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DIPLOMOVÁ PRÁCE

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Finitistic dimensions of rings

Katedra algebry
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Prohlašuji, že jsem svou diplomovou práci napsal samostatně a výhradně s použitím citovaných pramenů. Souhlasím se zapůjčováním práce.

V Praze dne 9.4.2004 Jan Šťovíček

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Abstrakt: Buď R okruh a označme $\mathcal{P}^{<\infty}$ třídu všech konečně prezentovaných modulů konečné projektivní dimenze. Potom malá finitistická dimenze okruhu R je konečná, právě když Ext ortogonální třída $(P^{<\infty})^{\perp}$ je vychylující. Jako prostředek pro další důkazy uvedeme obecnou verzi Auslander-Reitenovy formule. Následně popíšeme strukturu vychylujícího modulu odpovídajícího třídě $(P^{<\infty})^{\perp}$ nad algebrou podle Igusy, Smalø a Todorova. Uvedeme příklady modulů třídy $(P^{<\infty})^{\perp}$, které nejsou dosažitelné z konečně prezentovaných modulů této třídy pomocí direktních limit. Popíšeme také svaz všech vychylujících tříd konečného typu nad touto algebrou. Nakonec nastíníme možný přístup k hypotéze konečného typu pro 1-vychylující třídy. Specielně ukážeme, že konečný typ 1-vychylující třídy není možné poznat pouze z jejích čistě injektivních modulů.
Klíčová slova: finitistická dimenze, artinovská algebra, vychylující třída, vychylující modul

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Abstract: Let R be a ring and denote $\mathcal{P}^{<\infty}$ the class of all finitely presented modules of a finite projective dimension. Then the little finitistic dimension of the ring R is finite if, and only if, the Ext orthogonal class $(P^{<\infty})^{\perp}$ is tilting. As a means for the following proofs, we show a general version of the Auslander-Reiten formula. Then we describe a structure of a tilting module corresponding to the class $(P^{<\infty})^{\perp}$ in the case of the Igusa-Smalø-Todorov algebra. We give examples of modules from $(P^{<\infty})^{\perp}$ that are not reachable from finitely generated modules of $(P^{<\infty})^{\perp}$ by direct limits. We also give a description of the lattice of all tilting classes of a finite type over this algebra. Finally, we outline a possible approach for investigating a finite type conjecture for 1-tilting classes. In particular, we show that a finite type of a 1-tilting class cannot be determined by looking only at its pure-injective modules.
Keywords: finitistic dimension, artin algebra, tilting class, tilting module

1 Introduction

The general idea behind introducing homological dimensions was a wish to find a measure for a deviation of the module category over a given ring R from the "ideal" categories of vector spaces, where every object is projective. The classical Wedderburn-Artin theorem reveals that the same property of every object being projective arises exactly when R is a finite product of matrix rings over skewfields. The notion of the global dimension of a ring reflects this fact and gives us other characterisation of these so called semisimple rings.

Although this approach was successful in many cases, there are examples where it does not meet up the expectations. So the finitistic dimensions appeared. The principle is the same, but they measure a complexity only for particular subcategories of modules.

The topic of the interest here are the finitisic dimension conjectures. Namely the second one, which is a variant of a problem publicized by Bass already in 1960, but is not completely solved so far. It states that the little finitistic dimension is finite whenever R is a finite dimensional algebra. Of course, there is a lot of partial results in this direction achieved by various means, and covering many situations.

The idea pursued in this text is the connection between the second finitistic dimension conjecture and tilting theory.

In section 2, there is an overview of theoretical results employed later. The proofs are often omitted, since many of them are well-known, and they are not meant to be the point of this text. The connection between finitistic dimensions and tilting classes is made precise by theorem 2.30.

The necessary results specific to artin algebras are given in section 3. Although the statements in this section are implicitly used eg. in [17], the proofs seem to be hard to find. So the complete proofs derived from the more special ones in [3] are given here.

An example of a finite dimensional algebra given by Igusa, Smalø and Todorov is studied in detail in section 4. The point here is that although the existence of a tilting module for the tilting class $(\mathcal{P}^{<\infty})^{\perp}$ is ensured by the theory, this does not clarify its inner structure. A description of it, or more precisely of its corresponding linear representation, is done; so it is possible to determine values of linear maps on suitably chosen base vectors. On the way, there are "side effect" results, such as the characterisation of indecomposable modules from $\mathcal{P}^{<\infty}$, the description of the lattice of tilting classes of a finite type, or the example of modules from $(\mathcal{P}^{<\infty})^{\perp}$ that are not direct limits of finitely presented modules thereof.

The last section focuses on the conjecture that every 1-tilting class is of a finite type. It translates the conjecture using the definition of bundles, and shows that if R is a left coherent, then a finite type of a tilting class cannot be determined only by its pure-injective modules.

2 Preliminaries

In the following text, let R denote (associative unital) ring, R-Mod (Mod-R) the category of left (right) R-modules respectively. Let R-mod and mod-R be the corresponding full subcategories of finitely presented modules. For convenience, the word module itself will mean a left R-module.

2.1 Some homological facts

Let us recall some basic facts from homological algebra which will be used freely throughout this text. For the most part, proofs are omitted in this section, since they are well-known and usually quite long and technical. The details could be found eg. in [15].

The key notion here is the Ext *functor*. Let M, N be modules and

$$\cdots \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \to M \to 0$$

be a projective resolution of M; that is, a long exact sequence with P_i's projective, $i < \omega$. Applying the functor $\mathrm{Hom}_R(-, N)$ to this resolution, we will get a complex:

$$0 \to \mathrm{Hom}_R(P_0, N) \xrightarrow{d_1'} \mathrm{Hom}_R(P_1, N) \xrightarrow{d_2'} \mathrm{Hom}_R(P_2, N) \to \cdots$$

The n-th homology group $\mathrm{Ker}\, d_{n+1}' / \mathrm{Im}\, d_n'$ is denoted $\mathrm{Ext}_R^n(M, N)$.

Then $\mathrm{Ext}_R^n(M, N)$ depends functorially on both M, N in a natural way. As a consequence, if S is another ring and ${}_R M_S$ is an R-S-bimodule, then $\mathrm{Ext}_R^n(M, N)$ is also a left S-module. And similarly for N. Further, it turns out that $\mathrm{Ext}_R^n(M, N)$ does not depend on a praticular choice of a projective resolution of M, since the groups constructed using different resolutions are naturally isomorphic.

The following fact is important. It shows that Ext "describes a right non-exactness of the Hom functor".

Proposition 2.1. *Let M be a module and $0 \to X \to Y \to Z \to 0$ be a short exact sequence of modules. Then there are long exact sequences:*

$$\begin{aligned} 0 \to \mathrm{Hom}_R(M, X) \to \mathrm{Hom}_R(M, Y) \to \mathrm{Hom}_R(M, Z) \to \\ \to \mathrm{Ext}_R^1(M, X) \to \mathrm{Ext}_R^1(M, Y) \to \mathrm{Ext}_R^1(M, Z) \to \cdots \\ \cdots \to \mathrm{Ext}_R^n(M, X) \to \mathrm{Ext}_R^n(M, Y) \to \mathrm{Ext}_R^n(M, Z) \to \cdots \end{aligned}$$

and

$$\begin{aligned}0 \to \operatorname{Hom}_R(Z,M) \to \operatorname{Hom}_R(Y,M) \to \operatorname{Hom}_R(X,M) \to\\ \to \operatorname{Ext}^1_R(Z,M) \to \operatorname{Ext}^1_R(Y,M) \to \operatorname{Ext}^1_R(X,M) \to \cdots\\ \cdots \to \operatorname{Ext}^n_R(Z,M) \to \operatorname{Ext}^n_R(Y,M) \to \operatorname{Ext}^n_R(X,M) \to \cdots \quad \square\end{aligned}$$

There is also an equivalent definition of Ext^1_R. Two short exact sequences $0 \to N \to X \to M \to 0$ and $0 \to N \to Y \to M \to 0$ are said to be equivalent if there is a commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d, "e"'] & M \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & N \arrow[r] & Y \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

Necessarily, any map e in such a diagram must be an isomorphism. Let us denote $E(M,N)$ the set of equivalence classes of short exact sequences in the shape $0 \to N \to X \to M \to 0$. For a given exact sequence δ denote $[\delta]$ its equivalence class.

Take an exact sequence $\delta : 0 \to N \to X \to M \to 0$ and a map $f : M' \to M$. Then we have a pull-back diagram:

$$\begin{tikzcd} \delta_f : & 0 \arrow[r] & N \arrow[r] \arrow[d, equal] & X_f \arrow[r] \arrow[d] & M' \arrow[r] \arrow[d, "f"'] & 0 \\ \delta : & 0 \arrow[r] & N \arrow[r] & X \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

It can be shown that $[\delta] = [\varepsilon]$ implies $[\delta_f] = [\varepsilon_f]$. Thus, we have a map $E(f,N) : E(M,N) \to E(M',N)$. It can be easily checked that $E(-,N)$ is actually a contravariant functor; that is $E(\mathrm{id}_M, N) = \mathrm{id}_{E(M,N)}$ and $E(fg,N) = E(g,N) \circ E(f,N)$. Similarly, from a map $g : N \to N'$ we obtain a map $E(M,g) : E(M,N) \to E(M,N')$ using a push-out. Again, $E(M,-)$ is actually a covariant functor. There is an important proposition connecting these two functors:

Proposition 2.2. *Let $f : M' \to M$ and $g : N \to N'$ be homomorphisms of left R-modules. Then $E(f,N') \circ E(M,g) = E(M',g) \circ E(f,N)$.* $\square$

Given homomorphisms $f : M' \to M$ and $g : N \to N'$, this fact allows us to define a map $E(f,g) : E(M,N) \to E(M',N')$ as $E(f,g) = E(f,N') \circ E(M,g) = E(M',g) \circ E(f,N)$.

Now we can define an addition in $E(M,N)$, so called *Baer sum*; it makes $E(M,N)$ an abelian group. Let $\delta : 0 \to N \to X \to M \to 0$ and $\varepsilon : 0 \to N \to$

$Y \to M \to 0$ be exact sequences. Then we put $[\delta] + [\varepsilon] = E(\Delta, \sigma)([\delta \oplus \varepsilon])$, where $\delta \oplus \sigma$ is an exact sequence

$$\delta \oplus \varepsilon : \quad 0 \longrightarrow N \oplus N \longrightarrow X \oplus Y \longrightarrow M \oplus M \longrightarrow 0,$$

$\Delta : M \to M \oplus M$ is a diagonal map ($\Delta(x) = (x, x)$), and $\sigma : N \oplus N \to N$ is a map adding up components of the direct sum.

It can be shown that $E(M, N)$ is isomorphic to $\mathrm{Ext}^1_R(M, N)$ as an abelian group, and these isomorphisms are functorial in both variables M, N. The zero in $\mathrm{Ext}^1_R(M, N)$ corresponds to an equivalence class of split exact sequences. Thus, we have the following useful lemma:

Lemma 2.3. *Let M, N be left R-modules. Then $\mathrm{Ext}^1_R(M, N) = 0$ if, and only if, every extension of N by M splits.* □

The Ext groups are also closely related to the notions of projective and injective modules. We have the following characterisation:

Lemma 2.4. *Let M be a left R-module. Then:*

1. *M is injective if, and only if, $\mathrm{Ext}^1_R(X, M) = 0$ for all X if, and only if, $\mathrm{Ext}^n_R(X, M) = 0$ for all X and all $n \geq 1$,*

2. *M is projective if, and only if, $\mathrm{Ext}^1_R(M, X) = 0$ for all X if, and only if, $\mathrm{Ext}^n_R(M, X) = 0$ for all X and all $n \geq 1$.* □

The last lemma is not very convenient when we need to test a projectivity or injectivity, since we have to show that $\mathrm{Ext}^1_R(M, -)$ or $\mathrm{Ext}^1_R(-, M)$ vanishes for a proper class of all modules. Fortunately, the statement can be refined in many cases in such a way that vanishing of Ext needs to be tested only for a set of modules. In the case of injectivity, Baer lemma and proposition 2.1 imply that cyclic modules are sufficient. In the case of projectivity we need R to be left perfect, and then we only need to be concerned about simple modules [22, 2.2]. If R is non-perfect, however, the non-existence of such a testing set of modules is consistent with a set theory (ZFC+GCH) (cf. [24, 2.5]).

Proposition 2.5. *Let M be a left R-module. Then:*

1. *M is injective if, and only if, $\mathrm{Ext}^1_R(C, M) = 0$ for all cyclic left R-modules,*

2. *for R left perfect, M is projective if, and only if, $\mathrm{Ext}^1_R(M, S) = 0$ for all simple left R-modules.* □

Next, let M be a module and $p : P \to M$ be an epimorphism with P projective. Then the kernel of p is called a *syzygy* of M, and is denoted $\Omega^1(M)$. Higher syzygies of M are defined by induction, $(n+1)$-th syzygy $\Omega^{n+1}(M)$ is defined as $\Omega^1(\Omega^n(M))$. Similarly, if $i : M \to I$ is a monomorphism with I injective, the cokernel of i is called a *cosyzygy* of M and is denoted $\Omega^{-1}(M)$. Higher cosyzygies of M are defined as well: $\Omega^{-(n+1)}(M) = \Omega^{-1}(\Omega^{-n}(M))$. For the sake of completeness put $\Omega^0(M) = M$.

Of course, the syzygies and cosyzygies of M are not determined uniquely. But they offen behave in the same manner with respect to Ext. As an example we can state the so called dimension shifting theorem. It is useful eg. for turning Ext^n of something to Ext^1 of something else.

Theorem 2.6. *Let M, N be left R-modules and $n > 1$. Then:*

1. $\mathrm{Ext}^n_R(M, N) \cong \mathrm{Ext}^{n-1}_R(\Omega^1(M), N)$,
2. $\mathrm{Ext}^n_R(M, N) \cong \mathrm{Ext}^{n-1}_R(M, \Omega^{-1}(N))$. □

Now we can define other principal notions for this text.

Definition 2.7. The *projective dimension* $\mathrm{pd}\, M$ of a module M is the smallest $n < \omega$, such that there is a projective resolution:

$$0 \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0$$

If no such n exist, we say $\mathrm{pd}\, M = \infty$.

Dually, the *injective dimension* $\mathrm{id}\, M$ is the smallest $n < \omega$, such that there is a following injective coresolution; that is, a long exact sequence with I_i's injective:

$$0 \to M \to I_0 \to I_1 \to \cdots \to I_n \to 0$$

If no such n exist, we say $\mathrm{id}\, M = \infty$.

Clearly, $\mathrm{pd}\, M = 0$ if, and only if, M is projective, and $\mathrm{id}\, M = 0$ if, and only if, M is injective.

Proposition 2.8. *Let M be a left R-module and $n < \omega$. The following statements are equivalent:*

1. $\mathrm{pd}\, M \leq n$,
2. $\Omega^n(M)$ *is projective,*
3. $\mathrm{Ext}^{n+k}_R(M, X) = 0$ *for all X and all $k \geq 1$,*
4. $\mathrm{Ext}^{n+1}_R(M, X) = 0$ *for all X.*

Moreover, if R is left perfect, there is one more equivalent condition:

5. $\operatorname{Ext}_R^{n+1}(M, S) = 0$ *for all simple modules S.* $\square$

Proposition 2.9. *Let M be a left R-module and $n < \omega$. The following statements are equivalent:*

1. $\operatorname{id} M \leq n$,
2. $\Omega^{-n}(M)$ *is injective,*
3. $\operatorname{Ext}_R^{n+k}(X, M) = 0$ *for all X and all $k \geq 1$,*
4. $\operatorname{Ext}_R^{n+1}(X, M) = 0$ *for all X,*
5. $\operatorname{Ext}_R^{n+1}(C, M) = 0$ *for all cyclic modules C.* $\square$

Definition 2.10. A *left global dimension* of the ring R is defined as:

$$\operatorname{gl.dim} R = \sup\{\operatorname{pd} M | M \in R\text{-Mod}\}$$

The global dimension $\operatorname{gl.dim} R$ should in some sense "measure" a complexity of the module category R-Mod. A ring R is semisimple if, and only if, $\operatorname{gl.dim} R = 0$, and R is hereditary if, and only if, $\operatorname{gl.dim} R \leq 1$. So for example $\operatorname{gl.dim} \mathbb{Z} = 1$ and it can be shown that $\operatorname{gl.dim} k[x_1, \ldots, x_n] = n$ for the polynomial ring in n variables, with coefficients in a field k.

It could be easily seen that if $\operatorname{gl.dim} R$ is finite, it is equal to the smallest $n < \omega$, such that $\operatorname{Ext}^{n+1}(M, N) = 0$ for all M, N. Therefore, also $\operatorname{gl.dim} R = \sup\{\operatorname{id} M | M \in R\text{-Mod}\}$. Further, put $k = \sup\{\operatorname{pd} C | C \text{ is cyclic}\}$. Then $\operatorname{Ext}^{k+1}(C, X) = 0$ for all X and all cyclic modules C if k is finite. In this case $\operatorname{Ext}^{k+1}(C, X) = 0$ for all X and *all* C by proposition 2.9. So we have the following fundamental theorem:

Theorem 2.11. $\operatorname{gl.dim} R = \sup\{\operatorname{pd} C | C$ *is cyclic*$\}$. *In other words, a global dimension of R is attained on cyclic modules.* $\square$

Another functor which is used many times below is Tor. It is constructed in a similar fashion as Ext, but this time using a tensor product. Let M be a left R-module and N a right R-module, and take a projective resolution of M:

$$\cdots \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \to M \to 0$$

Applying the functor $N \otimes_R -$ to this resolution, we will get a complex:

$$\cdots \xrightarrow{d_3^*} N \otimes_R P_2 \xrightarrow{d_2^*} N \otimes P_1 \xrightarrow{d_1^*} N \otimes P_0 \to 0$$

The n-th homology group $\operatorname{Ker} d_n^* / \operatorname{Im} d_{n+1}^*$ is denoted $\operatorname{Tor}_n^R(M,N)$.

It turns out that $\operatorname{Tor}_n^R(M,N)$ depends functorially on both M,N, and it does not depend on a particular choice of a projective resolution of M. Moreover, if we took a projective resolution of the right module N and applied the functor $- \otimes_R M$, we would get the same Tor_n^R groups as well. It is not hard to see that $\operatorname{Tor}_n^R(M,-)$ and $\operatorname{Tor}_n^R(-,N)$ are covariant additive functors in a natural way.

There are analogues of properties of Ext stated below for Tor. Since both the variables M,N in $\operatorname{Tor}_n^R(M,N)$ have essentially the same role and the only difference is we have to change left modules for right ones and vice versa, the assertions below will be stated only for one variable, and the reader is left to write down the symmetric statements if necessary.

Proposition 2.12. *Let M be a left R-module and $0 \to X \to Y \to Z \to 0$ be a short exact sequence of right R-modules. Then there is a long exact sequence:*

$$\begin{aligned} \cdots \to \operatorname{Tor}_n^R(X,M) \to \operatorname{Tor}_n^R(Y,M) \to \operatorname{Tor}_n^R(Z,M) \to \cdots \\ \cdots \to \operatorname{Tor}_1^R(X,M) \to \operatorname{Tor}_1^R(Y,M) \to \operatorname{Tor}_1^R(Z,M) \to \\ \to X \otimes_R M \to Y \otimes_R M \to Z \otimes_R M \to 0 \quad \square \end{aligned}$$

Proposition 2.13. *Let M be a left R-module. The following conditions are equivalent*

1. *M is flat,*
2. *$\operatorname{Tor}_n^R(X,M) = 0$ for all X and all $n \geq 1$,*
3. *$\operatorname{Tor}_1^R(X,M) = 0$ for all X,*
4. *$\operatorname{Tor}_1^R(C,M) = 0$ for all cyclic right R-modules C.* $\square$

Proposition 2.14. *Let M be a left R-module, N be a right R-module and $n > 1$. Then $\operatorname{Tor}_n^R(N,M) \cong \operatorname{Tor}_{n-1}^R(N, \Omega^1(M))$.* $\square$

Inspired by projective and injective dimensions, we can define a *weak dimension* $\operatorname{wd} M$ of a module M as the smallest $n < \omega$, such that there is a flat resolution:

$$0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$$

If no such n exist, we put $\operatorname{wd} M = \infty$.

Proposition 2.15. *Let M be a left R-module and $n < \omega$. The following statements are equivalent:*

1. $\operatorname{wd} M \leq n$,

2. $\Omega^n(M)$ *is flat,*

3. $\operatorname{Tor}^R_{n+k}(X, M) = 0$ *for all* X *and all* $k \geq 1$,

4. $\operatorname{Tor}^R_{n+1}(X, M) = 0$ *for all* X,

5. $\operatorname{Tor}^R_{n+1}(C, M) = 0$ *for all cyclic right* R*-modules* C. $\square$

Let S be a ring, ${}_RM$ be a left R-module, ${}_SN_R$ be an S-R-bimodule and ${}_SX$ be a left S-module. It is well-known that there is an isomorphism functorial in all three variables:

$$\operatorname{Hom}_S(N \otimes_R M, X) \cong \operatorname{Hom}_R(M, \operatorname{Hom}_S(N, X))$$

If ${}_RM$ is a finitely presented left R-module, ${}_RN'_S$ is an R-S-bimodule and X'_S is an injective right S-module, there is also a functorial isomorphism [10, 3.2.11]:

$$\operatorname{Hom}_S(N', X') \otimes_R M \cong \operatorname{Hom}_S(\operatorname{Hom}_R(M, N'), X')$$

Slightly adjusted, these isomorphisms are also valid substituting Hom_R by Ext^n_R and the tensor product by Tor^R_n. But first, we need a generalization of the notion of a finitely presented module. We say a module M is FP_n provided it has a projective resolution

$$\cdots \to P_i \to \cdots \to P_1 \to P_0 \to M \to 0$$

with projective modules P_i finitely generated for $i \leq n$. Then FP_0 modules are precisely the finitely generated ones, and FP_1 modules are precisely the the finitely presented ones. Note that all the classes FP_i, $i \geq 1$, coincide over coherent rings, and all the classes FP_i, $i \geq 0$, coincide over noetherian rings.

Proposition 2.16. *Let* R, S *be rings and* $n \geq 1$. *Then:*

1. *if* ${}_RM$ *is a left* R*-module,* ${}_SN_R$ *is an* S*-*R*-bimodule and* ${}_SX$ *is an injective left* S*-module, there is an isomorphism functorial in all three variables:*

$$\operatorname{Hom}_S(\operatorname{Tor}^R_n(N, M), X) \cong \operatorname{Ext}^n_R(M, \operatorname{Hom}_S(N, X))$$

2. *if* ${}_RM$ *is an* FP_{n+1} *left* R*-module,* ${}_RN'_S$ *is an* R*-*S*-bimodule and* X'_S *is an injective right* S*-module, there is an isomorphism functorial in all three variables:*

$$\operatorname{Tor}^R_n(\operatorname{Hom}_S(N', X'), M) \cong \operatorname{Hom}_S(\operatorname{Ext}^n_R(M, N'), X') \quad \square$$

Finally, we will focus on the relation of Ext and Tor with limits and colimits of families of modules. It is well-known that a covariant Hom functor commutes with limits and a contravariant Hom functor sends colimits to limits in an arbitrary category. There are additional relations in categories of modules, namely $\mathrm{Hom}_R(M, -)$ commutes with direct limits for any $M \in R\text{-mod}$ and $N \otimes_R -$ commutes with direct sums and direct limits for any $N \in \text{Mod-}R$. These statements can be partially extended to Ext and Tor, as we can see in the following lemmas. Notice that unlike a covariant Hom functor, $\mathrm{Ext}^n_R(M, -)$ does not commute with inverse limits in general.

Lemma 2.17. *Let M be a module, (N_α) an arbitrary family of modules and $(L_\beta, f_{\beta\gamma})$ an arbitrary directed system of modules. Then the following holds for each $n < \omega$:*

1. $\mathrm{Ext}^n_R(M, \prod_\alpha N_\alpha) \cong \prod_\alpha \mathrm{Ext}^n_R(M, N_\alpha)$,
2. $\mathrm{Ext}^n_R(M, \varinjlim L_\beta) \cong \varinjlim \mathrm{Ext}^n_R(M, L_\beta)$ *provided that M is FP_{n+1},*
3. $\mathrm{Ext}^n_R(\bigoplus_\alpha N_\alpha, M) \cong \prod_\alpha \mathrm{Ext}^n_R(N_\alpha, M)$,
4. $\mathrm{Ext}^n_R(\varinjlim L_\beta, M) \cong \varprojlim \mathrm{Ext}^n_R(L_\beta, M)$ *provided that M is pure-injective.* □

Lemma 2.18. *Let M be a module, (N_α) an arbitrary family of right modules and $(L_\beta, f_{\beta\gamma})$ an arbitrary directed system of right modules. Then the following holds for each $n < \omega$:*

1. $\mathrm{Tor}^R_n(\bigoplus_\alpha N_\alpha, M) \cong \bigoplus_\alpha \mathrm{Tor}^R_n(N_\alpha, M)$,
2. $\mathrm{Tor}^R_n(\varinjlim L_\beta, M) \cong \varinjlim \mathrm{Tor}^R_n(L_\beta, M)$. □

2.2 Cotorsion pairs and approximations of modules

This section gives an overview of basic properties of cotorsion pairs and their relation to the approximation theory of modules. The notion of cotorsion pair, originally called "cotorsion theory", was introduced by Salce in the case of abelian groups [19]. More detailed and exhaustive overview of related state-of-the-art results could be found in [23].

First, we fix a notation. Let $n < \omega$. We will denote $\mathcal{P}_n$ ($\mathcal{I}_n$, $\mathcal{F}_n$) the class of all modules of projective (injective, weak) dimension at most n. The class of all modules of finite projective (injective, weak) dimension will be denoted by $\mathcal{P}$ ($\mathcal{I}$, $\mathcal{F}$). The class of all pure-injective modules will be denoted $\mathcal{PI}$. If $\mathcal{C}$ is an arbitrary class of modules, we will denote $\mathcal{C}^{<\infty} = \mathcal{C} \cap R\text{-mod}$.

Let M be a left or right R-module. Then the *character module* $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ will be denoted as M^c. Note that pure-injective modules are precisely the direct summands of character modules.

Let $\mathcal{C}$ be a class of modules. We will define Ext^1 orthogonal classes: $\mathcal{C}^\perp = \{M \in R\text{-Mod} | \operatorname{Ext}^1_R(X, M) = 0 \text{ for all } X \in \mathcal{C}\}$ and ${}^\perp\mathcal{C} = \{M \in R\text{-Mod} | \operatorname{Ext}^1_R(M, X) = 0 \text{ for all } X \in \mathcal{C}\}$.

Definition 2.19. A *cotorsion pair* is a pair $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ of classes of modules, such that $\mathcal{A} = {}^\perp\mathcal{B}$ and $\mathcal{B} = \mathcal{A}^\perp$. The class $\mathcal{A} \cap \mathcal{B}$ is called a *kernel* of $\mathfrak{C}$.

The cotorsion pairs are analogues of torsion pairs, where Hom_R is substituted by Ext^1_R. This allows us to easily derive some basic properties of cotorsion pairs. But one still should be cautious, since for example a kernel of cotorsion pair need not to be trivial, unlike an intersection of torsion and torsion-free classes of a torsion pair. In fact, the kernel is non-trivial whenever the cotorsion pair is complete in the sense set up below, and this is a very usual case.

If $\mathcal{C}$ is an arbitrary class of modules, there are cotorsion pairs, such that the second or the first class of the pair is the least one containing $\mathcal{C}$; namely $({}^\perp\mathcal{C}, ({}^\perp\mathcal{C})^\perp)$ and $({}^\perp(\mathcal{C}^\perp), \mathcal{C}^\perp)$. The former is called a cotorsion pair *generated* by $\mathcal{C}$, the latter is a cotorsion pair *cogenerated* by $\mathcal{C}$. If $\mathcal{C}$ has a representative set of elements $\mathcal{S}$, then the first pair is generated by the single module $\prod_{X \in \mathcal{S}} X$, while the second one is cogenerated by the single module $\bigoplus_{X \in \mathcal{S}} X$.

The class of cotorsion pairs could be partially ordered by an inclusion in the first component. Using this ordering, the least cotorsion pair is $(\mathcal{P}_0, R\text{-Mod})$, and the greatest pair is $(R\text{-Mod}, \mathcal{I}_0)$. These cotorsion pairs are called *trivial*.

In fact, the ordering on cotorsion pairs forms a complete lattice; we will denote it $\mathfrak{L}_{\mathrm{Ext}}$. Given a sequence of cotorsion pairs $(\mathcal{A}_i, \mathcal{B}_i)$, $i \in I$, the infimum in $\mathfrak{L}_{\mathrm{Ext}}$ is the pair $(\bigcap_{i \in I} \mathcal{A}_i, (\bigcap_{i \in I} \mathcal{A}_i)^\perp)$ generated by $\bigcup_{i \in I} \mathcal{B}_i$, and the supremum is $({}^\perp(\bigcap_{i \in I} \mathcal{B}_i), \bigcap_{i \in I} \mathcal{B}_i)$. In general, $\mathfrak{L}_{\mathrm{Ext}}$ could be very large. For example, in the case $R = \mathbb{Z}$ every partially ordered set embeds into $\mathfrak{L}_{\mathrm{Ext}}$ by [11], $\mathfrak{L}_{\mathrm{Ext}}$ then being a proper class.

For a module of M and a class of modules $\mathcal{C}$, a *$\mathcal{C}$-filtration* of M is an increasing sequence $(M_\alpha)_{\alpha < \kappa}$, such that $M_0 = 0$, $M = \bigcup_{\alpha < \kappa} M_\alpha$, $M_\beta = \bigcup_{\alpha < \beta} M_\alpha$ for every limit ordinal $\beta < \kappa$, and the factors $M_{\alpha+1}/M_\alpha$ are all isomorphic to elements of $\mathcal{C}$. A module M is called *$\mathcal{C}$-filtered* provided it has a $\mathcal{C}$-filtration. The class $\mathcal{C}$ is *closed under filtrations*, if every $\mathcal{C}$-filtered module again belongs to $\mathcal{C}$.

Except from the closure under filtrations, the following lemmas are easy consequences of basic properties of the Ext functor. The closure under filtrations is proved by induction on the length of a filtration [9].

Lemma 2.20. *Let $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a cotorsion pair. Then:*

1. *$\mathcal{A}$ is closed under extensions, direct sums, direct summands and filtrations,*

2. *$\mathcal{A}$ is closed under direct limits provided that $\mathfrak{C}$ is generated by a class of pure-injective modules,*

3. *$\mathcal{A}$ is a torsion-free class provided that $\mathfrak{C}$ is generated by a class of modules of injective dimension at most* 1. □

Lemma 2.21. *Let $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a cotorsion pair. Then:*

1. *$\mathcal{B}$ is closed under extensions, products and direct summands,*

2. *$\mathcal{B}$ is closed under direct limits, direct sums and pure submodules provided that $\mathfrak{C}$ is cogenerated by a class of FP_2 modules,*

3. *$\mathcal{B}$ is a torsion class provided that $\mathfrak{C}$ is cogenerated by a class of modules of projective dimension at most* 1. □

Replacing Ext by Tor in the definition 2.19, we can define a Tor-*torsion pair* as the pair $(\mathcal{A}, \mathcal{B})$ of classes of modules, such that $\mathcal{A} = \{A \in \text{Mod-}R |\ \text{Tor}_1^R(A, B) = 0 \text{ for all } B \in \mathcal{B}\}$ and $\mathcal{B} = \{B \in R\text{-Mod} |\ \text{Tor}_1^R(A, B) = 0 \text{ for all } A \in \mathcal{A}\}$.

Similarly, we can define Tor-torsion pairs generated or cogenerated by a class of modules. Again, Tor-torsion pairs form a complete lattice, this time using a partial ordering by an inclusion in the second component. Let us denote the lattice $\mathfrak{L}_{\text{Tor}}$. It could be proven that the cardinality of $\mathfrak{L}_{\text{Tor}}$ is bounded by 2^{2^κ}, where $\kappa = \text{card}(R) + \aleph_0$. The least Tor-torsion pair in $\mathfrak{L}_{\text{Tor}}$ is $(\text{Mod-}R, \mathcal{F}_0)$, the greatest one is $(\mathcal{F}_0, R\text{-Mod})$.

$\mathfrak{L}_{\text{Tor}}$ actually embeds into $\mathfrak{L}_{\text{Ext}}$, mapping a Tor-torsion pair $(\mathcal{A}, \mathcal{B})$ to a cotorsion pair $(\mathcal{B}, \mathcal{B}^\perp)$. The latter cotorsion pair is easily seen to be generated by the class $\{X^c | X \in \mathcal{A}\}$ (cf. prop 2.16).

Lemma 2.22. *Let $(\mathcal{A}, \mathcal{B})$ be a* Tor*-torsion pair. Then both $\mathcal{A}$ and $\mathcal{B}$ are closed under extensions, direct sums and direct limits.* □

Now we will look at the connection of cotorsion pairs to approximations of modules. The notions used here are special precovers and special preenvelopes.

Definition 2.23. Let M be a module and $\mathcal{C}$ be a class of modules. An homomorphism $f : C \to M$ is a *special $\mathcal{C}$-precover* provided that f is an epimorphism and its kernel lies in $\mathcal{C}^{\perp}$. $\mathcal{C}$ is a *special precovering* class if every module has a special $\mathcal{C}$-precover.

Dually, an homomorphism $g : M \to C$ is a *special* $\mathcal{C}$*-preenvelope* provided that g is a monomorphism and its cokernel lies in ${}^{\perp}\mathcal{C}$. The class $\mathcal{C}$ is *special preenveloping* if every module has a special $\mathcal{C}$-preenvelope.

The terminology comes from the fact that special precovers and preenvelopes are special instances of the more general notions defined eg. in [10]:

Let M be a module and $\mathcal{C}$ be a class of modules. Then an homomorphism $f : C \to M$ is called a $\mathcal{C}$*-precover* provided that for each homomorphism $f' : C' \to M$ with $C' \in \mathcal{C}$ there is an homomorphism $g : C' \to C$, such that $f' = fg$. The $\mathcal{C}$-precover f is a $\mathcal{C}$*-cover*, if every endomorphism g of C satisfying $fg = f$ is an automorphism. The class $\mathcal{C}$ is said to be *precovering* (*covering*) provided that all modules have $\mathcal{C}$-precover ($\mathcal{C}$-cover).

The notions of $\mathcal{C}$*-preenvelope*, $\mathcal{C}$*-envelope*, *preenveloping* class and *enveloping* class are defined dually.

The preenvelopes and precovers are generalizations of the notions of reflections and coreflections of categories, we drop the requirement that factoring maps g from the definition need to be unique. Although preenvelopes and precovers actually need not to be unique, it is easily seen that envelopes and covers are unique up to isomorphism. Moreover, the $\mathcal{C}$-cover ($\mathcal{C}$-envelope) of M is isomorphic to a direct summand of any $\mathcal{C}$-precover ($\mathcal{C}$-preenvelope).

It is well known that $\mathcal{I}_0$ and $\mathcal{PI}$ are enveloping classes, and in case R is perfect $\mathcal{P}_0$ is a covering class. More examples will be provided below.

The following lemma connects cotorsion pairs to approximations of modules and provides a homological tie between the dual notions of special precover and special preenvelope (cf. [23]):

Lemma 2.24. *Let M be a module and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a cotorsion pair. Then:*

1. *Assume M has an $\mathcal{A}$-cover f. Then f is a special $\mathcal{A}$-precover. Thus, if $\mathcal{A}$ is covering then $\mathcal{A}$ is special precovering.*

2. *Assume M has a $\mathcal{B}$-envelope f. Then f is a special $\mathcal{B}$-preenvelope. Thus, if $\mathcal{B}$ is enveloping then $\mathcal{B}$ is special preenveloping.*

3. *$\mathcal{A}$ is special precovering if, and only if, $\mathcal{B}$ is special preenveloping. In this case $\mathfrak{C}$ is called a* complete *cotorsion pair.* $\square$

The following theorem shows that a cotorsion pair is complete under rather weak assumptions [9]:

Theorem 2.25. *Let $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a cotorsion pair cogenerated by a set of modules $\mathcal{S}$. Then $\mathfrak{C}$ is complete, and $\mathcal{A}$ is the class of all direct summands of all $\mathcal{S} \cup \{R\}$-filtered modules.* □

It can be shown for a lot of the well-known cotorsion pairs that they are cogenerated by a set of modules, thus they are complete. For example, this is the case of (see [23]):

1. $(\mathcal{P}_n, (\mathcal{P}_n)^{\perp})$,
2. $({}^{\perp}\mathcal{I}_n, \mathcal{I}_n)$,
3. $(\mathcal{B}, \mathcal{B}^{\perp})$, where $(\mathcal{A}, \mathcal{B})$ is a Tor-torsion pair; in particular $(\mathcal{F}_n, (\mathcal{F}_n)^{\perp})$ are cogenerated by sets of modules,
4. $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$, such that $\mathfrak{C}$ is generated by a class of pure-injective modules.

If $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ is a cotorsion pair, such that $\mathcal{A}$ is covering and $\mathcal{B}$ is enveloping, then $\mathfrak{C}$ is called *perfect*. It has been shown in [10] that if $\mathfrak{C}$ is complete, the class $\mathcal{A}$ being closed under direct limits is a sufficient condition for $\mathfrak{C}$ to be perfect.

Thus, every cotorsion pair induced by a Tor-torsion pair or generated by a class of pure-injective modules is perfect. In particular, $(\mathcal{F}_n, (\mathcal{F}_n)^{\perp})$ are perfect cotorsion pairs.

Note that a cotorsion pair *generated* by a set of modules need not to provide for special approximations—the result may even be dependent on the extension of set theory (ZFC) we are working in. For more details, see [23].

To conclude this section, we remark that the notions of preenveloping and precovering class could be extended to a more general setting of an arbitrary category $\mathcal{K}$ and its full subcategory $\mathcal{C}$. In particular, if $\mathcal{K} = R$-mod, we will say that $\mathcal{C}$ is *covariantly finite* (*contravariantly finite*) provided that $\mathcal{C}$ is preenveloping (precovering) in R-mod.

2.3 Tilting modules and classes

The classical tilting theory generalizes Morita theory of equivalence of module categories by providing equivalences of large subcategories of modules. If we drop the requirement that a representing tilting module needs to be finitely generated, the category equivalences disappear, but some other properties remain. For example tilting approximations of modules. Moreover, the generalized view allows us to connect the tilting theory to the finitistic dimension conjectures. The topic is studied eg. in [6], [5] or [23].

Definition 2.26. A module T is said to be *n-tilting* if it satisfies the following conditions

1. $\operatorname{pd} T \le n$,

2. $\operatorname{Ext}^i_R(T, T^{(\kappa)}) = 0$ for each $i \ge 1$ and cardinal κ,

3. there is a coresolution $0 \to R \to T_0 \to T_1 \to \cdots \to T_m \to 0$, where $T_j \in \operatorname{Add} T$ for $0 \le j \le m$.

Also, if T is n-tilting, we can always find a coresolution with $m \le \operatorname{pd} T$.

A class $\mathcal{T} \subseteq R$-Mod is said to be *n-tilting*, if there is an n-tilting module T, such that $\mathcal{T} = \{X \in R\text{-Mod} \mid \operatorname{Ext}^i_R(T, X) = 0 \text{ for all } i \ge 1\}$.

A cotorsion pair is said to be *n-tilting*, if it is of the form $({}^{\perp}\mathcal{T}, \mathcal{T})$, where $\mathcal{T}$ is an n-tilting class.

Since we will be interested mainly in 1-tilting modules and classes, the overview of available results below is focused to that case. Note that 1-tilting class is a torsion class and a right hand side class of a cotorsion pair at the same time.

The following theorem is crucial for the construction in section 4 (cf. [23]):

Theorem 2.27. *Let $\mathcal{T}$ be a class of modules. Then the following statements are equivalent:*

1. $\mathcal{T}$ is a 1-tilting class,

2. $\mathcal{T}$ is a special preenveloping torsion class,

3. $\mathcal{T} = \mathcal{S}^{\perp}$ for some subset $\mathcal{S}$ of $\mathcal{P}_1$, and $\mathcal{T}$ is closed under direct sums.

Moreover, in the case $\mathcal{T}$ is 1-tilting, one appropriate 1-tilting module T for $\mathcal{T}$ can be costructed in the following way: we take a special preenvelope $0 \to R \to T_0 \to T_1 \to 0$ and put $T = T_0 \oplus T_1$. □

A 1-tilting class $\mathcal{T}$ is said to be of a *finite type*, if there is a set $\mathcal{S} \subseteq R$-mod, such that $\mathcal{T} = \mathcal{S}^{\perp}$. It holds that $\mathcal{T}$ is of a finite type if, and only if, $\mathcal{T}$ is *definable* [5]; that is, $\mathcal{T}$ is closed under direct limits, products and pure submodules.

So far, there are no known tilting classes which would not be of a finite type. It is conjectured that all the tilting classes are of finite type at least over any finite dimensional algebra. But the best result at the moment is that any 1-tilting class over an arbitrary ring is of *countable type* [5]; that is, $\mathcal{T} = \mathcal{S}^{\perp}$ for a set $\mathcal{S}$ of countably presented modules from $\mathcal{P}_1$.

Corollary 2.28. *Let $\mathcal{T}$ be a class of modules. Then $\mathcal{T}$ is a 1-tilting class of a finite type if, and only if, there is a subset $\mathcal{S}$ of $\mathcal{P}_1^{<\infty}$, such that $\mathcal{T} = \mathcal{S}^\perp$.*

Proof. An immediate consequence of theorem 2.27 and lemma 2.21. ☐

The theorem and the corollary show that an intersection of a family of 1-tilting classes (of a finite type) is again a 1-tilting class (of a finite type). Thus, both the 1-tilting classes and 1-tilting classes of a finite type form a lattice.

2.4 Finitistic dimension conjectures

While the global dimension provides an effective "measure of a complexity" of module categories in many cases, there are also examples where it does not meet the expectations.

Let $R = k[x]/\langle x^2 \rangle$, k being a field. Then R is local, so every projective R-module is free by [1, 26.7]. Namely, every projective module has an even dimension over k; and every module M of an odd dimension has $\operatorname{pd} M = \infty$. For example, $\operatorname{pd} R/\langle x \rangle = \infty$ and every syzygy of the module $R/\langle x \rangle$ could be taken isomorphic to $R/\langle x \rangle$. Thus, $\operatorname{gl.dim} R = \infty$. On the other hand, the category of R-modules is very simplistic—every module M is isomorphic to a direct sum of copies of R and $R/\langle x \rangle$.

This motivates a definition of so called finitistic dimensions of a ring R.

Definition 2.29. A (left) *little finitistic dimension* of R is defined as

$$\operatorname{fdim} R = \sup\{\operatorname{pd} M | M \text{ is a fin. gen. left } R\text{-module with } \operatorname{pd} M < \infty\}$$

A (left) *big finitistic dimension* of R is defined as

$$\operatorname{Fdim} R = \sup\{\operatorname{pd} M | M \text{ is an arbitrary left } R\text{-module with } \operatorname{pd} M < \infty\}$$

Looking back at the previous example, $\operatorname{fdim} k[x]/\langle x^2 \rangle = \operatorname{Fdim} k[x]/\langle x^2 \rangle = 0$. Also, if R is a ring with $\operatorname{gl.dim} R < \infty$, then $\operatorname{fdim} R = \operatorname{Fdim} R = \operatorname{gl.dim} R$ by theorem 2.11. So the interesting situation arises, when $\operatorname{gl.dim} R = \infty$. Moreover, we can construct a ring, such that $\operatorname{gl.dim} R = \infty$ and $\operatorname{fdim} R = n$ for any prescribed $n < \omega$ [15].

The questions were, whether the little and big finitistic dimensions coincide, and whether they are finite. The aswers are negative for commutative noetherian rings. If R is commutative noetherian local, then the $\operatorname{fdim} R$ is shown to be equal to the depth of R by Auslander and Buchsbaum. As to the $\operatorname{Fdim} R$, it is equal to the Krull dimension of R for commutative noetherian rings. The inequality $\geq$ was estabilished by Bass in 1962, the other one was completed by Gruson and Raynaud in 1973. So in particular, a commutative noetherian local ring has a coinciding little and big finitistic dimensions if, and only if, it is Cohen-Macaulay. And commutative noetherian rings with an infinite Krull dimension do exist, first examples given by Nagata. So the finitistic dimensions need not to be finite.

However, no examples settling these questions were available for non-commutative artinian situation at that time. The following assertions were considered by Bass in 1960, and they were later promoted to conjectures after being restricted to the case of finite dimensional algebras:

Finitistic Dimension Conjectures. *Let R be a finite dimensional algebra over a filed k. Then:*

I. $\operatorname{fdim} R = \operatorname{Fdim} R$*,*

II. $\operatorname{fdim} R < \infty$*.*

It turns out that the conjecture I. fails, even for so called monomial relation algebras. Examples of these algebras with $\operatorname{fdim} R = n$ and $\operatorname{Fdim} R = n+1$ for any $n \geq 2$ were given in [26]. Algebras with an arbitrary large difference between the finitistic dimensions were constructed in [20]. In the same paper it has been shown that the finitistic dimensions coincide, whenever $\mathcal{P}^{<\infty}$ is contravariantly finite in R-mod. Note, however, that the condition of the contravatiant finiteness of $\mathcal{P}^{<\infty}$ is not necessary for conjecture I. to be valid, an example given by Igusa, Smalø and Todorov in [14].

The second conjecture still remains open, although there are many results taking care of more special cases:

- If R is a monomial relation algebra, then $\operatorname{fdim} R \leq \operatorname{Fdim} R < \infty$. Moreover, explicit bounds on the finitistic dimensions were given.
- Let $\mathfrak{r}$ be Jacobson radical of R. Then conjecture II. holds whenever R is a finite dimensional algebra with $\mathfrak{r}^3 = 0$. Even a weaker condition $\operatorname{pd} \mathfrak{r}^3 < \infty$ is sufficient.
- Conjecture II. holds, if $\mathcal{P}^{<\infty}$ is contravariantly finite in R-mod. In this case, $\operatorname{fdim} R$ is equal to a supremum of projective dimensions of $\mathcal{P}^{<\infty}$-precovers of simple modules.

There is a following connection between tilting classes and the conjecture II., which is pursued further in this text [13]:

Theorem 2.30. *Let R be a left noetherian ring and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a cotorsion pair cogenerated by $\mathcal{P}^{<\infty}$. Then the conjecture II. holds if, and only if, $\mathfrak{C}$ is tilting. Moreover, if T is a corresponding tilting module, then* $\operatorname{fdim} R = \operatorname{pd} T$*.*

Proof. If $\operatorname{fdim} R < \infty$, then $\mathcal{P}^{<\infty} \subseteq \mathcal{P}_n$ and $\mathcal{B} = (\mathcal{P}^{<\infty})^{\perp}$ is a tilting class [23, 4.2]. Conversely, if $\mathfrak{C}$ is n-tilting then $\mathcal{P}^{<\infty} \subseteq \mathcal{P}_n$, so the conjecture II. holds. Since $\operatorname{fdim} R$ is the least m, such that $\mathcal{A} \subseteq \mathcal{P}_m$, we infer that $\operatorname{fdim} R = \operatorname{pd} T$. $\square$

For more complete overview of existing results and a historical account of finitistic dimension conjectures, we refer to the paper [25]. An overview of newer results is to be found in [23].

3 Artin algebras

The aim of this section is to give a characterisation of modules in the class $\mathcal{P}_1^{<\infty}$ by means of Auslander-Reiten translation DTr (proposition 3.3), and to prove a general version of Auslander-Reiten formulas (theorem 3.9). Then we will see that for a finitely generated module X of projective dimension at most 1, we have $X^\perp = \operatorname{Ker}\operatorname{Hom}_R(-, DTrX)$. Under this assuption, we can substitute a kernel of a covariant Ext functor by a kernel of a contravariant Hom functor, which is often easier to tackle with. The dual statement is also avialable for a finitely generated module X with $\operatorname{id} X \leq 1$.

Throughout the whole section we will assume that R is an *artin algebra* over a commutative artinian ring S; that is, we have a ring homomorphism $\psi : S \to R$ with an image in the centre of R, and R is finitely generated as a left S-module.

The notion of an artin algebra is an extension of that of a finite dimensional algebra over a field; and it is thoroughly studied in [3]. The key properties of artin algebras are:

1. Krull-Schmidt theorem for the category of finitely presented R-modules,

2. existence of contravariant functors D between R-Mod and Mod-R forming a duality between finitely generated left and right modules—these functors are obtained as $\operatorname{Hom}_S(-, J)$, where J is a minimal injective cogenerator for S; this is a generalization of the functors $\operatorname{Hom}_k(-, k)$ for finite dimensional algebras over a field k,

3. Auslander-Reiten formulas binding together the functors Hom and Ext.

A useful construction by which one can get a lot of examples of finite dimensional algebras is that of path algebras.

A *quiver* is an oriented graph with the possibility of having more arrows connecting the same pairs of vertices. We will assume that all quivers are finite. A *path* in a quiver is either an ordered sequence of arrows $p = \alpha_n \dots \alpha_i$, or the symbol e_i for each vertex i. We call the paths e_i *trivial paths*; they are formally both starting and ending at corresponding vertices i and are of length 0. For a path p, denote $s(p)$ the starting vertex of p and $e(p)$ the ending vertex of p.

Given a field k and a quiver Γ, we will denote $k\Gamma$ the k-vector space with the basis formed by all paths in Γ. We can define a multiplication of paths

p, q in $k\Gamma$:

$$p \cdot q = \begin{cases} pq & \text{if } p, q \text{ are non-trivial and } e(q) = s(p) \\ p & \text{if } q = e_{s(p)} \\ q & \text{if } p = e_{e(p)} \\ 0 & \text{otherwise} \end{cases}$$

Extending this multiplication linearly to the whole $k\Gamma$, we get a structure of k-algebra; it is called a *path algebra* of Γ over k. Obviously, $k\Gamma$ is finite dimensional if, and only if, Γ have no oriented cycles.

A *relation* on a quiver Γ is a k-linear combination of paths $\sum_{i=1}^{n} a_i p_i$ with $a_i \in k$, $s(p_1) = \cdots = s(p_n)$, $e(p_1) = \cdots = e(p_n)$, and all the paths being of length at least 2. Given a set ρ of relations, we will denote $k(\Gamma, \rho) = k\Gamma/\langle\rho\rangle$ and call it a *path algebra with relations ρ*.

Since R is artinian, it decomposes as a left module into a direct sum $P_1 \oplus \cdots \oplus P_n$ of indecomposable projectives. We will call R *basic*, if P_i's are mutually non-isomorphic. It could be seen that every artin algebra is Morita equivalent to a basic one (cf. [3, II.2.6]). The next theorem shows that artin algebras over an algebraically closed field are nearly only the path algebras:

Theorem 3.1. *Let R be a basic finite dimensional algebra over an algebraically closed field k. Then R is isomorphic to $k(\Gamma, \rho)$ for some quiver Γ with relations ρ.*

Proof. See [3, III.1.10]. □

$k(\Gamma, \rho)$-modules could be viewed as *k-linear representations* of the quiver Γ with relations ρ. That is, we take a vector space V_i for every vertex i of Γ, and a linear map $f_\alpha : V_i \to V_j$ for every arrow from a vertex i to a vertex j, in such a way that linear combinations of compositions of f_α's corresponding to the combinations of paths from ρ are vanishing.

Homomorphisms of representations (V_i, f_α) and (V_i', f_α') are n-tuples of linear maps $(h_i : V_i \to V_i')$, n being the number of vertices in Γ, such that $h_{e(\alpha)} f_\alpha = f_\alpha' h_{s(\alpha)}$.

For an arbitrary artin algebra R over S and $X \in R$-mod, denote $X^* = \mathrm{Hom}_R(X, R)$. Let $P_1 \xrightarrow{f} P_0 \xrightarrow{p} X \to 0$ be a *minimal projective presentation* of the module X; that is, p is a projective cover of X and f is a projective cover of $\operatorname{Ker} p$. Then the *transpose TrX* of X is defined as a cokernel of $P_0^* \xrightarrow{f^*} P_1^*$. The composition DTr is called *Auslander-Reiten translation* and TrD is its “inverse”.

In fact, DTr and TrD are not functors of module categories in general. We need to move to *stable categories* modulo projective and injective modules. Denote $\overline{\mathrm{Hom}}_R(X,Y)$ a factor of the S-module $\mathrm{Hom}_R(X,Y)$ by a submodule of homomorphisms from X to Y which factors through an injective module. Similarly, let $\underline{\mathrm{Hom}}_R(X,Y)$ be a factor of $\mathrm{Hom}_R(X,Y)$ by a submodule of homomorphisms which factor through a projective.

We will denote $\overline{R\text{-mod}}$ a stable category of finitely generated R-modules modulo injectives; this means, its objects are finitely generated R-modules and its morphisms are exactly the factors $\overline{\mathrm{Hom}}_R(X,Y)$. Similarly, we denote $\underline{R\text{-mod}}$ a stable category of R-mod modulo projectives, with morphisms $\underline{\mathrm{Hom}}_R(X,Y)$. Then $DTr : \underline{R\text{-mod}} \to \overline{R\text{-mod}}$ is a category equivalence with inverse equivalence TrD (cf. [3, IV.1.9]).

3.1 Modules of homological dimensions at most 1

It is obvious from the definition that for a finitely generated projective module P we have $TrP = 0$. For finitely generated non-projectives we have:

Lemma 3.2. *Let $X \in R$-mod be an indecomposable non-projective and let $P_1 \xrightarrow{f} P_0 \xrightarrow{p} X \to 0$ be its minimal projective presentation. Then $P_0^* \xrightarrow{f^*} P_1^* \xrightarrow{q} TrX \to 0$ is a minimal projective presentation of TrX.*

Proof. (See also [3], chapter IV) The functor $(-)^*$ is an equivalence of categories of finitely generated projective R-modules and finitely generated projective R^{op}-modules.

If f^* would not be a projective cover of $\operatorname{Ker} q$, there is a non-trivial decomposition $P_0^* = P \oplus Q$ such that $f^* \restriction Q = 0$. But then $P_0 \cong P^* \oplus Q^*$ and $\operatorname{Im} f \subseteq P^*$. Thus, $X = \operatorname{Coker} f^*$ has a non-zero projective direct summand isomorfic to Q^*, which is impossible.

Let us now assume that q is not a projective cover of TrX. Again, there is a non-trivial decomposition $P_1^* = P' \oplus Q'$ such that $q \restriction Q' = 0$. So the module Q' is a direct summand of $\operatorname{Ker} q$. Since f^* is a projective cover of $\operatorname{Ker} q$, P_0^* have a direct summand isomorfic to Q' and a homomorphism f^* can be decomposed into $f_{P'} \oplus f_{Q'}$, where $f_{Q'}$ is an isomorphism onto Q'. Thus, both P_0 i P_1 have a non-zero projective direct summand isomorphic to $(Q')^*$ and the map f can be decomposed into $f_{(P')^*} \oplus f_{(Q')^*}$, where $f_{(Q')^*}$ is an isomorphism between these direct summands. This decomposition yields a contradiction with the minimality of the projective presentation of X again. $\square$

The former lemma implies that for any finitely generated module X, TrX has no non-zero projective direct summand and $Tr(TrX) \cong X$ if, and only if, X has no non-zero projective direct summand. Now we give a characterisation of finitely generated modules of projective or injective dimensions at most 1.

Proposition 3.3. *Let R be an artin algebra and $X \in R$-mod. Then:*

1. $\operatorname{pd} X \leq 1$ *if and only if* $\operatorname{Hom}_R(D(R), DTrX) = 0$.
2. $\operatorname{id} X \leq 1$ *if and only if* $\operatorname{Hom}_R(TrDX, R) = 0$.

Proof. (See also [3, IV.1.16]) Let $X \in R$-mod. Then $X = Y \oplus P$, where P is projective and Y has no non-zero projective direct summand. Take a minimal projective presentation $P_1 \to P_0 \to TrX \to 0$ of the module TrX. Then $Tr(TrX) \cong Y$, and there is an exact sequence:

$$0 \to (TrX)^* \to P_0^* \to P_1^* \to Y \to 0$$

But $P_0^* \to P_1^* \to Y \to 0$ is a minimal projective presentation of Y by lemma 3.2. Thus, $\operatorname{Hom}_{R^{op}}(TrX, R) = (TrX)^* = 0$ if, and only if, $\operatorname{pd} Y \leq 1$ if, and only if, $\operatorname{pd} X \leq 1$. The first statement follows just by applying the duality D.

Moreover, $\operatorname{id} X \leq 1$ if, and only if, $\operatorname{pd} DX \leq 1$ if, and only if, $\operatorname{Hom}_R(TrDX, R) = 0$ by the former; this yields the second part. □

Corollary 3.4. *Let R be an artin algebra and $X \in R$-mod. Then:*

1. $\operatorname{pd} X \leq 1$ *if, and only if,* $\operatorname{Hom}_R(I, DTrX) = 0$ *for every injective module I.*
2. $\operatorname{id} X \leq 1$ *if, and only if,* $\operatorname{Hom}_R(TrDX, P) = 0$ *for every projective module P.*

As an application, we will show the following lemma, which will be of use later.

Lemma 3.5. *A finitely generated module M is in $(\mathcal{P}_1^{<\infty})^{\perp} \cap R$-mod if, and only if, it is filtered by a factors of an injective cogenerator I.*

Proof. See [2]. Or an alternative proof could be given as follows. The if part is obvious, since $(\mathcal{P}_1^{<\infty})^{\perp} \cap R$-mod is closed under factors and extensions. For the only if part, it is enough to prove that $\operatorname{Hom}_R(I, M) \neq 0$ for each $M \in (\mathcal{P}_1^{<\infty})^{\perp} \cap R$-mod. Moreover, it is sufficient to prove this only for M indecomposable non-injective. Assume to the contrary that $\operatorname{Hom}_R(I, M) = 0$. Then $\operatorname{pd} TrDM \leq 1$ by proposition 3.3. So $\operatorname{Ext}_R^1(TrDM, M) = 0$, a contradiction to the existence of an almost split sequence [3, V.1.15]. □

3.2 The relation between functors Hom and Ext

It is well-known that morphisms $\psi_{X,Z} : X^* \otimes_R Z \to \mathrm{Hom}_R(X,Z)$ defined by formulas $\psi_{X,Z}(f \otimes b)(a) = f(a)b$ are functorial in both variables, and $\psi_{P,Z}$ are isomorphisms for finitely generated projectives P.

Lemma 3.6. *Let X, Z be R-modules, X finitely generated, and $P_1 \to P_0 \to X \to 0$ a minimal projective presentation of X. Then there is an exact sequence* $0 \to \mathrm{Hom}_R(X,Z) \to \mathrm{Hom}_R(P_0,Z) \to \mathrm{Hom}_R(P_1,Z) \to TrX \otimes Z \to 0$ *which is functorial in Z.*

Proof. (See also [3, IV.4.2]) We have an exact sequence $P_0^* \to P_1^* \to TrX \to 0$ from the definition of the transpose. The left-exactness of $\mathrm{Hom}_R(-,Z)$ and the right-exactness of $- \otimes Z$ give rise to the commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Hom}_R(X,Z) & \longrightarrow & \mathrm{Hom}_R(P_0,Z) & \longrightarrow & \mathrm{Hom}_R(P_1,Z) & & \\
 & & & & {\scriptstyle \psi_{P_0,Z}}\Big\uparrow & & {\scriptstyle \psi_{P_1,Z}}\Big\uparrow & & \\
 & & & & P_0^* \otimes Z & \longrightarrow & P_1^* \otimes Z & \longrightarrow & TrX \otimes Z \longrightarrow 0
\end{array}$$

But $\psi_{P_0,Z}$ and $\psi_{P_1,Z}$ are isomorphisms, and all the morphisms in the diagram are functorial in Z. $\square$

Definition 3.7. Let $\delta : 0 \to A \to B \to C \to 0$ be an exact sequence and X any R-module. Then the *covariant defect* $\delta_*(X)$ and the *contravariant defect* $\delta^*(X)$ of the sequence δ are defined in the sense of [3] as the folowing cokernels:

$$0 \to \mathrm{Hom}_R(C,X) \to \mathrm{Hom}_R(B,X) \to \mathrm{Hom}_R(A,X) \to \delta_*(X) \to 0$$
$$0 \to \mathrm{Hom}_R(X,A) \to \mathrm{Hom}_R(X,B) \to \mathrm{Hom}_R(X,C) \to \delta^*(X) \to 0$$

Proposition 3.8. *Let $\delta : 0 \to A \to B \to C \to 0$ be an exact sequence and X a finitely generated R-module. Then $\delta_*(DTrX) \cong D(\delta^*(X))$.*

Proof. Take a minimal projective presentation $P_1 \to P_0 \to X \to 0$ of X.

The previous lemma gives rise to the following exact commutative diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & \\
 & 0 \arrow[r] & \mathrm{Hom}_R(X,A) \arrow[r] \arrow[d] & \mathrm{Hom}_R(X,B) \arrow[r] \arrow[d] & \mathrm{Hom}_R(X,C) \arrow[r] \arrow[d] & \delta^*(X) \arrow[r] & 0 \\
 & 0 \arrow[r] & \mathrm{Hom}_R(P_0,A) \arrow[r] \arrow[d] & \mathrm{Hom}_R(P_0,B) \arrow[r] \arrow[d] & \mathrm{Hom}_R(P_0,C) \arrow[r] \arrow[d] & 0 & \\
 & 0 \arrow[r] & \mathrm{Hom}_R(P_1,A) \arrow[r] \arrow[d] & \mathrm{Hom}_R(P_1,B) \arrow[r] \arrow[d] & \mathrm{Hom}_R(P_1,C) \arrow[r] \arrow[d] & 0 & \\
0 \arrow[r] & \mathrm{Ker}\, i' \arrow[r] & TrX \otimes A \arrow[r, "i'"] \arrow[d] & TrX \otimes B \arrow[r] \arrow[d] & TrX \otimes C \arrow[r] \arrow[d] & 0 & \\
 & & 0 & 0 & 0 & &
\end{tikzcd}$$

Put $K_A = \mathrm{Ker}(\mathrm{Hom}_R(P_1,A) \to TrX \otimes A)$ and similarly K_B and K_C. First, take a look at the upper part of the diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \mathrm{Hom}_R(X,A) \arrow[r, "i_X"] \arrow[d, "\iota_A"'] & \mathrm{Hom}_R(X,B) \arrow[r, "p_X"] \arrow[d, "\iota_B"'] & \mathrm{Hom}_R(X,C) \arrow[r, "p"] \arrow[d, "\iota_C"'] & \delta^*(X) \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Hom}_R(P_0,A) \arrow[r, "i_{P_0}"] \arrow[d, "\pi_A"'] & \mathrm{Hom}_R(P_0,B) \arrow[r, "p_{P_0}"] \arrow[d, "\pi_B"'] & \mathrm{Hom}_R(P_0,C) \arrow[r] \arrow[d, "\pi_C"'] & 0 & \\
0 \arrow[r] & K_A \arrow[r, "i_K"] \arrow[d] & K_B \arrow[r, "p_K"] \arrow[d] & K_C \arrow[r] \arrow[d] & 0 & \\
 & 0 & 0 & 0 & &
\end{tikzcd}$$

It has exact columns and the first two rows. Moreover, i_K is a monomorphism, since i_K is a restriction of the monomorphism $\mathrm{Hom}_R(P_1,A) \to \mathrm{Hom}_R(P_1,B)$ to the kernel K_A. Similarly, p_K is an epimorphism since the modules K_B and K_C are the cokernels of the homomorphisms ι_B a ι_C. And obviously $p_K i_K = 0$. Denote $H = \mathrm{Ker}\, p_K / \mathrm{Im}\, i_K$. In general, $H \neq 0$.

Define a map $f : \delta^*(X) \to H$ by the formula

$$f(x) = \pi_B(p_{P_0}^{-1}(\iota_C(p^{-1}(x)))) + \mathrm{Im}\, i_K$$

where $p_{P_0}^{-1}$ and p^{-1} mean some preimage by the given map. We will prove that the value $f(x)$ is independent of a choice of preimages above and the map f is a well-defined isomorphism of $Z(R)$-modules.

First, prove that f is a well-defined map. Let $b_1, b_2 \in \mathrm{Hom}_R(P_0, B)$, such that $p_{P_0}(b_1) = p_{P_0}(b_2)$. Then $b_1 - b_2 \in \mathrm{Ker}\, p_{P_0} = \mathrm{Im}\, i_{P_0}$, so $\pi_B(b_1 - b_2) \in \mathrm{Im}\, i_K$. Therefore, the value $f(x)$ is independent of a choice of a preimage by p_{P_0}. Next, let $c_1, c_2 \in \mathrm{Hom}_R(X, C)$, such that $p(c_1) = p(c_2)$. Then $c_1 - c_2 \in \mathrm{Ker}\, p = \mathrm{Im}\, p_X$. So we can take $b = p_{P_0}^{-1}(\iota_C(c_1 - c_2))$, such that $b \in \mathrm{Im}\, \iota_B$. But then $\pi_B(b) = 0$ and $f(x)$ is independent of a choice of a preimage by p either. The map f is clearly an homomorphism. Next, take $y = i_K(z) \in \mathrm{Im}\, i_K$. Let $b = \pi_B^{-1}(y)$ and $a = \pi_A^{-1}(z)$ are arbitrary preimages. Then $\pi_B(b - i_{P_0}(a)) = \pi_B(b) - \pi_B i_{P_0}(a) = y - i_K \pi_A(a) = 0$. So $b - i_{P_0}(a) \in \mathrm{Ker}\, \pi_B = \mathrm{Im}\, \iota_B$. Let w be arbitrary, such that $\iota_B(w) = b - i_{P_0}(a)$. We have $p_{P_0}(b) = p_{P_0}(\iota_B(w) + i_{P_0}(a)) = p_{P_0}\iota_B(w) = \iota_C p_X(w)$. Since ι_C is a monomorphism, the only preimage of $p_{P_0}(b)$ by ι_C is $p_X(w)$. But $p(p_X(w)) = 0$. Thus, f is a monomorphism. Finally, take an arbitrary $y \in \mathrm{Ker}\, p_K$. Then for every $c = p_{P_0}\pi_B^{-1}(y)$, we have $c \in \mathrm{Ker}\, \pi_C = \mathrm{Im}\, \iota_C$. We can put $x = p\iota_C^{-1}(c)$, and x is a preimage of $y + \mathrm{Im}\, i_K$ by f. And so f is an isomorphism.

Similarly, we can take the lower part of the big diagram on the page 27:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & 0 \\
 & & & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & \to & K_A & \xrightarrow{i_K} & K_B & \xrightarrow{p_K} & K_C & \to 0 \\
 & & & & {\scriptstyle\theta_A}\downarrow & & {\scriptstyle\theta_B}\downarrow & & {\scriptstyle\theta_C}\downarrow \\
 & & 0 & \to & \mathrm{Hom}_R(P_1, A) & \xrightarrow{i_{P_1}} & \mathrm{Hom}_R(P_1, B) & \xrightarrow{p_{P_1}} & \mathrm{Hom}_R(P_1, C) & \to 0 \\
 & & & & {\scriptstyle\sigma_A}\downarrow & & {\scriptstyle\sigma_B}\downarrow & & {\scriptstyle\sigma_C}\downarrow \\
0 & \to & \mathrm{Ker}\, i' & \xrightarrow{i} & TrX \otimes A & \xrightarrow{i'} & TrX \otimes B & \xrightarrow{p'} & TrX \otimes C & \to 0 \\
 & & & & \downarrow & & \downarrow & & \downarrow \\
 & & & & 0 & & 0 & & 0
\end{array}
$$

Again, this gives us a commutative diagram with exact columns and rows with one exception — a non-exactness of the first row by K_B. And as before, we define a map $g : \mathrm{Ker}\, i' \to H$ by the formula:

$$g(x) = \theta_B^{-1}(i_{P_1}(\sigma_A^{-1}(i(x)))) + \mathrm{Im}\, i_K$$

We will prove that g is a well-defined isomorphism too. Let $x \in \mathrm{Ker}\, i'$ and choose arbitrary $a = \sigma_A^{-1} i(x)$. Then $\sigma_B i_{P_1}(a) = i'\sigma_A(a) = i'(i(x)) = 0$, and therefore $i_{P_1}(a) \in \operatorname{Ker}\sigma_B = \operatorname{Im}\theta_B$. So the value of $g(x)$ is always defined. Let $a_1, a_2 \in \operatorname{Hom}_R(P_1, A)$, such that $\sigma_A(a_1) = \sigma_A(a_2)$. Then $a_1 - a_2 \in \operatorname{Ker}\sigma_A = \operatorname{Im}\theta_A$ and $\theta_B^{-1} i_{P_1}(a_1 - a_2) = i_K\theta_A^{-1}(a_1 - a_2) \in \operatorname{Im} i_K$. This implies that $g(x)$ is independent of a choice of a preimage by σ_A and g is a well-defined map. Obviously g is an homomorphism. Take $y \in \operatorname{Im} i_K$ arbitrary. Then $\theta_B(y) \in \operatorname{Im} i_{P_1}$ and $\sigma_A i_{P_1}^{-1}\theta_B(y) = \sigma_A\theta_A i_K^{-1}(y) = 0$. Because i is a monomorphism, the only preimage of zero by i is zero again, and g must be a monomorphism. Finally, take $y \in \operatorname{Ker} p_K$ arbitrary. It follows from the diagram that $\theta_B(y) \in \operatorname{Ker} p_{P_1} = \operatorname{Im} i_{P_1}$. Put $a = i_{P_1}^{-1}\theta_B(y)$. Then $i'\sigma_A(a) = \sigma_B i_{P_1}(a) = \sigma_B\theta_B(y) = 0$, and therefore $\sigma_A(a) \in \operatorname{Ker} i' = \operatorname{Im} i$. Thus, $x = i^{-1}\sigma_A(a)$ is a preimage of $y + \operatorname{Im} i_K$ by g and g is an isomorphism.

Putting it together, we have just shown that $\operatorname{Ker} i' \cong H \cong \delta^*(X)$. Now, it is enough to use a natural equivalence of functors $D(TrX \otimes -) \cong \operatorname{Hom}_R(-, DTrX)$ and we have the commutative diagram:

$$
\begin{tikzcd}
D(TrX \otimes B) \arrow[r] \arrow[d, "\cong"'] & D(TrX \otimes A) \arrow[r] \arrow[d, "\cong"'] & D(\operatorname{Ker} i') \arrow[r] \arrow[d] & 0 \\
\operatorname{Hom}_R(B, DTrX) \arrow[r] & \operatorname{Hom}_R(A, DTrX) \arrow[r] & \delta_*(DTrX) \arrow[r] & 0
\end{tikzcd}
$$

This yields the desired isomorphism $\delta_*(DTrX) \cong D(\operatorname{Ker} i') \cong D(\delta^*(X))$ $\square$

Theorem 3.9 (Auslander-Reiten formulas). *Let R be an artin algebra and let $X, Y \in R$-Mod, X finitely generated. Then:*

1. $D\operatorname{Ext}_R^1(X, Y) \cong \overline{\operatorname{Hom}}_R(Y, DTrX)$
2. $\operatorname{Ext}_R^1(Y, X) \cong D\underline{\operatorname{Hom}}_R(TrDX, Y)$

Proof. Take an injective envelope $\delta : 0 \to Y \to I \to C \to 0$ of the module Y. We have the following exact sequences:

$$\operatorname{Hom}_R(X, I) \to \operatorname{Hom}_R(X, C) \to \operatorname{Ext}_R^1(X, Y) \to 0$$
$$\operatorname{Hom}_R(I, DTrX) \to \operatorname{Hom}_R(Y, DTrX) \to \overline{\operatorname{Hom}}_R(Y, DTrX) \to 0$$

The previous proposition implies

$$D\operatorname{Ext}_R^1(X, Y) = D(\delta^*(X)) \cong \delta_*(DTrX) = \overline{\operatorname{Hom}}_R(Y, DTrX)$$

which is exactly the first statement.

Similarly, let $\epsilon : 0 \to K \to P \to Y \to 0$ be a projective cover of the module Y. Then we have exact sequences:

$$\operatorname{Hom}_R(X, P) \to \operatorname{Hom}_R(X, Y) \to \underline{\operatorname{Hom}}_R(X, Y) \to 0$$
$$\operatorname{Hom}_R(P, DTrX) \to \operatorname{Hom}_R(K, DTrX) \to \operatorname{Ext}_R^1(Y, DTrX) \to 0$$

And $D\underline{\mathrm{Hom}}_R(X,Y) = D(\epsilon^*(X)) \cong \epsilon_*(DTrX) = \mathrm{Ext}^1_R(Y, DTrX)$. Substituting a finitely generated module $TrDX$ instead of X into this formula we see that:

$$D\underline{\mathrm{Hom}}_R(TrDX, Y) \cong \mathrm{Ext}^1_R(Y, DTr(TrDX))$$

Take a decomposition $X = Z \oplus I'$ of X, such that I' is injective and Z has no non-zero injective direct summand. Then $DX = DZ \oplus DI'$, DI' is projective and DZ has no non-zero projective direct summand. But this means that $Tr(TrDX) \cong DZ$, and thus $DTr(TrDX) \cong Z$. Since I' is injective, it is

$$\mathrm{Ext}^1_R(Y, DTr(TrDX)) \cong \mathrm{Ext}^1_R(Y, Z) \cong \mathrm{Ext}^1_R(Y, X)$$

and this concludes the proof of the second statement. □

Corollary 3.10. *Let $X, Y \in R$-Mod, X finitely generated. It holds:*

1. *If $\mathrm{pd}\, X \leq 1$, then $D\,\mathrm{Ext}^1_R(X, Y) \cong \mathrm{Hom}_R(Y, DTrX)$*
2. *If $\mathrm{id}\, X \leq 1$, then $\mathrm{Ext}^1_R(Y, X) \cong D\,\mathrm{Hom}_R(TrDX, Y)$*

Proof. We have $\overline{\mathrm{Hom}}_R(Y, DTrX) = \mathrm{Hom}_R(Y, DTrX)$ for a module X with $\mathrm{pd}\, X \leq 1$ by the corollary 3.4. And similarly, for X with $\mathrm{id}\, X \leq 1$ it is $\underline{\mathrm{Hom}}_R(TrDX, Y) = \mathrm{Hom}_R(TrDX, Y)$. □

Corollary 3.11. *Let $X \in R$-mod. It holds:*

1. *If $\mathrm{pd}\, X \leq 1$, then $X^{\perp} = \mathrm{Ker}\,\mathrm{Hom}_R(-, DTrX)$.*
2. *If $\mathrm{id}\, X \leq 1$, then ${}^{\perp}X = \mathrm{Ker}\,\mathrm{Hom}_R(TrDX, -)$.*

4 On an example of Igusa, Smalø and Todorov

Fix an algebraically closed field k and let R be an algebra introduced in [14] by Igusa, Smalø and Todorov, from now on shortly IST-algebra. It is a path algebra over k over the quiver

$$1\cdot \quad \begin{array}{c} \xrightarrow{\gamma} \\ \xleftarrow{\beta} \\ \xleftarrow{\alpha} \end{array} \quad \cdot 2$$

with relations $\alpha\gamma = \beta\gamma = \gamma\alpha = 0$. We will work only with modules over this artin algebra below, if not stated otherwise.

The basic properties of R-modules are summarized in [12, section 5]. Denote $\Lambda = R/\langle\gamma\rangle$; then Λ is isomorphic to the *Kronecker algebra*, ie. the hereditary algebra $k\Gamma'$ with the following quiver Γ':

$$1\cdot \quad \begin{array}{c} \xleftarrow{\beta} \\ \xleftarrow{\alpha} \end{array} \quad \cdot 2$$

R-modules M with $\gamma M = 0$ will be called *Kronecker modules*, since they are also Λ-modules. Denote P_i, I_i and S_i the indecomposable projective, injective and simple module corresponding to the vertex i ($i = 1, 2$) respectively. Then $\dim_k P_1 = 2$, $\dim_k P_2 = 4$ and $\dim_k I_1 = \dim_k I_2 = 3$. Let $R_{\lambda,i}$ and R_λ be an indecomposable regular Kronecker module of regular length i and simple regular Kronecker module respectively, given $\lambda \in k \cup \{\infty\}$. Corresponding Prüfer modules are denoted $R_{\lambda,\infty}$.

Let $\mathcal{P}^{<\infty}$ be a full subcategory of all finitely generated R-modules of finite projective dimension and $\mathcal{KP}^{<\infty}$ a full subcategory of all Kronecker modules from $\mathcal{P}^{<\infty}$.

4.1 Kronecker modules

We will briefly recall basic facts about Λ-modules. The finite dimensional indecomposable Λ-modules are divided into three families, *preprojective*, *preinjective* and *regular* modules:

1. The preprojectives Q_n, $n \geq 1$, are modules with the representation $V_1 = k^n$, $V_2 = k^{n-1}$, $f_\beta = (E, 0)$ and $f_\alpha = (0, E)$, where E is a unit matrix $(n-1) \times (n-1)$.

2. The preinjectives J_n, $n \geq 1$, are modules with the representation $V_1 = k^{n-1}$, $V_2 = k^n$, $f_\beta = (E, 0)^T$ and $f_\alpha = (0, E)^T$.

3. For the simple regulars R_λ, there is $V_1 = V_2 = k$. For $\lambda \in k$, f_β is a multiplication by λ and f_α is the identity. When $\lambda = \infty$ we have f_β the identity and $f_\alpha = 0$.

4. Every simple regular module R_λ, $\lambda \in k \cup \{\infty\}$, defines a *tube*; that is, a chain of indecomposable modules

$$R_\lambda = R_{\lambda,1} \subseteq R_{\lambda,2} \subseteq R_{\lambda,3} \subseteq \dots$$

connected by almost split sequences $0 \to R_{\lambda,n} \to R_{\lambda,n-1} \oplus R_{\lambda,n+1} \to R_{\lambda,n} \to 0$. Any finite length indecomposable regular module occures in this way.

Note, that there are no non-zero homomorphisms from preinjectives to preprojectives or regulars, and no non-zero homomorphisms from regulars to preprojectives. Moreover, $\dim_k \operatorname{Hom}(R_\lambda, R_\mu) = \delta_{\lambda,\mu}$ for any $\lambda, \mu \in k \cup \{\infty\}$.

The Prüfer modules $R_{\lambda,\infty}$ are defined as direct limits of ascending chains:

$$R_{\lambda,1} \subseteq R_{\lambda,2} \subseteq R_{\lambda,3} \subseteq \dots$$

It is $\operatorname{Hom}(R_{\lambda,\infty}, R_\mu) = 0$ and $\dim_k \operatorname{Hom}(R_\mu, R_{\lambda,\infty}) = \delta_{\lambda,\mu}$ for every $\lambda, \mu \in k \cup \{\infty\}$.

The description of finite dimensional Λ-modules is done in [3]. For more detailes about infinite dimensional Λ-modules we refer to the papers [18] or [17].

4.2 Simple modules and composition series in $\mathcal{P}^{<\infty}$

For every finitely generated R-module M, there is an exact sequence

$$0 \to P_1^n \to M \to \overline{M} \to 0$$

where $n < \omega$ and $\overline{M}$ is a Kronecker module.

Now, let $M \in \mathcal{P}^{<\infty}$. Then, clearly, $\overline{M} \in \mathcal{P}^{<\infty}$ too. But a finitely generated Kronecker module of finite projective dimension over R have a projective dimension at most 1, and this is exactly when M is a direct sum of finitely many regular Kronecker modules $R_{\lambda_1,i_1}, \dots, R_{\lambda_m,i_m}$ with $\lambda_1, \dots, \lambda_m \in k$ [12, 5.1]. Therefore, every module $M \in \mathcal{P}^{<\infty}$ is finitely filtered by modules P_1 and R_λ, $\lambda \in k$. Conversely, every such module is clearly an object of $\mathcal{P}^{<\infty}$. Moreover, modules P_1 and R_λ, $\lambda \in k$, have no proper submodules from $\mathcal{P}^{<\infty}$, thus they are simple in the category $\mathcal{P}^{<\infty}$. So every module M from $\mathcal{P}^{<\infty}$ has a composition series

$$0 = M_0 \subset M_1 \subset \dots \subset M_l = M$$

in $\mathcal{P}^{<\infty}$ where the factors M_j/M_{j-1} are isomorphic either to P_1 or to R_λ for some $\lambda \in k$.

4.3 The (non-)uniqueness of the composition series

In general, the analogy of Jordan-Hölder theorem is not true in $\mathcal{P}^{<\infty}$. Take for example short exact sequences $0 \to P_1 \xrightarrow{\iota_\lambda} P_2 \to R_\lambda \to 0$. These exist for all $\lambda \in k$.

But the number of factors isomorphic to P_1 is unique. Consider the function $f : \mathcal{P}^{<\infty} \to \omega$ set up by a formula:

$$f(U) = \dim_k \operatorname{Hom}_R(U, R_\infty)$$

Since P_1 is projective, it is $\operatorname{Ext}^1_R(P_1, R_\infty) = 0$. The module R_∞ has no submodule isomorphic to S_2, so $\operatorname{Ext}^1_R(R_\lambda, R_\infty) = \operatorname{Ext}^1_\Lambda(R_\lambda, R_\infty) = 0$ for each $\lambda \in k$ by [12, 5.3]. Thus, $\operatorname{Ext}^1_R(U, R_\infty) = 0$ for every $U \in \mathcal{P}^{<\infty}$ and $f(V) = f(U) + f(W)$ for each exact sequence $0 \to U \to V \to W \to 0$ of modules from $\mathcal{P}^{<\infty}$. Further, $f(P_1) = 1$ and $f(R_\lambda) = 0$ for each $\lambda \in k$. The function f "counts" factors isomorphic to P_1 in a composition series of the module M and its definition is independent of the particular composition series.

If we are concerned only in modules from $\mathcal{KP}^{<\infty}$, then the composition series is unique in the sense of Jordan-Hölder. It could be seen by a similar reasoning as for P_1, this time using functions:

$$g_\mu(U) = \dim_k \operatorname{Hom}_R(U, R_{\mu,\infty}), \quad \mu \in k$$

Again, $\operatorname{Ext}^1_R(R_\lambda, R_{\mu,\infty}) = 0$ for every $\lambda, \mu \in k$ and $g_\mu(R_\lambda) = \delta_{\lambda,\mu}$. The function g_μ counts factors isomorphic to R_μ and its definition is independent of the particular composition series.

4.4 Determining modules from $\mathcal{KP}^{<\infty}$ by matrices

Let $M \in \mathcal{KP}^{<\infty}$. Then we can write

$$M \cong R_{\lambda_1,i_1} \oplus \cdots \oplus R_{\lambda_m,i_m}$$

for some Kronecker regular modules $R_{\lambda_1,i_1}, \ldots, R_{\lambda_m,i_m}$ with $\lambda_1, \ldots, \lambda_m \in k$. In particular, the linear map $x \mapsto \alpha x$ is a bijective map $e_2M \to e_1M$, since this is true for every R_{λ_j,i_j}. Denote α_M^{-1} the inverse map and for a given module M, set up a map $\chi_M \in \operatorname{End}_k(e_1M)$ by the formula $\chi_M(x) = \beta \cdot \alpha_M^{-1}(x)$.

Let us focus on the matrix A_M of the linear map χ_M in the Jordan canonical form, with respect to some suitable k-basis of the vector space

e_1M. When $M \cong R_{\lambda,i}$, then A_M is a Jordan cell of the size $i \times i$ with an eigennumber λ, ie.

$$A_M = \begin{pmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix}$$

In the general case, A_M is block-diagonal, built of Jordan cells corresponding to the direct summands $R_{\lambda_1,i_1}, \ldots, R_{\lambda_m,i_m}$ of M. Ie. $A_M = diag(A_{R_{\lambda_1,i_1}}, \ldots, A_{R_{\lambda_m,i_m}})$

Let N be some other module from $\mathcal{KP}^{<\infty}$. It is easy to see that if the vector spaces e_1M and e_1N have the same dimension and matrices of the linear maps χ_M and χ_N, with respect to some bases are similar, then the modules M and N are isomorphic. Thus we can state:

Lemma 4.1. *Two modules M, N from $\mathcal{KP}^{<\infty}$ are isomorphic if, and only if, the Jordan canonical forms of the matrices of linear maps χ_M and χ_N are the same up to the order of Jordan cells.*

4.5 Special modules of finite projective dimension

Definition 4.2. Let us call a module M from $\mathcal{P}^{<\infty}$ *special* if each of its composition series in $\mathcal{P}^{<\infty}$ has exactly one factor isomorphic to P_1, and this factor is always on the first place (ie. as a submodule of M). Let us denote $\mathcal{SP}^{<\infty}$ the full subcategory of all special modules from $\mathcal{P}^{<\infty}$.

For example, the modules P_1 and P_2 are special. Clearly, if $M \in \mathcal{SP}^{<\infty}$ and M' is a submodule of M belonging to $\mathcal{P}^{<\infty}$, then $M' \in \mathcal{SP}^{<\infty}$ too. All modules from $\mathcal{SP}^{<\infty}$ have an even dimension, since the same is true for all modules from $\mathcal{P}^{<\infty}$. In the next few paragraphs we will show that for each non-zero even $n \in \omega$, there is exactly one isomorphism class of modules of dimension n in $\mathcal{SP}^{<\infty}$. We will start by proving the existence.

Lemma 4.3. *Let $M \in \mathcal{P}^{<\infty}$, $M \not\cong P_1$. Then the following conditions are equivalent:*

1. *$M \in \mathcal{SP}^{<\infty}$*
2. *There is an exact sequence $0 \to P_1 \xrightarrow{\iota} M \xrightarrow{\pi} N \to 0$, such that N is a Kronecker module and for each non-zero submodule $N' \subseteq N$ from $\mathcal{P}^{<\infty}$ the sequence $0 \to P_1 \xrightarrow{\iota} \pi^{-1}(N') \to N' \to 0$ does not split.*

Proof. Let $M \in \mathcal{SP}^{<\infty}$. By definition, there is some exact sequence $0 \to P_1 \xrightarrow{\iota} M \xrightarrow{\pi} N \to 0$ with N Kronecker. Let $N' \subseteq N$ is an arbitrary non-zero submodule from $\mathcal{P}^{<\infty}$. If the sequence $0 \to P_1 \xrightarrow{\iota} \pi^{-1}(N') \to N' \to 0$ would split, we have also $0 \to N' \to \pi^{-1}(N') \to P_1 \to 0$, and thus the composition series of $\pi^{-1}(N')$ in $\mathcal{P}^{<\infty}$ with the factor isomorfic to P_1 at the end. Extending this series to the composition series of M, we would have a contradiction with the definition of $\mathcal{SP}^{<\infty}$.

Conversely, let $0 \to P_1 \xrightarrow{\iota} M \xrightarrow{\pi} N \to 0$ be an exact sequence with the required properties. Then every composition series of M will have exactly one factor isomorphic to P_1. If there is a series with the P_1 factor not on the first place, there is a submodule $M' \subseteq M$ isomorphic to some R_λ. But then $\iota(P_1) \cap M' = 0$, since $\operatorname{soc} M' \cong S_1$ and $\operatorname{soc} P_1 \cong S_2$. Thus, $\pi \restriction M'$ is a monomorphism and the exact sequence $0 \to P_1 \xrightarrow{\iota} \iota(P_1) + M' \to \pi(M') \to 0$ splits. But this contradicts our assumption. $\square$

Lemma 4.4. *Let $\delta : 0 \to P_1 \to M \to R_\lambda \to 0$ be an exact sequence. Then either δ splits or $M \cong P_2$. Moreover, δ splits if, and only if, M has a submodule isomorphic to R_λ.*

Proof. There is always an exact sequence $0 \to P_1 \xrightarrow{\iota_\lambda} P_2 \to R_\lambda \to 0$, and because P_2 is projective, we have the following commutative diagram:

$$\begin{array}{ccccccccccc}
 & 0 & \longrightarrow & P_1 & \xrightarrow{\iota_\lambda} & P_2 & \longrightarrow & R_\lambda & \longrightarrow & 0 \\
 & & & {\scriptstyle f}\big\downarrow & & \big\downarrow & & \big\| & & \\
\delta: & 0 & \longrightarrow & P_1 & \longrightarrow & M & \longrightarrow & R_\lambda & \longrightarrow & 0
\end{array}$$

Since $\dim_k \operatorname{End}_R(P_1) = \dim_k e_1 P_1 = 1$, f is either a zero map or an isomorphism. In the first case δ splits, in the second case $M \cong P_2$. The second assertion holds, because P_2 has not a submodule isomorphic to R_λ. $\square$

Proposition 4.5. *Take $n \in \omega$ non-zero even. Then there is a module $M \in \mathcal{SP}^{<\infty}$ of dimension n.*

Proof. We have the module P_1 for $n = 2$. So let $n > 2$. Put $m = \frac{n}{2} - 1$ and choose m different elements $\lambda_1, \ldots, \lambda_m$ of the field k. For each λ_j, consider the exact sequence $0 \to P_1 \xrightarrow{\iota_j} P_2 \to R_{\lambda_j} \to 0$. We will construct the desired module M by following push-out, where $\sigma : P_1^m \to P_1$ is a map adding up components of the direct sum:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P_1^m & \xrightarrow{\bigoplus_j \iota_j} & P_2^m & \longrightarrow & \bigoplus_j R_{\lambda_j} & \longrightarrow & 0 \\
 & & {\scriptstyle \sigma}\big\downarrow & & \big\downarrow & & \big\| & & \\
0 & \longrightarrow & P_1 & \xrightarrow{\iota} & M & \xrightarrow{\pi} & \bigoplus_j R_{\lambda_j} & \longrightarrow & 0
\end{array}$$

Suppose that there is a module $N \in \mathcal{P}^{<\infty}$, $N \subseteq \bigoplus_j R_{\lambda_j}$, such that the sequence $0 \to P_1 \xrightarrow{\iota} \pi^{-1}(N) \to N \to 0$ splits. Without a loss of generality, we can assume $N \cong R_\lambda$ for some $\lambda \in k$. Because the sequence splits, we can perceive N as a submodule of $\pi^{-1}(N)$, and thus also as a submodule of M. But $\operatorname{soc} N \cong S_1$ and $\operatorname{soc} \iota(P_1) \cong S_2$, so $\iota(P_1) \cap N = 0$ and $\pi \restriction N$ is monic. The module $\pi(N)$ being a submodule of $\bigoplus_j R_{\lambda_j}$ and $\pi(N) \cong R_\lambda$, there must be an index j, such that $\lambda = \lambda_j$ and $\pi(N) = R_{\lambda_j}$. Then we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & P_1 \arrow[r, "\iota_j"] \arrow[d, "\sigma\restriction P_1"'] & P_2 \arrow[r] \arrow[d] & R_{\lambda_j} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & P_1 \arrow[r, "\iota"] & \iota(P_1) + N \arrow[r, "\pi\restriction\iota(P_1)+N"] & R_{\lambda_j} \arrow[r] & 0
\end{tikzcd}$$

The map in the left, and therefore also in the middle, column is an isomorphism. But the first row does not split and the second row does. This situation yields a contradiction.

So $M \in \mathcal{SP}^{<\infty}$ by lemma 4.3, and certainly $\dim_k M = n$. □

Next, we would like to prove that every two modules from $\mathcal{SP}^{<\infty}$ of the same dimension are isomorphic. For the dimension 2, this is clear directly from the definition. First, we will prove a lemma which poses a restriction to the shape of a cokernel of an inclusion of the module P_1 into a chosen module from $\mathcal{SP}^{<\infty}$.

Lemma 4.6. *Let $M \in \mathcal{SP}^{<\infty}$ and $0 \to P_1 \xrightarrow{\iota} M \xrightarrow{\pi} \bigoplus_{j=1}^m R_{\lambda_j, i_j} \to 0$ be an exact sequence. Then the elements $\lambda_1, \dots, \lambda_m$ are pairwise different.*

Proof. Assume for a contradiction that the converse is true. Without a loss of generality, put $\lambda = \lambda_1 = \lambda_2$. Then the module $\bigoplus_{j=1}^m R_{\lambda_j, i_j}$ has a submodule isomorphic to $R_\lambda \oplus R_\lambda$, and it gives rise to an exact sequence $0 \to P_1 \xrightarrow{\iota} M' \xrightarrow{\pi\restriction M'} R_\lambda \oplus R_\lambda \to 0$. Denote $M'_v = \pi^{-1}(R_\lambda)$ for the v-th component of $R_\lambda \oplus R_\lambda$, $v = 1, 2$. Since $0 \to P_1 \to M'_v \to R_\lambda \to 0$ does not split, it is $M'_v \cong P_2$ by lemma 4.4. Take $h \in \iota(P_1)$ a generator of $\iota(P_1)$ and g_1, g_2 generators of M'_1, M'_2 respectively, such that :

$$\beta g_v - \lambda \alpha g_v = h, \quad v = 1, 2$$

Denote $g = g_1 - g_2$. Then a submodule generated by g is isomorphic to R_λ. But this contradicts the assumption $M \in \mathcal{SP}^{<\infty}$. □

The core of the proof of uniqueness is the following proposition which says that there is no other restriction to the cokernel of the inclusion ι, apart from the one in the lemma. Then the uniqueness proof itself follows.

Proposition 4.7. *Let $M \in \mathcal{SP}^{<\infty}$, $M \ncong P_1$. Put $n = (\dim_k M)/2 - 1$. Then for arbitrary pairwise different elements $\mu_1, \ldots, \mu_q \in k$ and non-zero numbers $i'_1, \ldots, i'_q$, such that $i'_1 + \cdots + i'_q = n$, there is an inclusion $\iota : P_1 \to M$ with* $\operatorname{Coker} \iota \cong \bigoplus_{j=1}^{q} R_{\mu_j, i'_j}$.

Proof. For the beginning, let $\iota : P_1 \to M$ be any inclusion and put $C = \operatorname{Coker} \iota \cong \bigoplus_{j=1}^{m} R_{\lambda_j, i_j}$. Then the module C is determined up to isomorphism by a Jordan canonical form of a matrix of the linear map χ_C by lemma 4.1. But there is only one Jordan cell for each eigennumber of χ_C in the Jordan canonical form by lemma 4.6. Thus, the cokernel C is in fact determined only by the multiplicities of the eigennumbers of χ_C. Using the following construction, we can increase a multiplicity of a chosen $\mu \in k$ as an eigennumber by 1, or $\mu \in k$ will become an eigennumber if it has not been before. And we can do this at the cost of decreasing a multiplicity of an eigennumber λ_1 by 1. After applying this method a finite number of times, we can "change" the eigennumbers, and thus also the cokernel of an inclusion $P_1 \to M$, to any prescribed situation.

Take an exact sequence $0 \to P_1 \xrightarrow{\iota} M \xrightarrow{\pi} \bigoplus_{j=1}^{m} R_{\lambda_j, i_j} \to 0$. Denote $M_j = \pi^{-1}(R_{\lambda_j, i_j})$. Further, take canonical generators $\bar{g}_{j,v}$ of R_{λ_j, i_j} satisfying

$$\begin{aligned} \beta \bar{g}_{j,1} &= \lambda_j \alpha \bar{g}_{j,1} \\ \beta \bar{g}_{j,v} &= \lambda_j \alpha \bar{g}_{j,v} + \alpha \bar{g}_{j,v-1}, \quad 1 < v \le i_j \end{aligned}$$

Under the suitable choice of generators $\bar{g}_{j,v}$, we can also choose a generator h of the module $\iota(P_1)$ and generators $g_{j,v}$ of modules M_j, such that $\bar{g}_{j,v} = \pi(g_{j,v})$ and $h = \beta g_{j,1} - \lambda_j \alpha g_{j,1}$ for each $j \le m$ and $v \le i_j$.

Take a module $L \subseteq M_1$ generated by $g_{1,1}$. Then $L \cong P_2$ by lemma 4.4 and for any $\mu \in k$, there is an exact sequence $0 \to P_1 \xrightarrow{\vartheta} L \to R_\mu \to 0$. In fact, we also have the following exact sequence for some regular Kronecker module X:

$$0 \to P_1 \xrightarrow{\vartheta} M \xrightarrow{\sigma} X \to 0$$

Denote $\bar{f}_{j,v} = \sigma(g_{j,v})$ and let h' be a generator of $\vartheta(P_1)$, such that $h' = \beta g_{1,1} - \mu \alpha g_{1,1}$. Then

$$\beta \bar{f}_{j,v} = \sigma(\beta g_{j,v}) = \sigma(\lambda_j \alpha g_{j,v} + \alpha g_{j,v-1} + c_{j,v} h) = \lambda_j \alpha \bar{f}_{j,v} + \alpha \bar{f}_{j,v-1} + c_{j,v} \sigma(h)$$

where $c_{j,v} \in k$ is a suitable constant, and for convenience we consider $g_{j,0} = 0$ and $\bar{f}_{j,0} = 0$. Further:

$$h = \beta g_{1,1} - \lambda_1 \alpha g_{1,1} = h' + (\lambda_1 - \mu) \alpha g_{1,1}$$

So we have

$$c_{j,v}\sigma(h) = c_{j,v}(\lambda_1 - \mu)\alpha\sigma(g_{1,1}) = c_{j,v}(\lambda_1 - \mu)\alpha\bar{f}_{1,1}$$

and together

$$\beta\bar{f}_{j,v} = \lambda_j\alpha\bar{f}_{j,v} + \alpha\bar{f}_{j,v-1} + c_{j,v}(\lambda_1 - \mu)\alpha\bar{f}_{1,1}$$

The matrix of the linear endomorphism χ_X of the vector space e_1X, with respect to the basis $\alpha\bar{f}_{j,v}$, $j \leq m$, $v \leq i_j$ and the pairs (j, v) being ordered lexicographically, is of a form

$$\begin{pmatrix} \mu & * & * & * & * & * & \cdots \\ & \lambda_1 & \ddots & & & & \\ & & \ddots & 1 & & & \\ & & & \lambda_1 & & & \\ & & & & \lambda_2 & 1 & \\ & & & & & \ddots & \ddots \end{pmatrix}$$

where the symbols $*$ in the first row should be substituted by some suitable elements of k. Concerning the eigennumbers of χ_X, we have exactly changed one occurence of λ_1 for one occurence of μ against the eigennumbers of χ_C. $\square$

Proposition 4.8. *Let $n \in \omega$. Then any two modules from $\mathcal{SP}^{<\infty}$ of dimension n are isomorphic.*

Proof. It is enough to carry out the proof only for $n > 2$ even. Choose an arbitrary $M \in \mathcal{SP}^{<\infty}$ of dimension n. Put $m = \frac{n}{2} - 1$ and choose m pairwise different elements $\lambda_1, \ldots, \lambda_m$ of the field k. Then, by the former proposition, there is an exact sequence

$$0 \to P_1 \xrightarrow{\iota} M \xrightarrow{\pi} \bigoplus_{j=1}^{m} R_{\lambda_j} \to 0$$

Let N be a factor of the module $P_2^{(m)}$, with generators of the individual components $g_1, \ldots, g_m$, determined by relations $\beta g_i - \lambda_i\alpha g_i = \beta g_{i+1} - \lambda_i\alpha g_{i+1}$. Then the dimension of N is no more than $n = 2m + 2$, since $\dim_k e_1P_2^{(m)} = \dim_k e_2P_2^{(m)} = 2m$, and for both these vector spaces we have $m - 1$ k-independent relations. Further, there is clearly an epimorphism $N \to M$ which sends elements g_i onto suitably chosen generators of $\pi^{-1}(R_{\lambda_i})$ respectively. Thus, $\dim_k M = \dim_k N = 2m + 2 = n$ and $N \cong M$. And since the module N is independent of the choice of the module M, we have at most one isomorphism class of R-modules in $\mathcal{SP}^{<\infty}$ for each dimension. $\square$

For every $n \geq 1$, denote P_n one representative of objects of $\mathcal{SP}^{<\infty}$ of dimension $2n$. This notation is consistent with the former notation of the indecomposable projectives P_1 a P_2, because these two are the representatives of modules from $\mathcal{SP}^{<\infty}$ of dimensions 2 a 4 respectively.

4.6 Auslander-Reiten translation of modules from $\mathcal{P}^{<\infty}$

In a view of corollary 3.11, it is convenient to compute the Auslander-Reiten translations of modules from $\mathcal{P}^{<\infty}$. In this subsection, we will prove that the modules R_λ, $\lambda \in k$, are invariant with respect to the translation, while the modules from $\mathcal{SP}^{<\infty}$ are transformed to the Kronecker preprojective modules.

It is well-known that the functor $(-)^* = \operatorname{Hom}_R(-, R)$ transformes an indecomposable projective R-module $P_i = Re_i$ to an indecomposable projective R^{op}-module isomorphic to $R^{op}e_i^{op}$, $i = 1, 2$. And the latter isomorphism assigns to the path $p^{op} \in R^{op}e_i^{op}$ the following homomorphism from Re_i to R:

$$\begin{aligned} Re_i &\to R \\ e_i &\mapsto p\ (\in R) \end{aligned}$$

From now on we will identify the modules $R^{op}e_i^{op}$ and P_i^*. In particular, we will denote the above mentioned homomorphism in P_i^* as p^{op}.

It is also clear that a homomorphism $f \in P_i^*$ is determined by its value on e_i. Thus, if $f(e_i) = \sum_{j=1}^m a_j p_j$ for some paths $p_1, \ldots, p_m \in R$ and elements $a_1, \ldots, a_m \in k$, then $f = \sum_{j=1}^m a_j p_j^{op}$.

Lemma 4.9. *Let $\lambda \in k$. Then $DTrR_\lambda \cong R_\lambda$.*

Proof. The minimal projective presentation of the module R_λ is $0 \to P_1 \xrightarrow{\iota_\lambda} P_2 \to R_\lambda \to 0$, where $\iota_\lambda(e_1) = \beta - \lambda\alpha$. It holds for a map $\iota_\lambda^* : P_2^* \to P_1^*$:

$$\big(\iota_\lambda^*(e_2^{op})\big)(e_1) = e_2^{op}\iota_\lambda(e_1) = e_2^{op}(\beta - \lambda\alpha) = \beta - \lambda\alpha$$

Thus, $\iota_\lambda^*(e_2^{op}) = \beta^{op} - \lambda\alpha^{op}$. The module P_1^* has a basis e_1^{op}, α^{op}, β^{op} as a vector space. For $M = P_1^*/\operatorname{Im}\iota_\lambda^*$, we have $\dim_k e_1^{op}M = \dim_k e_2^{op}M = 1$ and $\gamma^{op}M = 0$. Therefore, DM must be a Kronecker simple regular module. Because $(\beta^{op} - \lambda\alpha^{op})M = 0$, it is also $(\beta - \lambda\alpha)DM = 0$, and thus $DM = DTrR_\lambda \cong R_\lambda$. $\square$

Denote Q_j the j-th indecomposable Kronecker preprojective module. Ie. $\dim_k e_1Q_j = j$ and $\dim_k e_2Q_j = j - 1$. For P_1 a P_2, obviously $DTrP_1 = DTrP_2 = 0$. There is a following lemma for the other modules from $\mathcal{SP}^{<\infty}$.

Lemma 4.10. *Let $n \in \omega$, $n \geq 3$. Then $DTrP_n \cong Q_{n-2}$.*

Proof. Examining the proof of the proposition 4.8, if we choose $n-1$ pairwise different elements $\lambda_1, \ldots, \lambda_{n-1}$ of the field k, we will get a projective presentation of the module P_n in the shape:

$$0 \to P_1^{n-2} \xrightarrow{\iota} P_2^{n-1} \to P_n \to 0$$

Moreover, if we denote f_j the residue of the trivial path e_1 in the j-th copy of P_1 and g_l the residue of the path e_2 in the l-th copy of P_2, we have

$$\iota(f_j) = (\beta g_j - \lambda_j \alpha g_j) - (\beta g_{j+1} - \lambda_{j+1}\alpha g_{j+1}), \quad 1 \leq j \leq n-2$$

For arbitrary R-modules M, N and non-zero natural numbers m, v, there is a canonical bijection between the elements of $\mathrm{Hom}_R(M^m, N^v)$ and matrices $v \times m$ over $\mathrm{Hom}_R(M, N)$. Let us denote $i_j : M \to M^m$ the j-th inclusion and $p_l : N^v \to N$ the l-th projection. Then this bijection assigns to an homomorphism $h \in \mathrm{Hom}_R(M^m, N^v)$ the matrix $(p_l h\, i_j)_{l \leq v, j \leq m}$. Moreover, $i_j^* : (M^*)^m \to M^*$ is the j-th projection, $p_l^* : N^* \to (N^*)^v$ is the l-th inclusion, and by similar canonical bijection in R^{op}, the element $h^* \in \mathrm{Hom}_R((N^*)^v, (M^*)^m)$ corresponds to the matrix $(i_j^* h^* p_l^*)_{j \leq m, l \leq v}$.

Now put $M = P_1$, $N = P_2$, $m = n-2$ and $v = n-1$. Then the map ι corresponds to a matrix (ι_{lj}), where $\iota_{lj} = p_l \iota\, i_j$. It holds:

$$\iota_{lj}(e_1) = p_l \iota(f_j) = \begin{cases} \beta - \lambda_l \alpha & \text{for } l = j \\ -(\beta - \lambda_l \alpha) & \text{for } l = j+1 \\ 0 & \text{otherwise} \end{cases}$$

This means that:

$$\big(\iota_{jj}^*(e_2^{op})\big)(e_1) = e_2^{op}\iota_{jj}(e_1) = e_2^{op}(\beta - \lambda_j\alpha) = \beta - \lambda_j \alpha$$
$$\big(\iota_{j+1,j}^*(e_2^{op})\big)(e_1) = e_2^{op}\iota_{j+1,j}(e_1) = e_2^{op}\big(-(\beta - \lambda_{j+1}\alpha)\big) = -(\beta - \lambda_{j+1}\alpha)$$

Thus:

$$\iota_{lj}^*(e_2^{op}) = \begin{cases} \beta^{op} - \lambda_l \alpha^{op} & \text{for } l = j \\ -(\beta^{op} - \lambda_l \alpha^{op}) & \text{for } l = j+1 \\ 0 & \text{otherwise} \end{cases}$$

For the map $\iota^* : (P_2^*)^{n-1} \to (P_1^*)^{n-2}$, let us denote by g_l' the residue of the path e_2^{op} in the l-th copy of P_2^*, and by f_j' the residue of the path e_1^{op} in the

j-th copy of P_1^*. We attain the following formulas by composing the results of the former computations:

$$\iota^*(g_l') = \begin{cases} \beta^{op} f_l' - \lambda_l \alpha^{op} f_l' & \text{for } l = 1 \\ (\beta^{op} f_l' - \lambda_l \alpha^{op} f_l') - (\beta^{op} f_{l-1}' - \lambda_l \alpha^{op} f_{l-1}') & \text{for } 1 < l < n-1 \\ -(\beta^{op} f_{l-1}' - \lambda_l \alpha^{op} f_{l-1}') & \text{for } l = n-1 \end{cases}$$

Since $\lambda_1, \ldots, \lambda_{n-1}$ are pairwise different, we have $\dim_k e_2^{op}(\operatorname{Im} \iota^*) = n-1$. Clearly $e_1^{op}(\operatorname{Im} \iota^*) = 0$. And we have $\dim_k e_1^{op} P_1^* = 1$, $\dim_k e_2^{op} P_1^* = 2$. So for the module $L = (P_1^*)^{n-2} / \operatorname{Im} \iota^*$ we have $\dim_k e_1^{op} L = n-2$ and $\dim_k e_2^{op} L = 2(n-2) - (n-1) = n-3$. Then $\dim_k e_1 DL = n-2$ and $\dim_k e_2 DL = n-3$. Moreover, $DL = DTrP_n$ must be an indecomposable Kronecker module, and by a characterisation of such modules it is $DL \cong Q_{n-2}$. $\square$

4.7 Indecomposable modules in $\mathcal{P}^{<\infty}$

Now, we can characterise the indecomposable modules in $\mathcal{P}^{<\infty}$.

Proposition 4.11. *Let $0 \neq M \in \mathcal{P}^{<\infty}$ be indecomposable. It arises one of the following cases then:*

1. *$M \cong R_{\lambda,i}$ for some $\lambda \in k$ and $i \geq 1$,*
2. *$M \cong P_n$ for some $n \geq 1$.*

Before we prove the proposition, we need some auxiliary lemmas.

Lemma 4.12. *Let $M \in \mathcal{P}^{<\infty}$, such that M has no submodule isomorphic to R_λ for any $\lambda \in k$. Then M is $\mathcal{SP}^{<\infty}$-filtered.*

Proof. We will prove the lemma by induction on the number n of the composition factors isomorphic to P_1 in the composition series of M in $\mathcal{P}^{<\infty}$. There is nothing to prove for $n = 1$. Let $n > 1$. Take a composition series

$$0 = M_0 \subset M_1 \subset \cdots \subset M_l = M$$

of M, such that the last index j, for which $M_{j+1}/M_j \cong P_1$, is the greatest possible. It is $M/M_j \in \mathcal{SP}^{<\infty}$ by the definition and M_j is $\mathcal{SP}^{<\infty}$-filtered by an induction hypothesis. Thus, M is $\mathcal{SP}^{<\infty}$-filtered too. $\square$

Lemma 4.13. *Let M be $\mathcal{SP}^{<\infty}$-filtered. Then M is a direct sum of modules from $\mathcal{SP}^{<\infty}$.*

Proof. The modules P_1 a P_2 are projective and every module P_n, $n \geq 3$ has a minimal projective presentation in the shape $0 \to P_1^{n-2} \to P_2^{n-1} \to P_n \to 0$. We can construct a simultaneous minimal projective presentation of the module M, which must be of the shape:

$$0 \to P_1^m \to P_1^u \oplus P_2^v \to M \to 0$$

The module TrM is a factor of $(P_1^*)^m$ by definition. Therefore, the module $DTrM$ is a submodule of $D(P_1^*)^m = I_1^m$. Since I_1 is a Kronecker module, so is $DTrM$.

Choose an arbitrary $\lambda \in k \cup \{\infty\}$. Then

$$D\operatorname{Ext}_R^1(P_n, R_\lambda) \cong \operatorname{Hom}_R(R_\lambda, DTrP_n) \cong \operatorname{Hom}_R(R_\lambda, Q_{n-2}) = 0$$

for all $n \geq 3$ by corollary 3.10 and lemma 4.10. In particular, $\operatorname{Ext}_R^1(M, R_\lambda) = 0$, and so $\operatorname{Hom}_R(R_\lambda, DTrM) = 0$, again by corollary 3.10. Thus, module $DTrM$ is preprojective, ie. $DTrM \cong \bigoplus_{j=1}^m Q_{i_j}$ for some $i_1, \ldots, i_m$. So it is

$$M \cong P \oplus TrD(DTrM) \cong P \oplus \bigoplus_{j=1}^m P_{i_j+2}$$

for some finitely generated projective P. $\square$

Proof of proposition 4.11. Let $M \in \mathcal{P}^{<\infty}$ be indecomposable. If M is a Kronecker module, we are in the case number 1.

Suppose M is not a Kronecker module and L a maximal Kronecker regular submodule of M. Since Kronecker regular modules are closed under extensions, M/L has no submodule isomorphic to R_λ, $\lambda \in k$. Then M/L is $\mathcal{SP}^{<\infty}$-filtered by lemma 4.12. Further, we have

$$D\operatorname{Ext}_R^1(P_n, R_\lambda) \cong \operatorname{Hom}_R(R_\lambda, Q_{n-2}) = 0$$

for all $\lambda \in k$ and $n \geq 3$ by corollary 3.10 and lemma 4.10. In particular, $\operatorname{Ext}_R^1(M/L, L) = 0$ and $M = L \oplus M/L$. Thus, $L = 0$ and $M \in \mathcal{SP}^{<\infty}$ by lemma 4.13. $\square$

4.8 Lattice of tilting classes of finite type

Since $\operatorname{Fdim} R = 1$ by [14], every tilting R-module is 1-tilting. Thus, all tilting classes of a finite type could be obtained as $\mathcal{S}^\perp$, where $\mathcal{S}$ is a subset of representatives of isomorphism classes of $\mathcal{P}^{<\infty}$. Moreover, we can limit ourselves only to indecomposables from $\mathcal{P}^{<\infty}$. Denote a representative subset of such modules as $\operatorname{ind} \mathcal{P}^{<\infty}$. They were characterised in the proposition 4.11.

Proposition 4.14. *The class $\mathcal{T} \subseteq R$-Mod is a tilting class of finite type if, and only if, there is a subset $\mathcal{S} \subseteq \operatorname{ind} \mathcal{P}^{<\infty}$, such that $\mathcal{S}^\perp = \mathcal{T}$.* $\square$

Let $\mathcal{S} \subseteq \operatorname{ind} \mathcal{P}^{<\infty}$. Denote $\bar{\mathcal{S}} = {}^\perp(\mathcal{S}^\perp) \cap \operatorname{ind} \mathcal{P}^{<\infty}$. It is easy to see that $\mathcal{S}^\perp = \bar{\mathcal{S}}^\perp$. We will call the set $\mathcal{S}$ *closed*, if $\mathcal{S} = \bar{\mathcal{S}}$. Clearly, the lattice of 1-tilting classes of a finite type is anti-isomorphic to the lattice of closed subsets of $\operatorname{ind} \mathcal{P}^{<\infty}$. And the description of closed subsets follows.

Theorem 4.15. *A subset $\mathcal{S} \subseteq \operatorname{ind} \mathcal{P}^{<\infty}$ is closed if, and only if, it satisfies the following conditions:*

1. *$P_1 \in \mathcal{S}$, $P_2 \in \mathcal{S}$.*
2. *If $R_{\lambda,i} \in \mathcal{S}$ for some $\lambda \in k$ and $i \geq 1$, then $R_{\lambda,j} \in \mathcal{S}$ for every $j \geq 1$.*
3. *If $R_{\lambda,i} \in \mathcal{S}$ fore some $\lambda \in k$ and $i \geq 1$, then $P_j \in \mathcal{S}$ for every $j \geq 1$.*
4. *If $P_n \in \mathcal{S}$ for some $n \geq 3$, then $P_j \in \mathcal{S}$ for every $j \leq n$.*

Proof. First, assume $\mathcal{S} \subseteq \operatorname{ind} \mathcal{P}^{<\infty}$ is closed. The necessity of the condition 1 is obvious. For Kronecker regular modules, we have the exact sequences:

$$0 \to R_{\lambda,i} \to R_{\lambda,i-1} \oplus R_{\lambda,i+1} \to R_{\lambda,i} \to 0$$

Thus, if $R_{\lambda,i} \in \mathcal{S}$, then also $R_{\lambda,i-1}, R_{\lambda,i+1} \in \mathcal{S}$. The condition 2 follows by induction. Further, by proposition 4.7 we have

$$0 \to P_1 \to P_j \to R_{\lambda,j-1} \to 0,$$

for each $j \geq 3$. This implies the condition 3. Let $n \geq 3$ and $M \in P_n^\perp$. Then $\operatorname{Hom}_R(M, Q_{n-2}) = 0$ by corollary 3.11 and lemma 4.10. Therefore, $\operatorname{Hom}_R(M, Q_{j-2}) = 0$ for each $3 \leq j \leq n$, since Q_{n-2} has submodules isomorphic to Q_{j-2}. This means that $M \in P_j^\perp$, and $P_j \in {}^\perp(P_n^\perp)$ for each $3 \leq j \leq n$. This results in the condition 4.

Conversely, let $\mathcal{S} \subseteq \operatorname{ind} \mathcal{P}^{<\infty}$ satisfy the conditions 1–4. Assume that there is some $M \in \bar{\mathcal{S}} \setminus \mathcal{S}$. If $M = R_{\lambda,i}$ for some λ and i, then $R_{\lambda,j} \notin \mathcal{S}$ for each $j \geq 1$ by the condition 2. But this implies $R_\lambda \in \mathcal{S}^\perp$, so $R_\lambda \in \bar{\mathcal{S}}^\perp$ is a contradiction. Thus, it remains only $M = P_n$ for some $n \geq 3$. But then $R_{\lambda,i} \notin \mathcal{S}$ for each $\lambda \in k$, $i \geq 1$ and $P_j \notin \mathcal{S}$ for each $j \geq n$ by the conditions 3 and 4. So $\mathcal{S}$ consists only from some of the modules $P_1, \ldots,$ P_{n-1}. But this means $Q_{n-2} \in \mathcal{S}^\perp = \bar{\mathcal{S}}^\perp$. This is a contradiction again, since $D \operatorname{Ext}_R^1(P_n, Q_{n-2}) \cong \operatorname{Hom}_R(Q_{n-2}, Q_{n-2}) \neq 0$. $\square$

Corollary 4.16. $(\mathcal{P}^{<\infty})^\perp = \{R_\lambda | \lambda \in k\}^\perp = \bigcap_{\lambda \in k} \operatorname{Ker} \operatorname{Hom}_R(-, R_\lambda)$.

Proof. For the first equality, see [12, 5.4]. Or alternatively, let us take $\mathcal{S} = \{R_\lambda | \lambda \in k\}$. Then $\bar{\mathcal{S}} = \operatorname{ind} \mathcal{P}^{<\infty}$ by the former theorem. Thus $S^\perp = (\operatorname{ind} \mathcal{P}^{<\infty})^\perp = (\mathcal{P}^{<\infty})^\perp$. The second equality follows from corollary 3.11 and lemma 4.9. $\square$

4.9 Impossibility of reconstructing a tilting class from finitely generated modules by direct limits

This section is inspired by a dual case, where every 1-cotilting class $\mathcal{C}$ over a noetherian ring could by reconstructed from its finitely generated modules by direct limits. Ie. $\mathcal{C} = \varinjlim(\mathcal{C} \cap R\text{-mod})$, $\mathcal{C}$ being closed under direct limits, since every 1-cotilting module is pure-injective by Bazzoni [4]. So there is a bijective correspondence between 1-cotilting classes and a torision-free classes of finitely generated modules containing R (cf. [23]).

But an analogous proposition with direct limits is not true for 1-tilting classes over IST-algebra. Take $\mathcal{T} = (\mathcal{P}^{<\infty})^{\perp}$ and $\mathcal{T}^{<\infty} = \mathcal{T} \cap R$-mod. Then $\mathcal{T} = \varinjlim \mathcal{T}^{<\infty}$ implies that $\varinjlim \mathcal{T}^{<\infty}$ is closed under direct products. This is equivalent to a covariant finiteness of $\mathcal{T}^{<\infty}$ in R-mod by [12], and thus to a contravariant finiteness of $\mathcal{P}^{<\infty}$ in R-mod by [19]. But this is not true for IST-algebra. The aim of this subsection is to exhibit an example of a concrete module from $\mathcal{T} \setminus \varinjlim \mathcal{T}^{<\infty}$.

Proposition 4.17. *Let $\mathcal{T} = (\mathcal{P}^{<\infty})^{\perp}$ and $\mathcal{T}^{<\infty} = \mathcal{T} \cap R$-mod. Then a Prüfer module $R_{\lambda,\infty}$ is a member of $\mathcal{T}$ for each $\lambda \in k$, but $\mathrm{Hom}_R(M, R_{\lambda,\infty}) = 0$ for all $M \in \mathcal{T}^{<\infty}$.*

Proof. It is well-known that $\mathrm{Hom}_R(R_{\lambda,\infty}, R_\mu) = 0$ for each $\mu \in k$. Therefore, $R_{\lambda,\infty} \in \mathcal{T}$ by corollary 4.16.

We have $\mathrm{Hom}_R(I, R_\lambda) = 0$ for an injective cogenerator $I = I_1 \oplus I_2$ by corollary 3.4 and lemma 4.9. Then also $\mathrm{Hom}_R(I, R_{\lambda,\infty}) = 0$ and $\mathrm{Hom}_R(M, R_{\lambda,\infty}) = 0$ for every factor M of I. Thus, $\mathrm{Hom}_R(M, R_{\lambda,\infty}) = 0$ for each $M \in \mathcal{T}^{<\infty}$ by lemma 3.5. □

Corollary 4.18. *$R_{\lambda,\infty} \in \mathcal{T} \setminus \varinjlim \mathcal{T}^{<\infty}$ for each $\lambda \in k$.*

4.10 Constructing more complex preenvelopes

Now we are on the way to show an explicit structure of a tilting module for the class $(\mathcal{P}^{<\infty})^{\perp}$. First, we need a following general proposion which is valid for any ring. Let us remind that if X is an FP_2 module over an arbitrary ring, then $X^{\perp}$ is then closed under direct limits, thus also under filtrations and arbitrary direct sums.

Proposition 4.19. *Let R be an arbitrary ring and $\mathcal{S}$ be a set of FP_2 modules such that $\mathrm{Ext}^1_R(X, Y) = 0$ for any pair of different modules $X, Y \in \mathcal{S}$. Further, let $M \in R$-Mod be any module and*

$$0 \to M \to J_X \to C_X \to 0$$

be special $X^\perp$-preenvelopes with X-filtered cokernels C_X for each $X \in \mathcal{S}$. Then the second row of the following push-out diagram (the map σ just adding up components of the direct sum) is a special $\mathcal{S}^\perp$-preenvelope of M:

$$\begin{tikzcd}
0 \arrow[r] & M^{(\mathcal{S})} \arrow[r] \arrow[d, "\sigma"'] & \bigoplus_{X\in\mathcal{S}} J_X \arrow[r] \arrow[d] & \bigoplus_{X\in\mathcal{S}} C_X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M \arrow[r] & J \arrow[r] & \bigoplus_{X\in\mathcal{S}} C_X \arrow[r] & 0
\end{tikzcd}$$

Proof. It is sufficient to prove that $J \in \mathcal{S}^\perp$ and $C = \bigoplus_{X\in\mathcal{S}} C_X \in {}^\perp(\mathcal{S}^\perp)$. But the latter is clear, since the module C is a direct sum of $\mathcal{S}$-filtered modules.

Choose an arbitrary $Y \in \mathcal{S}$. If we take only a component corresponding to the module Y in the first row of the commutative diagram above, and if we denote σ' a restriction of the map σ to that component, we will get an induced diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M \arrow[r] \arrow[d, "\sigma'"'] & J_Y \arrow[r] \arrow[d] & C_Y \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r] & J \arrow[r] \arrow[d] & \bigoplus_{X\in\mathcal{S}} C_X \arrow[r] \arrow[d] & 0 \\
 & & \bigoplus_{X\in\mathcal{S}\setminus\{Y\}} C_X \arrow[r, equal] \arrow[d] & \bigoplus_{X\in\mathcal{S}\setminus\{Y\}} C_X \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

By assumption, $X \in Y^\perp$ for each $X \in \mathcal{S} \setminus \{Y\}$ and $Y^\perp$ is closed under filtrations and direct sums, thus $\bigoplus_{X\in\mathcal{S}\setminus\{Y\}} C_X \in Y^\perp$. But also $J_Y \in Y^\perp$, therefore $J \in Y^\perp$. And this is true for any $Y \in \mathcal{S}$, so $J \in \mathcal{S}^\perp$. $\square$

4.11 Structure of tilting modules for $R_\lambda^\perp$

Construction 4.20 *($R_\lambda^\perp$-preenvelopes of P_1 and P_2).* Let $\lambda \in k$. By proposition 4.7, there is an exact sequence $0 \to P_1 \to P_{n+1} \xrightarrow{\sigma} R_{\lambda,n} \to 0$ for each $n \geq 1$. If we take an inclusion $j : R_{\lambda,n-1} \to R_{\lambda,n}$ for any $n \geq 2$, then the module $M = \sigma^{-1}(\operatorname{Im} j)$ is clearly an object of $\mathcal{SP}^{<\infty}$ (cf. def. 4.2), thus $M \cong P_n$ by proposition 4.8. Moreover, $P_{n+1}/M \cong R_{\lambda,n}/\operatorname{Im} j \cong R_\lambda$. So we have a following exact sequence for any $n \geq 1$:

$$0 \longrightarrow P_n \xrightarrow{\iota_{n+1,n}} P_{n+1} \longrightarrow R_\lambda \longrightarrow 0$$

Denote $\iota_{m,n} = \iota_{m,m-1} \dots \iota_{n+2,n+1}\iota_{n+1,n}$ and $\iota_{n,n} = 1_{P_n}$ for every $m > n \geq 1$. There are obviously commutative squares for $n \geq 2$:

$$\begin{tikzcd} P_1 \arrow[r, "\iota_{n,1}"] \arrow[d, equal] & P_n \arrow[d, "\iota_{n+1,n}"'] \\ P_1 \arrow[r, "\iota_{n+1,1}"] & P_{n+1} \end{tikzcd}$$

Further, $\operatorname{Coker} \iota_{n,1}$ is R_λ-filtered, thus $\operatorname{Coker} \iota_{n,1} \cong R_{\lambda,n-1}$ by lemma 4.6. Therefore, we have exact commutative diagrams with monomorphisms in columns:

$$\begin{tikzcd} 0 \arrow[r] & P_1 \arrow[r, "\iota_{n,1}"] \arrow[d, equal] & P_n \arrow[r, "\pi_n"] \arrow[d, "\iota_{n+1,n}"'] & R_{\lambda,n-1} \arrow[r] \arrow[d, "j_n"'] & 0 \\ 0 \arrow[r] & P_1 \arrow[r, "\iota_{n+1,1}"] & P_{n+1} \arrow[r, "\pi_{n+1}"] & R_{\lambda,n} \arrow[r] & 0 \end{tikzcd}$$

Denote T_λ the direct limit of modules P_n, $n \geq 1$ with inclusions $\iota_{m,n}$, $m \geq n \geq 1$. Then we obtain an exact sequence:

$$\delta_1 : \quad 0 \longrightarrow P_1 \xrightarrow{\iota} T_\lambda \xrightarrow{\pi} R_{\lambda,\infty} \longrightarrow 0$$

Next, take a commutative diagram with canonical inclusions in columns:

$$\begin{tikzcd} 0 \arrow[r] & P_1 \arrow[r, "\iota_{2,1}"] \arrow[d, equal] & P_2 \arrow[r, "\pi_2"] \arrow[d, "\iota'"'] & R_\lambda \arrow[r] \arrow[d, "j'"'] & 0 \\ 0 \arrow[r] & P_1 \arrow[r, "\iota"] & T_\lambda \arrow[r, "\pi"] & R_{\lambda,\infty} \arrow[r] & 0 \end{tikzcd}$$

Then $\operatorname{Coker} \iota' \cong \operatorname{Coker} j' \cong R_{\lambda,\infty}$, so we have an exact sequence

$$\delta_2 : \quad 0 \longrightarrow P_2 \xrightarrow{\iota'} T_\lambda \xrightarrow{\pi'} R_{\lambda,\infty} \longrightarrow 0$$

Proposition 4.21. *Let us adopt the notation from the preceding construction. Then the short exact sequence δ_i is a special $R_\lambda^\perp$-preenvelope of the indecomposable projective P_i, $i = 1, 2$.*

Proof. It is sufficient to prove that $T_\lambda \in R_\lambda^\perp$ and $R_{\lambda,\infty} \in {}^\perp(R_\lambda^\perp)$. The latter is clear, since the Prüfer module $R_{\lambda,\infty}$ is R_λ-filtered.

It is enough to show that $\mathrm{Hom}_R(T_\lambda, R_\lambda) = 0$ by corollary 3.11 and lemma 4.9. Take an arbitrary $f \in \mathrm{Hom}_R(T_\lambda, R_\lambda)$. If we apply the functor $\mathrm{Hom}_R(-, R_\lambda)$ to the exact sequence $0 \to P_1 \xrightarrow{\iota_{2,1}} P_2 \to R_\lambda \to 0$, we obtain

$$0 \longrightarrow \mathrm{Hom}_R(R_\lambda, R_\lambda) \longrightarrow \mathrm{Hom}_R(P_2, R_\lambda) \xrightarrow{\mathrm{Hom}_R(\iota_{2,1}, R_\lambda)} \mathrm{Hom}_R(P_1, R_\lambda)$$

But $\dim_k \mathrm{Hom}_R(R_\lambda, R_\lambda) = 1$, and also $\dim_k \mathrm{Hom}_R(P_i, R_\lambda) = \dim_k e_i R_\lambda = 1$ for $i = 1, 2$. This implies $\mathrm{Hom}_R(\iota_{2,1}, R_\lambda) = 0$. So $f\iota = f\iota'\iota_{2,1} = 0$. Therefore, there is a map $\bar{f}$, such that $f = \bar{f}\pi$. But now $\bar{f} \in \mathrm{Hom}_R(R_{\lambda,\infty}, R_\lambda) = 0$, and thus $f = 0$. □

Theorem 4.22. *Let $\lambda \in k$ and T_λ be a module as in the contruction 4.20. Then $T_\lambda \oplus R_{\lambda,\infty}$ is a tilting module corresponding to the tilting class $R_\lambda^\perp$.*

Proof. We have $R \cong P_1 \oplus P_2$, and by the former proposition, there is a special preenvelope of R in the shape

$$0 \to R \to T_\lambda \oplus T_\lambda \to R_{\lambda,\infty} \oplus R_{\lambda,\infty} \to 0$$

So the module $T = T_\lambda \oplus T_\lambda \oplus R_{\lambda,\infty} \oplus R_{\lambda,\infty}$ is a tilting module corresponding to the class $R_\lambda^\perp$ by theorem 2.27. If T' is a module, such that $T' \in \mathrm{Add}T$ and $T \in \mathrm{Add}T'$, then T' is tilting too, and $T^\perp = (T')^\perp$. Putting $T' = T_\lambda \oplus R_{\lambda,\infty}$ gives us the desired result. □

Remark 4.23. Let us write down a linear representation corresponding to the module T_λ. Since we are over IST-algebra, the representations are of the shape

$$V_1 \quad \begin{array}{c} \xrightarrow{f_\gamma} \\ \xleftarrow{f_\beta} \\ \xleftarrow{f_\alpha} \end{array} \quad V_2$$

with linear maps satisfying equations $f_\alpha f_\gamma = f_\beta f_\gamma = f_\gamma f_\alpha = 0$.

In this case the vector spaces are of countable dimensions, ie. $V_1 = V_2 = k^{(\omega)}$, and the linear maps could by given by following matrices (the blank spaces are zeroes):

$$f_\alpha = \begin{pmatrix} 0 & 0 & 0 & \dots \\ 0 & 1 & & \\ 0 & & 1 & \\ \vdots & & & \ddots \end{pmatrix}, \quad f_\beta = \begin{pmatrix} 0 & 1 & & \\ 0 & \lambda & 1 & \\ 0 & & \lambda & \ddots \\ \vdots & & & \ddots \end{pmatrix}, \quad f_\gamma = \begin{pmatrix} 1 & 0 & 0 & \dots \\ 0 & 0 & & \\ 0 & & 0 & \\ \vdots & & & \ddots \end{pmatrix}$$

Just for comparison, a representation of the Prüfer module $R_{\lambda,\infty}$:

$$f_\alpha = \begin{pmatrix} 1 & 0 & 0 & \dots \\ 0 & 1 & & \\ 0 & & 1 & \\ \vdots & & & \ddots \end{pmatrix}, \quad f_\beta = \begin{pmatrix} \lambda & 1 & & \\ 0 & \lambda & 1 & \\ 0 & & \lambda & \ddots \\ \vdots & & & \ddots \end{pmatrix}, \quad f_\gamma = \begin{pmatrix} 0 & 0 & 0 & \dots \\ 0 & 0 & & \\ 0 & & 0 & \\ \vdots & & & \ddots \end{pmatrix}$$

Another thing we can notice is that in contrast to proposition 4.8, modules T_λ and T_μ are non-isomorphic for $\lambda \neq \mu$. Otherwise, there would be an inclusion $i : P_1 \to T_\mu$ with a cokernel isomorphic to $R_{\lambda,\infty}$. But this is not possible, since cokernel of any inclusion $i : P_1 \to T_\mu$ is isomorphic to $R_{\mu,\infty} \oplus M$, where M is a suitable finitely generated Kronecker regular module.

4.12 Structure of tilting module for $(\mathcal{P}^{<\infty})^\perp$

Theorem 4.24. *Let $A \subseteq k$ be non-empty and put $\mathcal{S} = \{R_\lambda | \lambda \in A\}$. For each $\lambda \in A$, take a special preenvelope $0 \to P_1 \xrightarrow{\iota_\lambda} T_\lambda \to R_{\lambda,\infty} \to 0$ from construction 4.20. And take the following push-out diagram with an adding map σ:*

$$\begin{tikzcd}
0 \arrow[r] & P_1^{(A)} \arrow[r] \arrow[d, "\sigma"'] & \bigoplus_{\lambda\in A} T_\lambda \arrow[r] \arrow[d] & \bigoplus_{\lambda\in A} R_{\lambda,\infty} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & P_1 \arrow[r] & T_A \arrow[r] & \bigoplus_{\lambda\in A} R_{\lambda,\infty} \arrow[r] & 0
\end{tikzcd}$$

Then $T = T_A \oplus \bigoplus_{\lambda\in A} R_{\lambda,\infty}$ is a tilting module corresponding to the tilting class $\mathcal{S}^\perp$.

Proof. The set $\mathcal{S}$ fit the assumptions of proposition 4.19. Thus, an exact sequence $0 \to P_1 \to T_A \to \bigoplus_{\lambda\in A} R_{\lambda,\infty} \to 0$ is a special $\mathcal{S}^\perp$-preenvelope of the projective P_1.

Take an arbitrary $\mu \in A$. Then we have a following commutative diagram with isomorphisms in the first and monomorphisms in the other columns:

$$\begin{tikzcd}
0 \arrow[r] & P_1 \arrow[r] \arrow[d, equal] & P_2 \arrow[r] \arrow[d, "\iota'"'] & R_\mu \arrow[r] \arrow[d, "j'"'] & 0 \\
0 \arrow[r] & P_1 \arrow[r, "\iota_\mu"] \arrow[d, "\sigma'"'] & T_\mu \arrow[r] \arrow[d, "\iota''"'] & R_{\mu,\infty} \arrow[r] \arrow[d, "j''"'] & 0 \\
0 \arrow[r] & P_1 \arrow[r] & T_A \arrow[r] & \bigoplus_{\lambda\in A} R_{\lambda,\infty} \arrow[r] & 0
\end{tikzcd}$$

Then $T_A/\operatorname{Im}\iota''\iota' \cong \bigoplus_{\lambda\in A} R_{\lambda,\infty}/\operatorname{Im} j''j' \cong \bigoplus_{\lambda\in A} R_{\lambda,\infty}$. So we have a short exact sequence, which is necesarilly a special $\mathcal{S}^{\perp}$-preenvelope of the module P_2:

$$0 \longrightarrow P_2 \xrightarrow{\iota''\iota'} T_A \longrightarrow \bigoplus_{\lambda\in A} R_{\lambda,\infty} \longrightarrow 0$$

Since $R \cong P_1 \oplus P_2$, the module $T \oplus T$ is tilting, corresponding to the tilting class $\mathcal{S}^{\perp}$, and so is T itself. □

Corollary 4.25. *Adapting the notation from the theorem, $T_k \oplus \bigoplus_{\lambda\in k} R_{\lambda,\infty}$ is a tilting module corresponding to $(\mathcal{P}^{<\infty})^{\perp}$.*

Proof. Cf. corollary 4.16. □

Remark 4.26. The linear representation of a module T_k could not be written in terms of matrices, since its dimension is equal to a cardinality of k, uncountable in general. But the push-out construction from the theorem is actually "glueing" the modules T_λ, $\lambda \in k$ together alongside a "common copy of P_1" included in them. Thus, its only a matter of a straightforward computation to give an explicit behavior of the linear maps on suitably chosen bases of the vector spaces.

5 Bundles of 1-tilting classes

This section points out to a possible approach of an investigation of the conjecture that every 1-tilting class over a finite dimensional algebra is of a finite type.

Most of the section is devoted to analysing torsion and cotorsion pairs over artin algebras, since 1-tilting classes are special instances of a torsion and cotorsion classes at the same time. But the main idea of the section, which is more general, is expressed in lemma 5.14, definition 5.20, proposition 5.22, and their corollaries.

Throughout this section, R will be an arbitrary ring if not stated otherwise. Let $\mathcal{C}$ be a class od R-modules. Then denote $\varinjlim \mathcal{C}$ the class of all direct limits of modules of $\mathcal{C}$, $\underrightarrow{\cup}\, \mathcal{C}$ the class of all directed unions of modules of $\mathcal{C}$ and $\operatorname{filt} \mathcal{C}$ the class of all $\mathcal{C}$-filtered modules. Let us remind the notation $\mathcal{C}^{<\infty} = \mathcal{C} \cap R\text{-mod}$ for a class of modules $\mathcal{C}$.

5.1 The dual

First, we notice that the following functors have very much in common:

1. the functor $(-)^c = \operatorname{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$,
2. the functor $D(-) = \operatorname{Hom}_S(-, J)$ if R is an artin algebra over S.

For the sake of summarizing basic properties of functors like these, we now introduce a slightly more general setting which encompasses both the examples. Let S be a commutative ring and $\psi : S \to R$ be a ring homomorphism into the centre of R. Further, let J be an injective cogenerator of S-Mod. Then every left or right R-module is also an R-S-bimodule in a natural way. Moreover, the groups of moprhisms in R-Mod or Mod-R are S-modules too. Put $D(-) = \operatorname{Hom}_S(-, J)$ and let us call this functor S-Mod $\to S$-Mod a *dual*. It is easy to see that D is also a functor from left to right R-modules and vice versa. As to the notation, put $D\mathcal{C} = \{DM | M \in \mathcal{C}\}$ for any class of modules $\mathcal{C}$.

Lemma 5.1 (the dual and exactness). *Let $K, L, M \in S$-Mod (or R-Mod or Mod-R). Then*

1. *$K = 0$ if, and only if, $DK = 0$,*
2. *$K \to L \to M$ is exact if, and only if, $DM \to DL \to DK$ is exact.*

Proof. [1, par. 18]. $\square$

Lemma 5.2 (the dual and homology). *Let $K, L, M \in R$-Mod and $N \in$ Mod-R (or $K, L, M \in$ Mod-R and $N \in R$-Mod). Then*

1. $\mathrm{Hom}_R(K, DN) \cong \mathrm{Hom}_R(N, DK)$ *and the isomorphism is functorial in both variables (thus, $D : R$-Mod $\leftrightarrow$ Mod-R is a duality also in a categorical sense—as a contravariant adjoint functor),*
2. *for any homomorphism $f : K \to L$, it is $\mathrm{Im}\, Df \cong D(\mathrm{Im}\, f)$,*
3. *for $f : K \to L$ and $g : L \to M$, such that $gf = 0$, it is $\mathrm{Ker}\, Df / \mathrm{Im}\, Dg \cong D(\mathrm{Ker}\, g / \mathrm{Im}\, f)$,*
4. $\mathrm{Ext}^i_R(K, DN) \cong \mathrm{Ext}^i_R(N, DK)$ *and the isomorphisms are functorial in both variables for any $i \geq 1$.*

Proof.

1. Using the adjunction formula, we have:

$$\mathrm{Hom}_R(K, DN) \cong D(N \otimes_R K) \cong \mathrm{Hom}_R(N, DK)$$

2. From the exactness lemma, it follows that the unique epi-mono factorisation of f is transformed to epi-mono factorisation of Df by D.
3. We have $\mathrm{Ker}\, g / \mathrm{Im}\, f = \mathrm{Im}(p \restriction \mathrm{Ker}\, g)$, where p is a cokernel of f. Similarly for $\mathrm{Ker}\, Df / \mathrm{Im}\, Dg$, so we can use 2.
4. Either by 1. and 3. or by proposition 2.16, we have:

$$\mathrm{Ext}^i_R(K, DN) \cong D\, \mathrm{Tor}^R_i(N, K) \cong \mathrm{Ext}^i_R(N, DK)$$

□

Lemma 5.3 (the dual and purity). *Let $K, L, M \in R$-Mod (or Mod-R). Then*

1. *DK is pure-injective,*
2. *$0 \to K \to L \to M \to 0$ is pure exact if, and only if, $0 \to DM \to DL \to DK \to 0$ splits,*
3. *the map $\eta_K : K \to D^2K$ defined as $\eta(x)(f) = f(x)$ (a unit of adjunction) is a pure monomorphism.*

Proof.

1. DK is pure-injective if, and only if, $\operatorname{Hom}_R(-, DK)$ is exact on pure exact sequences if, and only if, $- \otimes_R K$ (or $K \otimes_R -$) is exact on pure exact sequences. But this is one of equivalent definitions of pure exact sequences.

2. By the adjunction formula, $0 \to K \to L \to M \to 0$ is pure if, and only if, $\operatorname{Hom}_R(N, -)$ is exact on $0 \to DM \to DL \to DK \to 0$ for each R-module N. But this is equivalent to $0 \to DM \to DL \to DK \to 0$ being split.

3. It is a straightforward computation that $D\eta_K \circ \eta_{DK} = 1_{DK}$. Then use 2.

□

Lemma 5.4 (homology and the double dual). *Let $K, N \in R$-Mod (or Mod-R) and let K be finitely presented. Then*

1. $\operatorname{Hom}_R(K, D^2N) \cong D^2 \operatorname{Hom}_R(K, N)$ *and the isomorphism is functorial in both variables,*

2. *for R left (or right) coherent,* $\operatorname{Ext}^i_R(K, D^2N) \cong D^2 \operatorname{Ext}^i_R(K, N)$ *and the isomorphisms are functorial in both variables for any $i \geq 1$.*

Proof.

1. Let us assume the case of left modules. Then there is a functorial isomorphism $D \operatorname{Hom}_R(K, N) \cong DN \otimes_R K$ (cf. [10, th. 3.2.11]). Thus,

$$D^2 \operatorname{Hom}_R(K, N) \cong D(DN \otimes_R K) \cong \operatorname{Hom}_R(K, D^2N)$$

2. It follows either by 1. and lemma 5.2 or by proposition 2.16.

□

Lemma 5.5 (dualizing dimensions). *Let M be left (or right) R-module. Then*

1. $\operatorname{wd} M = \operatorname{id} DM$,

2. *for R left (or right) noetherian, also* $\operatorname{id} M = \operatorname{wd} DM$.

Proof.

1. Using lemma 5.2, we obtain $\operatorname{wd} M \leq n$ if, and only if, $\operatorname{Tor}^R_{n+1}(X, M) = 0$ for each X if, and only if, $\operatorname{Ext}^{n+1}_R(X, DM) = 0$ for each X if, and only if, $\operatorname{id} DM \leq n$ for any left R-module M.

2. R being noetherian, all cyclic modules are finitely presented. Applying lemma 5.4, we obtain $\operatorname{Ext}^i_R(C, M) = 0$ if, and only if, $\operatorname{Ext}^i_R(C, D^2M) = 0$ for any cyclic module C and $i \geq 1$. Using Baer lemma, this implies $\operatorname{id} M = \operatorname{id} D^2M$, and $\operatorname{id} D^2M = \operatorname{wd} DM$ by the first assertion.

□

If R is an artin algebra, we will often need the crucial property of the classical D functor—that the unity of adjunction is an R-isomorphism for all finitely presented left and right R-modules (cf. [3]). So, if state anything namely for artin algebras below, we silently assume that D is this particular dual.

5.2 Coherent functors and artin algebras

An additive functor $F : R\text{-Mod} \to \mathrm{Ab}$ is said to be *coherent* if it commutes with direct limits and products. The following description of coherent functors is given in [7, 2.1]. In fact, the proof there is only for artin algebras, but it can be generalised in a straightforward manner.

Proposition 5.6. *Let R be a ring. Then $F : R\text{-Mod} \to \mathrm{Ab}$ is a coherent functor if, and only if, there is an homomorphism $\vartheta : X \to Y$, $X, Y \in R\text{-mod}$ such that F is naturally equivalent to* $\operatorname{Coker}(\operatorname{Hom}_R(Y, -) \to \operatorname{Hom}_R(X, -))$. □

Lemma 5.4 implies the following relation between coherent functors and a dual:

Corollary 5.7. *Let R be a ring, $M \in R\text{-Mod}$ and F a coherent functor. Then $F(M) = 0$ if, and only if, $F(D^2M) = 0$.*

Proof. We can assume $F = \operatorname{Coker} \operatorname{Hom}_R(\vartheta, -)$ for some $\vartheta : X \to Y$, $X, Y \in R$-mod. Then $F(D^2M) = 0$ if, and only if, $\operatorname{Hom}_R(\vartheta, D^2M)$ is epic if, and only if, $D^2 \operatorname{Hom}_R(\vartheta, M)$ is epic if, and only if, $F(M) = 0$. □

In case R is an artin algebra, coherent functors are closely related to a model theory and definable classes.

Proposition 5.8. *Let R be an artin algebra over an algebraically closed field. Then*

1. *for every $M \in R$-Mod, M and D^2M are elementarily equivalent,*

2. *every definable class is closed under an application of D^2.*

Proof. It has been shown in [7, 2.1] that M and M' are elementarily equivalent if, and only if, they are in the kernel of the same set of coherent functors. And definable classes are always closed under elementarily equivalence. $\square$

5.3 Extensions of torsion pairs in R-mod

Let $(\mathcal{R}, \mathcal{F})$ be a torsion pair in R-mod, $\mathcal{R}$ being the torsion and $\mathcal{F}$ being the torsion-free class. We will call a torsion pair $(\mathcal{R}', \mathcal{F}')$ in R-Mod an *extension* of $(\mathcal{R}, \mathcal{F})$ to R-Mod if $\mathcal{R} = \mathcal{R}' \cap R$-mod and $\mathcal{F} = \mathcal{F}' \cap R$-mod.

It is easy to see that for R noetherian and any torsion pair $(\mathcal{R}', \mathcal{F}')$ in R-Mod, $(\mathcal{R}' \cap R\text{-mod}, \mathcal{F}' \cap R\text{-mod})$ is a torsion pair in R-mod and $(\mathcal{R}', \mathcal{F}')$ is its extension. On the other hand, there can be many extensions of a torsion pair $(\mathcal{R}, \mathcal{F})$ in R-mod in general. Two of them are extremal:

1. $(\mathcal{R}_\infty, \mathcal{F}_\infty)$, where $\mathcal{F}_\infty = \operatorname{Ker}\operatorname{Hom}_R(\mathcal{R}, -)$ (thus the torsion-free class is the largest possible),

2. $(\hat{\mathcal{R}}, \hat{\mathcal{F}})$, where $\hat{\mathcal{R}} = \operatorname{Ker}\operatorname{Hom}_R(-, \mathcal{F})$ (thus the torsion class is the largest possible).

Lemma 5.9. *Let R be a left noetherian ring, $(\mathcal{R}, \mathcal{F})$ a torsion pair in R-mod, and $(\mathcal{R}_\infty, \mathcal{F}_\infty)$ its extension with the largest possible torsion-free class. Then*

1. $\mathcal{R}_\infty = \varinjlim \mathcal{R} = \underrightarrow{\cup}\, \mathcal{R} = \operatorname{gen} \mathcal{R} = \operatorname{filt} \mathcal{R}$,

2. $\mathcal{F}_\infty = \varinjlim \mathcal{F} = \underrightarrow{\cup}\, \mathcal{F}$.

Proof. $(\varinjlim \mathcal{R}, \varinjlim \mathcal{F})$ is a torsion pair by [8]. Then obviously $\mathcal{R}_\infty \supseteq \operatorname{gen} \mathcal{R} \supseteq \varinjlim \mathcal{R} \supseteq \mathcal{R}_\infty$. Let $M \in \mathcal{R}_\infty$ and $N \subseteq M$ finitely generated. Since $M \in \operatorname{gen} \mathcal{R}$, there is $N' \in \mathcal{R}$ and an homomorphism $f : N' \to M$, such that $N \subseteq \operatorname{Im} f$. This implies $\mathcal{R}_\infty = \underrightarrow{\cup}\, \mathcal{R}$. In particular, every module from $\mathcal{R}_\infty$ has a non-trivial submodule from $\mathcal{R}$. The factor is again from $\mathcal{R}_\infty$, since it is a torsion class. Thus, we can always construct an $\mathcal{R}$-filtration by induction. So $\mathcal{R}_\infty = \operatorname{filt} \mathcal{R}$.

Every module $L \in \mathcal{F}_\infty$ is a directed union of its submodules, therefore of submodules from $\mathcal{F}$. Thus $\varinjlim \mathcal{F} \supseteq \underrightarrow{\cup}\, \mathcal{F} \supseteq \mathcal{F}_\infty$. $\square$

Now we will try to give at least a partial description of the other extremal extension over an artin algebra.

Lemma 5.10. *Let R be an artin algebra, $(\mathcal{R},\mathcal{F})$ a torsion pair in R-mod and $(\hat{\mathcal{R}},\hat{\mathcal{F}})$ its extension to R-Mod with the largest possible torsion class. Then $(D\mathcal{F},D\mathcal{R})$ is a torsion pair in mod-R and for any $M \in$ Mod-R:*

1. *$DM \in \hat{\mathcal{R}}$ if, and only if, $M \in (D\mathcal{R})_\infty$,*
2. *$DM \in \hat{\mathcal{F}}$ if, and only if, $M \in (D\mathcal{F})_\infty$.*

Proof. It is clear from the properties of D that $(D\mathcal{F},D\mathcal{R})$ is a torsion pair in mod-R. Further, $\mathrm{Hom}_R(DM,\mathcal{F}) \cong \mathrm{Hom}_R(DM,D^2\mathcal{F}) = 0$ if, and only if, $\mathrm{Hom}_R(D\mathcal{F},D^2M) = 0$ if, and only if, $\mathrm{Hom}_R(D\mathcal{F},M) = 0$ by lemmas 5.2 and 5.4. This proves the first assertion.

Now let $M \in (D\mathcal{F})_\infty$. Then $M = \varinjlim M_\nu$ for some system of finitely presented modules M_ν from $D\mathcal{F}$. Then $DM = \varprojlim DM_\nu$ with $DM_\nu \in \mathcal{F}$, hence $DM \in \hat{\mathcal{F}}$. Conversely, let $DM \in \hat{\mathcal{F}}$. There is a unique exact sequence $0 \to M_\mathcal{F} \to M \to M_\mathcal{R} \to 0$ with $M_\mathcal{F} \in (D\mathcal{F})_\infty$ and $M_\mathcal{R} \in (D\mathcal{R})_\infty$. Dualizing this sequence, we obtain $0 \to DM_\mathcal{R} \to DM \to DM_\mathcal{F} \to 0$ with $DM_\mathcal{R} \in \hat{\mathcal{R}}$ and $DM_\mathcal{F} \in \hat{\mathcal{F}}$. But $DM \in \hat{\mathcal{F}}$, hence $DM_\mathcal{R} = 0$, $M_\mathcal{R} = 0$ and $M = M_\mathcal{F} \in (D\mathcal{F})_\infty$. □

Lemma 5.11. *Let R be an artin algebra and $M \in$ R-Mod (or Mod-R). Then M purely embeds into a product of its finitely presented factors.*

Proof. Let $\{p_\nu : M \to M_\nu\}$ be a representative set of all epimorphisms from M to finitely presented modules. Denote $p : M \to \prod M_\nu$ the corresponding product map. Take any $0 \neq x \in M$. Then the injective envelope $Rx \to E(Rx)$ factors through the inclusion $Rx \subseteq M$, and since $E(Rx)$ is finitely presented, we have $p(x) \neq 0$. Hence p is monic.

R being an artin algebra, it is enough to show that $\mathrm{Hom}(p,X)$ is epic for any $X \in R$-mod (or mod-R). But if $f \in \mathrm{Hom}_R(M,X)$, then $\mathrm{Im}\, f$ is finitely presented, so f clearly factors through p. □

Proposition 5.12. *Let R be an artin algebra, $(\mathcal{R},\mathcal{F})$ a torsion pair in R-mod and $(\hat{\mathcal{R}},\hat{\mathcal{F}})$ its extension to R-Mod with the largest possible torsion class. Then $M \in \hat{\mathcal{R}}$ if, and only if, M purely embeds into a product of modules from $\mathcal{R}$. In particular, $\hat{\mathcal{R}} \cap \mathcal{PI} = \mathrm{Prod}\mathcal{R}$.*

Proof. First, we prove that $\mathrm{Prod}\mathcal{R} \subseteq \hat{\mathcal{R}}$. Let (X_ν) by a family of modules from $\mathcal{R}$. Then $\prod X_\nu \cong D(\bigoplus DX_\nu) \in \hat{\mathcal{R}}$ if, and only if, $\bigoplus DX_\nu \in (D\mathcal{R})_\infty$ by lemma 5.10. But the latter holds, since $(D\mathcal{R})_\infty$ is closed under direct sums.

Now the only if part is proven, taking lemma 5.11 into account. The if part follows from a closure of $\hat{\mathcal{R}}$ under pure submodules. This is because $\hat{\mathcal{R}} = \mathrm{Ker}\,\mathrm{Hom}_R(-,\mathcal{F})$ and and the fact that all finitely presented modules over an artin algebra are pureinjective. The last assertion is clear. □

Remark 5.13. The last proposition, stating that modules in $\hat{\mathcal{R}}$ are precisely the "purely $\mathcal{R}$-cogenerated" ones, is in some sense a dualization of the lemma 5.9. There, modules from $\mathcal{F}_\infty$ are precisely the pure epimorphic images of direct sums of modules from $\mathcal{F}$ by Lenzing's characterisation of direct limits of finitely presented modules [16]. But, unlike lemma 5.9, this does not work in case of $\hat{\mathcal{F}}$, since this class contains at least *all* $\mathcal{F}$-cogenerated modules.

5.4 Extensions of cotorsion pairs in R-mod

Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair in R-mod. Similarly as in the last section, we will call a cotorsion pair $(\mathcal{A}', \mathcal{B}')$ in R-Mod an *extension* of $(\mathcal{A}, \mathcal{B})$ to R-Mod if $\mathcal{A} = \mathcal{A}' \cap R$-mod and $\mathcal{B} = \mathcal{B}' \cap R$-mod.

The situation is slightly more complicated here, since over general rings we need pure-injective modules rather than finitely presented ones when dealing with the class $\mathcal{B}$; these modules are the right ones for ${}^\perp\mathcal{B}$ to be closed under direct limits. This can be illustrated by the following example:

Lemma 5.14. *Let R be a ring and $(\mathcal{C}, \mathcal{D})$ be a cotorsion pair in R-Mod, such that $\mathcal{C} \subseteq \mathcal{P}_1$. Then $\mathcal{C}^\perp \cap \mathcal{PI} = (\mathcal{C}^{<\infty})^\perp \cap \mathcal{PI}$.*

Proof. Clearly, $\mathcal{C}^\perp \cap \mathcal{PI} \subseteq (C^{<\infty})^\perp \cap \mathcal{PI}$. On the other hand, if $M \in (C^{<\infty})^\perp$ is pure-injective, then $M \in (\varinjlim \mathcal{C}^{<\infty})^\perp \subseteq \mathcal{C}^\perp$, the last inclusion by [23, 4.4]. $\square$

Corollary 5.15. *Let R be a left coherent ring and $(\mathcal{C}_1, \mathcal{D}_1)$, $(\mathcal{C}_2, \mathcal{D}_2)$ be cotorsion pairs in R-Mod, such that $\mathcal{C}_1, \mathcal{C}_2 \subseteq \mathcal{P}_1$. Then the following conditions are equivalent:*

1. *$\mathcal{C}_1 \cap R$-mod $= \mathcal{C}_2 \cap R$-mod,*

2. *$\mathcal{D}_1 \cap \mathcal{PI} = \mathcal{D}_2 \cap \mathcal{PI}$.*

Proof. The former lemma yields that 1. implies 2. Conversely, let 2. hold. Then $\mathcal{D}_i \cap \mathcal{PI} = (\mathcal{C}_i^{<\infty})^\perp \cap \mathcal{PI}$ by the preceding lemma, and ${}^\perp(\mathcal{D}_i \cap \mathcal{PI}) = \varinjlim \mathcal{C}_i^{<\infty}$ by [12, 2.1] and [12, 2.3] for $i = 1, 2$. But this means that:

$$\mathcal{C}_1^{<\infty} = (\varinjlim \mathcal{C}_1^{<\infty}) \cap R\text{-mod} = (\varinjlim \mathcal{C}_2^{<\infty}) \cap R\text{-mod} = \mathcal{C}_2^{<\infty}$$

$\square$

But when dealing with modules over artin algebras, pure-injectives are precisely direct summands of products of finitely presented modules. We will focus on this case in the rest of this subsection.

There is a following analogy of lemma 5.10 for cotorsion pairs:

Lemma 5.16. *Let R be an artin algebra and $\mathcal{A} \subseteq R$-mod, such that $\mathcal{A}$ is closed under extensions and contains $D(R)$. Denote $(\mathcal{C}, \mathcal{D})$ the cotorsion pair in R-Mod generated by $\mathcal{A}$. Then:*

1. *$DM \in \mathcal{C}$ if, and only if, $M \in (D\mathcal{A})^{\perp}$,*

2. *$DM \in \mathcal{D}$ if, and only if, $M \in \varinjlim D\mathcal{A}$.*

Proof. We have $\mathrm{Ext}^1_R(DM, \mathcal{A}) \cong \mathrm{Ext}^1_R(DM, D^2\mathcal{A}) = 0$ if, and only if, $\mathrm{Ext}^1_R(D\mathcal{A}, D^2M) = 0$ if, and only if, $\mathrm{Ext}^1_R(D\mathcal{A}, M) = 0$ by lemmas 5.2 and 5.4. This proves the first assertion.

If $M \in \varinjlim D\mathcal{A}$, then M is a pure epimorphic image of a direct sum $\bigoplus M_\nu$ of modules from $D\mathcal{A}$. So DM is a direct summand of $\prod DM_\nu$, hence $DM \in \mathcal{D}$. Conversely, let $DM \in \mathcal{D}$ and take a ${}^{\perp}((D\mathcal{A})^{\perp})$-precover $0 \to C \to E \to M \to 0$ of M. Then $DC \in \mathcal{C}$, hence $0 \to DM \to DE \to DC \to 0$ splits, hence $0 \to C \to E \to M \to 0$ is pure. Thus $M \in \varinjlim D\mathcal{A}$ by [12, sec. 2]. $\square$

Let us now concentrate only on cotorsion pairs with the first cotorsion class of the pair contained in $\mathcal{P}_1$.

Lemma 5.17. *Let R be an artin algebra and $\mathcal{A} \subseteq \mathcal{P}_1^{<\infty}$. Then $\mathcal{A}$ is a left cotorsion class of a cotorsion pair in R-mod if, and only if, $R \in \mathcal{A}$ and $\mathcal{A}$ is closed under extensions and direct summands.*

In particular, if $(\mathcal{C}, \mathcal{D})$ is a cotorsion pair in R-Mod, such that $\mathcal{C} \subseteq \mathcal{P}_1$, then $(\mathcal{C}^{<\infty}, \mathcal{D}^{<\infty})$ is a cotorsion pair in R-mod.

Proof. Every module from $\mathcal{B}' = \mathcal{A}^{\perp}$ purely embeds into a product of modules from $\mathcal{B} = \mathcal{B}' \cap R$-mod by lemma 5.11. Hence, $\mathcal{B}' \cap \mathcal{PI} = \mathrm{Prod}\mathcal{B}$, and ${}^{\perp}\mathcal{B} = {}^{\perp}(\mathcal{B}' \cap \mathcal{PI}) = \varinjlim \mathcal{A}$ by [12, sec. 2]. So, ${}^{\perp}\mathcal{B} \cap R$-mod$= \mathcal{A}$. This proves the if part of the first statement, while the only if part is clear.

The second statement is a direct consequence of lemma 5.14, since we have $(C^{<\infty})^{\perp} \cap R\text{-mod} = \mathcal{D}^{<\infty}$. $\square$

So similarly as for torsion pairs, for any cotorsion pair $(\mathcal{A}, \mathcal{B})$ in R-mod, such that $\mathcal{A} \subseteq \mathcal{P}_1$, there are two extremal extensions:

1. $(\mathcal{A}_\infty, \mathcal{B}_\infty)$, where $\mathcal{B}_\infty = \mathcal{A}^{\perp}$,

2. $(\hat{\mathcal{A}}, \hat{\mathcal{B}})$, where $\hat{\mathcal{A}} = {}^{\perp}\mathcal{B}$.

Proposition 5.18. *Let R be an artin algebra and $(\mathcal{A}, \mathcal{B})$ a cotorsion pair in R-mod, such that $\mathcal{A} \subseteq \mathcal{P}_1$.*

1. *Let $(\mathcal{A}_\infty, \mathcal{B}_\infty) = ({}^\perp(\mathcal{A}^\perp), \mathcal{A}^\perp)$ be a cotorsion pair in R-Mod cogenerated by $\mathcal{A}$. Then*
   (a) *$\mathcal{A}_\infty$ are precisely direct summands of $\mathcal{A}$-filtered modules,*
   (b) *$\mathcal{B}_\infty$ are precisely pure submodules of products of modules from $\mathcal{B}$,*
   (c) *$\mathcal{B}$ is definable.*
2. *Let $(\hat{\mathcal{A}}, \hat{\mathcal{B}}) = ({}^\perp\mathcal{B}, ({}^\perp\mathcal{B})^\perp)$ be a cotorsion pair in R-Mod generated by $\mathcal{B}$. Then*
   (a) *$\hat{\mathcal{A}} = \varinjlim \mathcal{A}$,*
   (b) *$\hat{\mathcal{A}}$ is definable.*
3. *Let $(\mathcal{C}, \mathcal{D})$ be any extension of $(\mathcal{A}, \mathcal{B})$ to R-Mod. Then $\mathcal{D} \cap \mathcal{PI} = \mathrm{Prod}\mathcal{B}$.*

Proof.

1. The part (a) is just a part of theorem 2.25, while (c) is a straightforward consequence of lemma2.21. For (b), one inclusion follows by lemma 5.11 and the other by (c).
2. We have $\hat{\mathcal{A}} \subseteq \varinjlim \mathcal{A}$ by [23, 4.4]. On the other hand, $\varinjlim \mathcal{A}$ is a left cotorsion class by [12, sec. 2]. Thus (a) holds. $\hat{\mathcal{A}}$ is closed under direct limits, since $\mathcal{B} \subseteq R\text{-mod} \subseteq \mathcal{PI}$. Next, for any $B \in \mathcal{B}$ and $M \in R$-Mod, we have $\mathrm{Ext}^1_R(M, B) \cong D(\mathrm{Hom}_R(TrDB, M)/\mathcal{P}(TrDB, M))$ by theorem 3.9, where $\mathcal{P}(TrDB, M)$ is a subgroup of those homorphisms from $TrDB$ to M that factor through a projective module. But for any family of homomorphisms $f_\nu : TrDB \to M_\nu$ and the corresponding product homomorphism $f : TrDB \to \prod M_\nu$, it is $f \in \mathcal{P}(TrDB, \prod M_\nu)$ if, and only if, $f_\nu \in \mathcal{P}(TrDB, M_\nu)$ for each ν, since projectives are closed under arbitrary products, R being artinian. Thus, $\hat{\mathcal{A}}$ is closed under direct products. Finally, let $0 \to K \to L \to M \to 0$ be a pure exact sequence with $L \in \hat{\mathcal{A}}$. Choosing some $B \in \mathcal{B}$ and applying a functor $\mathrm{Hom}_R(-, B)$ to this sequence, we obtain

   $$0 \to \mathrm{Ext}^1_R(M, B) \to \mathrm{Ext}^1_R(L, B) \to \mathrm{Ext}^1_R(K, B) \to \mathrm{Ext}^2_R(M, B)$$

   Hence $M \in \hat{\mathcal{A}}$ and $\mathrm{Ext}^2_R(M, B) = 0$, $\mathcal{B}$ being closed under (finitely presented) cosyzygies. Thus, $K \in \hat{\mathcal{A}}$, and this concludes the proof of (b).
3. On the one hand, $\mathcal{D} \cap \mathcal{PI} \subseteq \mathcal{B}_\infty \cap \mathcal{PI} = \mathrm{Prod}\mathcal{B}$, using the description of $\mathcal{B}_\infty$. The other inclusion is clear.

□

Corollary 5.19. *If R is an artin algebra, the third equivalent condition could be added to corollary 5.15:*

3. $\mathcal{D}_1 \cap R$-mod $= \mathcal{D}_2 \cap R$-mod.

5.5 Tilting classes

It is an open question in general, whether any 1-tilting module T is of a finite type, ie. if there is $\mathcal{S} \subseteq R$-mod, such that $T^{\perp} = \mathcal{S}^{\perp}$. This is proven for Prürer domains, but only conjectured for finite dimensional algebras, or more generally for artin algebras. We will show, however, that any 1-tilting class is not far away from one of a finite type for any coherent ring. Moreover, we will see that a finite type of the particular 1-tilting class could not be determined by looking only at its pure-injective modules, or at its finitely presented modules when R is an artin algebra.

Definition 5.20. We will define an equivalence for 1-tilting classes, such that $\mathcal{T} \sim_f \mathcal{U}$ whenever $\mathcal{T} \cap \mathcal{PI} = \mathcal{U} \cap \mathcal{PI}$. We will call equivalence classes of $\sim_f$ *bundles* of 1-tilting classes.

Similarly, there is an equivalence on 1-tilting cotorsion pairs, such that $({}^{\perp}\mathcal{T}, \mathcal{T}) \sim_f ({}^{\perp}\mathcal{U}, \mathcal{U})$ whenever $\mathcal{T} \sim_f \mathcal{U}$. A *bundle* of 1-tilting cotorsion pairs is then again an equivalence class of $\sim_f$.

Let us remind that there is actually only a set of 1-tilting classes for a given ring R, since every 1-tilting class is of a countable type.

Clearly, there is the least element with respect to an inclusion in any bundle $\mathfrak{B}$ of 1-tilting classes. This follows from the fact that an intersection of any family of 1-tilting classes is again a 1-tilting class (see theorem 2.27). We will now show that there is also the greatest element in $\mathfrak{B}$ when R is a coherent ring. But first, we need the following lemma:

Lemma 5.21. *Let R be a ring and $(\mathcal{S}, \mathcal{T})$ be a 1-tilting cotorsion pair. Then $\mathcal{T}$ is closed under pure-injective hulls. Moreover, $\mathcal{T}$ is closed under taking D^2 if R is an artin algebra over an algebraically closed field.*

Proof. The class $\mathcal{T}$ is closed under products and direct limits, hence under ultraproducts. It is well-known that a module M is elementarily equivalent to its pure-injective hull $PE(M)$, thus $PE(M)$ elementarily embeds into an ultrapower of M by a theorem of Frayne. Moreover, $PE(M)$ being pure-injective, $PE(M)$ is a direct summand of an ultrapower of M. Hence $PE(M) \in \mathcal{T}$ if $M \in \mathcal{T}$. The same proof applies for D^2 by lemma 5.8. □

Proposition 5.22. *Let R be a coherent ring and $(\mathcal{S}, \mathcal{T})$ be a 1-tilting cotorsion pair. Denote $\mathcal{U}$ the class of all pure submodules of modules from $\mathcal{T}$. Then $\mathcal{U} = (\mathcal{S}^{<\infty})^{\perp}$, thus $\mathcal{U}$ is the least 1-tilting class of a finite type containing $\mathcal{T}$. Moreover, $\mathcal{T} \cap \mathcal{PI} = \mathcal{U} \cap \mathcal{PI}$.*

Proof. The class $(\mathcal{S}^{<\infty})^{\perp}$ is 1-tilting of a finite type by corollary 2.28, and $\mathcal{T} \cap \mathcal{PI} = (\mathcal{S}^{<\infty})^{\perp} \cap \mathcal{PI}$ by lemma 5.14.

It remains to prove that $\mathcal{U} = (\mathcal{S}^{<\infty})^{\perp}$. Clearly, $\mathcal{U} \subseteq (\mathcal{S}^{<\infty})^{\perp}$. On the other hand, if $M \in (\mathcal{S}^{<\infty})^{\perp}$, then $PE(M) \in (\mathcal{S}^{<\infty})^{\perp} \cap \mathcal{PI} = \mathcal{T} \cap \mathcal{PI}$. In particular $PE(M) \in \mathcal{U}$, thus $M \in \mathcal{U}$. $\square$

Corollary 5.23. *If R is a coherent ring, then every bundle $\mathfrak{B}$ of 1-tilting classes contains exactly one 1-tilting class $\mathcal{U}$ of a finite type, and $\mathcal{U}$ is the greatest element in $\mathfrak{B}$.*

Proof. Cf. corollary 5.15. $\square$

Thus, the conjecture of any 1-tilting class being of a finite type for a subclass of coherent rings translates to the condition that every bundle has exactly one element. Or equivalently that the least and the greatest elements of any bundle are the same. The problem now is, how to determine that a tilting class is the least element of a bundle. There is a simple lemma taking care of the case when a tilting module with a decomposition into pure-injectives exists:

Lemma 5.24. *Let R be a coherent ring, $\mathfrak{B}$ be a bundle of 1-tilting classes and $\mathcal{T}$ be an element of $\mathfrak{B}$. Then $\mathcal{T}$ is the least element of $\mathfrak{B}$, whenever there is a tilting module T, such that $\mathcal{T} = T^{\perp}$ and T is a direct sum of pure-injective modules.*

Proof. Suppose that we have such a T, and there is a 1-tilting class $\mathcal{U}$ in the same bundle $\mathfrak{B}$, such that $\mathcal{U} \subseteq \mathcal{T}$. This implies that $T \in \mathrm{Add}(\mathcal{T} \cap \mathcal{PI}) = \mathrm{Add}(\mathcal{U} \cap \mathcal{PI})$. Thus, $T \in \mathcal{U}$, and also $T \in {}^{\perp}\mathcal{T} \subseteq {}^{\perp}\mathcal{U}$. But then $\mathcal{U} = T^{\perp}$ by [6, 2.10]. $\square$

The rest of this section is devoted to 1-tilting classes over artin algebras. In the dual case of cotilting classes, there is a bijective corespondence between 1-cotilting classes and torsion-free classes of finitely presented modules containing R for any left noetherian ring R [23, 3.10]. We will show that there is a similar statement for tilting classes over artin algebras, but only on the level of 1-tilting classes of a finite type, or equivalently on the level of bundles.

Proposition 5.25. *Let R be an artin algebra. There is a bijective correspondence between 1-tilting classes $\mathcal{T}$ of a finite type, and torsion classes $\mathcal{R}$ in R-mod containing $D(R)$. The correspondence is given by the mutually inverse assignments $\mathcal{T} \mapsto \mathcal{T} \cap R$-mod and $\mathcal{R} \mapsto \operatorname{cog}^* \mathcal{R}$, where $\operatorname{cog}^* \mathcal{R}$ is a class of pure submodules of products of modules from $\mathcal{R}$.*

Proof. If $\mathcal{T}$ is a 1-tilting class of a finite type, then clearly $\mathcal{T} \cap R$-mod is a torsion class containing $D(R)$.

On the other hand, let $\mathcal{R} \subseteq R$-mod be such a torsion class. Notice that ${}^\perp\mathcal{R} \subseteq \mathcal{P}_1$, since $\mathcal{R}$ contains all finitely generated injectives, thus all cosyzygies of simple modules. If we denote $\mathcal{A} = ({}^\perp\mathcal{R}) \cap R$-mod, then $(\mathcal{A}, ({}^\perp\mathcal{R})^\perp \cap R\text{-mod})$ is a cotorsion pair in R-mod by lemma 5.17. But, for M finitely presented, $M \in ({}^\perp\mathcal{R})^\perp$ if, and only if, $DM \in \varinjlim D\mathcal{R}$ if, and only if, $DM \in DR$ if, and only if, $M \in \mathcal{R}$ by lemma 5.16 and Lenzing's characterisation of direct limits of finitely presented modules [16]. So, $(\mathcal{A}, \mathcal{R})$ is a cotorsion pair in R-mod, and $\mathcal{A}^\perp = \operatorname{cog}^* \mathcal{R}$ by proposition 5.18. Thus, $\operatorname{cog}^* \mathcal{R}$ is a 1-tilting class of a finite type.

The equalities $\mathcal{R} = \operatorname{cog}^* \mathcal{R} \cap R$-mod and $\mathcal{T} = \operatorname{cog}^*(\mathcal{T} \cap R\text{-mod})$ follow by lemma 5.17 and proposition 5.18 again. ☐

The last proposition yields that in a special case the duality works diretly on 1-tilting modules too.

Proposition 5.26. *Let R be an artin algebra over an algebraically closed field, and T a 1-tilting module. Then DT is a 1-cotilting module (of a cofinite type).*

Proof. We have $\operatorname{id} DT \leq 1$ by lemma 5.5, and a $\operatorname{Prod}DT$-coresolution of $D(R)$ could be obtained just by dualizing an $\operatorname{Add}T$-resolution of R. It only remains to show that $\operatorname{Ext}^1_R(DT^\kappa, DT) = 0$ for any cardinal κ. But $\operatorname{Ext}^1_R(DT^\kappa, DT) \cong \operatorname{Ext}^1_R(D(T^{(\kappa)}), DT) = 0$ if, and only if, $\operatorname{Ext}^1_R(T, D^2(T^{(\kappa)})) = 0$ by lemma 5.2. But the last condition is satisfied by lemma 5.21. Finally, every 1-cotilting module over any artinian ring is of a cofinite type, [23, 4.11]. ☐

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