \beta\gamma$, and $\gamma\alpha$. Calculate the global dimension of the algebra.

Reflection functors

Definition 1 (Reflections; section 3.2 in [1])**.** *Let Q be a quiver. The reflection with respect to a vertex $i \in Q_0$ is by definition the map:*

$$\sigma_i : \mathbb{Z}^n \to \mathbb{Z}^n \,\text{with}\, \sigma_i(x) = x - \frac{2(x, e_i)_Q}{(e_i, e_i)_Q} e_i$$

where e_i is the ith coordinate vector.

Recall that $(x, y)_Q = q_Q(x + y) - q_Q(x) - q_Q(y)$ where:

$$q_Q(x) = \sum_{i \in Q_0} x_i^2 - \sum_{\alpha : i \to j \in Q_1} x_i x_j$$

is the associated quadratic form.

Definition 2 (Reflection functors on objects; section 3.3 in [1])**.** *Let Q be a quiver, and let $i \in Q_0$ be a sink. We define a functor $S_i^+ : \mathrm{Rep}(Q, k) \to \mathrm{Rep}(\sigma_i Q, k)$ where $\sigma_i Q$ has the same vertices and arrows as Q, but for all $e \in Q_1$ with $t_Q(e) = i$ we set $t_{\sigma_i Q}(e) = s_Q(e)$ and $s_{\sigma_i Q}(e) = i$.*

We define $S_i^+(X)$ as follows:

(i) For $j \in Q_0$ such that $i \neq j$, we set $S_i^+(X)_j = X_j$.

(ii) For $\alpha \in Q_1$ such that $t_Q(\alpha) \neq i$, we set $S_i^+(X)_\alpha = X_\alpha$.

(iii) The vector space $S_i^+(X)_i$ and maps $S_i^+(X)_\alpha$ for arrows ending in i in Q are given by the following exact sequence:

$$0 \longrightarrow S_i^+(X)_i \xrightarrow{[S_i^+(X_\alpha)]^T} \bigoplus_{\alpha:j\to i\in Q_1} X_j \xrightarrow{[X_\alpha]} X_i$$

Proposition 3 (Part of Lemma 3.3.3 in [1])**.** *Let Q be a quiver, and let $i \in Q_0$ be a sink. Given X an indecomposable representation of Q, the followign are equivalent:*

(i) $X \not\cong S(i)$*;*

(ii) $S_i^+(X)$ *is non-zero indecomposable;*

(iii) $\sigma(\underline{\dim}\, X) = \underline{\dim}\, S_i^+(X)$.

Exercise 2. Let us have the following representation X of the Kronecker quiver:

$$k^2 \underset{\psi}{\overset{\varphi}{\rightrightarrows}} k^3$$

where φ is the inclusion on the first two coordinates, ψ is the inclusion on the last two coordinates, and vertices are labelled from left to right. Calculate $\sigma_2(\underline{\dim}\, X)$ and $S_2^+(X)$.

For the next two exercises, we will work with the following quiver Q:

$$\begin{tikzcd}
& 1 \arrow[d, "\alpha"'] & \\
& 2 & \\
3 \arrow[ur, "\beta"'] & & 4 \arrow[ul, "\gamma"']
\end{tikzcd}$$

Note that its underlying graph is the Dynkin diagram D_3.

Exercise 3. Given the representation $P(1)$:

$$\begin{tikzcd}
& k \arrow[d, "1"'] & \\
& k & \\
0 \arrow[ur] & & 0 \arrow[ul]
\end{tikzcd}$$

Calculate $\sigma_2(\underline{\dim}\, P(1))$ and $S_2^+(P(1))$, and then iterate with sinks 1, 3, and 4.

Exercise 4. Given the representation X:

$$\begin{tikzcd}
& k \arrow[d, "\begin{bmatrix}1\\0\end{bmatrix}"'] & \\
& k^2 & \\
k \arrow[ur, "\begin{bmatrix}1\\1\end{bmatrix}"'] & & k \arrow[ul, "\begin{bmatrix}1\\-1\end{bmatrix}"']
\end{tikzcd}$$

Calculate $\sigma_2(\underline{\dim}\, X)$ and $S_2^+(X)$, and then iterate with sinks 1, 3, and 4. The representation X is indecomposable assuming that $\operatorname{char} k \neq 2$. This was proved at the last session.

References

[1] Krause, H. Representations of quivers via reflection functors. *arXiv preprint arXiv:0804.1428* (2008).

Feel free to reach me at `jakub.kopriva@mff.cuni.cz`. Also, I am available for short consultations on problems from the exercise sessions after previous arrangment via e-mail.