Ext and Vanishing of Derived Functors of Inverse Limits

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Outline

1. Motivation
   - When Is Ext Ortoghonality Inherited by Pure Submodules?
   - Uncountable Inverse Systems
2. Exactness of Inverse Limits and Vanishing of Ext
   - Inverse Limits of Uncountable Inverse Systems
   - Vanishing of Ext for Uncountable Direct Systems
   - Application – Σ-cotorsion Modules

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Finitely Presented Modules.

- Well-known: If M is finitely presented, $\mathrm{Ext}^1(M, C) = 0$ and D is a pure submodule of C, then $\mathrm{Ext}^1(M, D) = 0$.
- Not true in general when M is not finitely presented.

A Result for Countable Direct Systems.

Theorem ([Bazzoni, Herbera '05])

If $(M_n)_{n<\omega}$ is a countable direct system of finitely presented module such that $\mathrm{Ext}^1(M_n, C) = 0$ for all M_n, then TFAE:

1. $\mathrm{Ext}^1(\varinjlim M_n, C^{(\omega)}) = 0$,
2. $(\mathrm{Hom}(M_n, C))_{n<\omega}$ *satisfies the Mittag-Leffler condition.*

Moreover, if either condition is satisfied, then also $\mathrm{Ext}^1(\varinjlim M_n, C^{(\omega)}) = 0$ *for any pure submodule $D \subseteq C$.*

Applications.

- Theorem [Bazzoni, Herbera '05] has been used to solve the following open problems:
  - All tilting classes are of finite type (Bazzoni, Eklof, Herbera, Š., Trlifaj '05).
  - Baer modules over domains (Angeleri Hügel, Bazzoni, Herbera '05).
- Important! In both cases, the problem was reduced to proving that $\operatorname{Ker}\operatorname{Ext}^1(\varinjlim M_n, -) = 0$ is closed under certain pure submodules using other methods (set theory).
- Hence, we only need to consider countable direct limits of finitely presented modules and Theorem applies.

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The Problem of Σ-cotorsion Modules.

Definition

An module C is cotorison if $\mathrm{Ext}^1(F, C) = 0$ for every flat module F. And C is Σ-cotorsion if $C^{(I)}$ is cotorsion for any set I.

Question

If C is Σ-cotorsion, is any pure submodule of C or any module elementarily equivalent to C again Σ-cotorsion?

Example

- True for countable rings [Asensio, Herzog '05].
- True for valuation domains [Bazzoni, Š. '06].
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No Hope for Reduction to Countable Direct Systems

- If we wanted to mimic the proofs for tilting modules or Baer modules, we would need:
  - For every flat module F, there is a filtration

$$0 = F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_\alpha \subseteq F_{\alpha+1} \subseteq \cdots \subseteq F_\kappa = F$$

    with $F_{\alpha+1}/F_\alpha$ flat for all α.
- But there is no hope to have this!
  - In this case, all the F_α's would be pure submodules of F.
  - But if R is a valuation domain and Q its quotient field, then Q is flat but does not have any non-trivial pure submodules.

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Ext and the Derived Functor of Inverse Limit

- [Jensen '72] For any inverse system $(H_i)_{i\in I}$, there is a complex

$$0 \to \varprojlim H_i \to \prod_{i_0\in I} H_{i_0} \xrightarrow{\Delta^0} \prod_{i_0<i_1} H_{i_0 i_1} \xrightarrow{\Delta^1} \prod_{i_0<i_1<i_2} H_{i_0 i_1 i_2} \xrightarrow{\Delta^2} \dots$$

where $H_{i_0 i_1 \dots i_n} = H_{i_0}$ for all $i_0 < i_1 < \cdots < i_n$ in I and
- The functor $\varprojlim^n H_i$ is defined as n-th homology.
- If $(M_i)_{i\in I}$ is a direct system and $\operatorname{Ext}^1(M_i, C) = 0$ for all M_i:

$$\operatorname{Ext}^1(\varinjlim M_i, C) \cong \varprojlim{}^1 \operatorname{Hom}(M_i, C)$$

- That is, we can examine when $\varprojlim^1 \operatorname{Hom}(M_i, C) = 0$.

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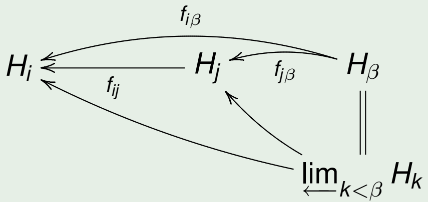
- That is, we can examine when $\varprojlim^1 \operatorname{Hom}(M_i, C) = 0$.

Continuous Inverse Systems.

Definition

An inverse system $(H_i, f_{ij})_{i<\kappa}$ indexed by a well-ordered set is **continuous** if $(H_\beta, f_{i\beta})$ is an inverse limit of the subsystem $(H_i, f_{ij})_{i<\beta}$ for each limit element $\beta < \kappa$.

Example



Reduced products.

Definition

For a module H and a cardinal κ, denote by $H^{<\kappa}$ the submodule of H^κ formed by the elements with supports of cardinality $< \kappa$

Example

$H^{<\omega} = H^{(\omega)}$.

The Result.

Theorem

Let

- κ *be an uncountable regular cardinal, and*
- $(H_i, f_{ij})_{i<\kappa}$ *be a continuous inverse system.*

TFAE:

1. $\varprojlim^1 H_i^{<\kappa} = 0$.
2. *There is a closed unbounded subset* $X \subseteq \kappa$ *such that* $f_{ij} : H_j \to H_i$ *is surjective for each* $i, j \in X$, $i < j$.

About the Proof.

- The proof is technical.
- Set-theoretic methods are used (stationary subsets, closed unbounded subsets).
- The fact that κ is uncountable is essential.

Summary for vanishing of $\varprojlim^1$.

- Plus: A general condition for vanishing of $\varprojlim^1$.
- Minus: We wanted direct sums instead of $H_i^{<\kappa}$.
- Minus: We have to translate the condition back to the language of Hom's and Ext's.

Outline

Continuous Direct Systems.

Definition (a generalization of the definition of a filtration)

A direct system $(M_i, g_{ij})_{i<\kappa}$ indexed by a well-ordered set is **continuous** if $(M_\beta, g_{i\beta})$ is a direct limit of the subsystem $(M_i, g_{ij})_{i<\beta}$ for each limit element $\beta < \kappa$.

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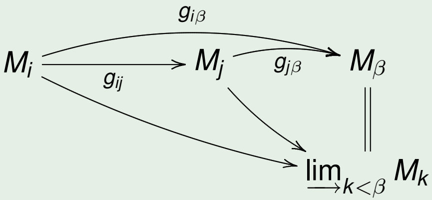

The Result.

Theorem (generalizes [Eklof, Mekler '02] and [Š., Trlifaj '06])

Let

- κ *be an uncountable regular cardinal,*
- $\mathcal{C}$ *be a class of modules closed under direct sums,*
- $(M_i, g_{ij})_{i<\kappa}$ *be a continuous direct system, and*
- *all* M_i *are* $< \kappa$*-generated modules and* $\operatorname{Ext}^1_R(M_\alpha, \mathcal{C}) = 0$.

TFAE:

1. $\operatorname{Ext}^1_R(\varinjlim M_i, C) = 0 \ \ (\forall C \in \mathcal{C})$.
2. *There is a closed unbounded subset* $X \subseteq \kappa$ *such that* $\operatorname{Hom}_R(g_{ij}, C)$ *is surjective* $(\forall C \in \mathcal{C})$ $(\forall i, j \in X, i < j)$.

About the Proof.

- The proof is technical.
- The homological part of the proof is different from [Eklof, Mekler '02] and [Š., Trlifaj '06].
- But the set-theoretic methods are used in a similar way.
- The fact that κ is uncountable is essential.

Summary for the vanishing of Ext.

- Plus: We have direct sums inside Ext now.
- Minus: More complicated statement, more assumptions.

Outline

The Result.

Theorem

Let C be a Σ-cotorsion module over a ring of cardinality $\leq \aleph_1$. Then all pure submodules of C as well as all modules elementarily equivalent to C are Σ-cotorsion.

Example (previous results)

- True for countable rings [Asensio, Herzog '05].
- True for valuation domains [Bazzoni, Š. '06].
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Sketch of the Proof.

Let $D \subseteq C$ be a pure submodule. The proof consists of 4 steps:

1. Show that C is cotorsion iff $\mathrm{Ext}^1(\varinjlim M_i, C) = 0$ for direct systems $(M_i)_{i \in I}$ of f. g. free modules with $|I| \leq \aleph_1$.
2. Show that $\mathrm{Ext}^1(\varinjlim M_i, D) = 0$ when I is countable.
3. Show that $\mathrm{Ext}^1(\varinjlim M_i, D) = 0$ when $|I| = \aleph_1$.
4. (Show that $\tilde{C}$ is Σ-cotorsion for every $\tilde{C} \equiv C$.)

We use the general idea from [Bazzoni, Herbera '05]:

- Translate $\mathrm{Ext}^1(\varinjlim M_i, C) = 0$ to a chain condition for $(\mathrm{Hom}(M_i, C))_{i \in I}$.
- Show that the condition is inherited for pure $D \subseteq C$.

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Step I – Reduction to Flat Modules of Cardinality $\leq \aleph_1$.

- Well-known: for any flat module F, there is a filtration

$$0 = F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_\alpha \subseteq F_{\alpha+1} \subseteq \cdots \subseteq F_\kappa = F$$

  such that $F_{\alpha+1}/F_\alpha$ are flat of cardinality $\leq \aleph_0 + |R|$.
- Well known:
  $(\forall \alpha)\, \mathrm{Ext}^1(F_{\alpha+1}/F_\alpha, C) = 0 \implies \mathrm{Ext}^1(F, C) = 0.$
- [Lazard '69] F flat of cardinality $\leq \aleph_1 \implies F = \varinjlim_{i \in I} M_i$ where M_i are f. g. free and $|I| \leq \aleph_1$.

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Step II – Countable Flat Modules.

Theorem (generalizes [Bazzoni, Herbera '05])

Let

- *M be a countably presented module (over any ring), and*
- *C be a module such that $\mathrm{Ext}^1(M, C^{(\omega)}) = 0$.*

Then:

- $\mathrm{Ext}^1(M, D) = 0$ *whenever $D \subseteq C$ is a pure submodule.*

Proof.

Fine tuning the proof in [Bazzoni, Herbera '05]. □

Step III – Flat Modules of Cardinality $\aleph_1$.

- Find a continuous inverse system $(M_i, g_{ij})_{i<\aleph_1}$ with M_i's flat countably generated and $F = \varinjlim M_i$.
- Theorem $\implies$ pass to a subsystem with all $\mathrm{Hom}(g_{ij}, C)$ surjective.
- By the countable case, show that all $\mathrm{Hom}(g_{ij}, D)$ are surjective whenever $D \subseteq C$ is pure.
- Theorem $\implies$ $Ext^1(F, D) = 0$.

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Summary

- We have found a criterion for vanishing of $\varprojlim^1$ on uncountable continuous inverse systems.
- We have found a similar criterion for vanishing of Ext on uncountable continuous direct systems.
- We have generalized results on Σ-cotorsion modules by Asensio and Herzog to rings of cardinality $\leq \aleph_1$.

- Outlook
  - Do similar results hold for Σ-cotorsion modules over arbitrary rings?
  - Most of the metods are not tied to flat and cotorsion modules – perhaps there are also other applications.

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