GENERALIZED HILL LEMMA, KAPLANSKY THEOREM FOR COTORSION PAIRS, AND SOME APPLICATIONS

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Dedicated to Luigi Salce on his 60th birthday

Abstract. We generalize Hill's lemma in order to obtain a large family of $\mathcal{C}$-filtered submodules from a single $\mathcal{C}$-filtration of a module. We use this to prove the following generalization of Kaplansky's structure theorem for projective modules: for any ring R, a cotorsion pair $(\mathcal{A}, \mathcal{B})$ in Mod-R is of countable type if and only if every module $M \in \mathcal{A}$ is $\mathcal{A}^{\leq \omega}$-filtered. We also prove rank versions of these results for torsion-free modules over commutative domains.

As an application, we solve a problem of Bazzoni and Salce [3] by showing that strongly flat modules over any valuation domain coincide with the extensions of free modules by divisible torsion-free modules. Another application yields a short proof of the structure of Matlis localizations of commutative rings.

Introduction

In [11], Hill invented an ingenious method of constructing a large family of subgroups from a single infinite continuous chain of abelian p-groups. Later on, Fuchs and Lee extended the method to the general setting of arbitrary modules over arbitrary rings (including a rank version for torsion-free modules over commutative domains), [9, XVI.§8], [7]. Similar constructions were used in connection with Shelah's Singular Compactness Theorem in [4] and [5].

More recently, Šaroch and the second author [14] noticed an extra property of the Hill method (see property (H3) below). In Theorems 6 and 7 of Section 1, we discover an additional feature: the family is always a complete sublattice of the submodule lattice.

Hill's method provides a powerful tool for extending structure theory of various classes of modules from the countable (rank) case to the arbitrary one. It is applied either directly or in conjunction with the Shelah's Singular Compactness Theorem, see e.g. [9, XVI.§8], [7], [14].

Here, we first apply Hill's method to extend a theorem of Kaplansky on projective modules to the setting of cotorsion pairs. Kaplansky's theorem says that any projective module is a direct sum of countably generated modules. Considering the cotorsion pair (Proj-R, Mod-R) cogenerated by R, we can rephrase the theorem by saying that each module $M \in$ Proj-R is (Proj-R)$^{\leq \omega}$-filtered (see below for unexplained terminology).

Theorem 10 in Section 2 shows that the same holds for an arbitrary cotorsion pair $(\mathcal{A}, \mathcal{B})$ in Mod-R cogenerated by a set of $< \kappa$ presented modules (where κ is a regular uncountable cardinal): each module $M \in \mathcal{A}$ is $\mathcal{A}^{<\kappa}$-filtered. We also prove a rank version of this result in Lemma 16.

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Section 3 deals with applications to strongly flat modules over valuation domains. Bazzoni and Salce [3] proved that any countable rank strongly flat module M has the following property: M contains a free submodule F such that M/F is torsion-free divisible. Theorem 17 shows that the property characterizes abitrary strongly flat modules. This answers in the positive a question raised in [3]. (The property was known to hold by [3, Theorem 3.15] for any valuation domain, but restricted to strongly flat modules M of rank $\leq \aleph_1$, and by [10, Theorem 3.3], for all strongly flat modules M, but restricted to Matlis valuation domains).

The application in Section 4 yields a short proof of the fact that the localization, $Q = RS^{-1}$, of a commutative ring R in a set S of regular elements is a Matlis localization if and only if Q/R decomposes (as R-module) into a direct sum of countably presented modules. This result, first proved in [1], extends Lee's characterization of Matlis domains [12] as well as its generalization to localizations of commutative domains by Fuchs and Salce [8].

Let R be a (unital associative) ring. Denote by Mod-R the category of all (right R-) modules, and by Proj-R the full subcategory of all projective modules.

A *filtration* is a continuous well-ordered chain of modules $(M_\alpha \mid \alpha \leq \sigma)$ with $M_0 = 0$. A filtration is called a $\mathcal{C}$*-filtration* for a class of modules $\mathcal{C}$ if in addition $M_{\alpha+1}/M_\alpha$ is isomorphic to an element of $\mathcal{C}$ for each $\alpha < \sigma$. A module M is $\mathcal{C}$*-filtered* if there is a $\mathcal{C}$-filtration $(M_\alpha \mid \alpha \leq \sigma)$ such that $M = M_\sigma$.

For an infinite cardinal κ and a class of modules $\mathcal{A}$, denote by $\mathcal{A}^{<\kappa}$ and $\mathcal{A}^{\leq\kappa}$ the subclass of all $<\kappa$-presented, and $\leq\kappa$-presented, respectively, modules from $\mathcal{A}$.

A pair of classes of modules $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ is a *cotorsion pair* provided that $\mathcal{A}$ and $\mathcal{B}$ are orthogonal with respect to the Ext^1-functor, and they are maximal with this property, that is, $\mathcal{A} = \{A \in \text{Mod-}R \mid \mathrm{Ext}^1_R(A, B) = 0 \ \forall B \in \mathcal{B}\}$ and $\mathcal{B} = \{B \in \text{Mod-}R \mid \mathrm{Ext}^1_R(A, B) = 0 \ \forall A \in \mathcal{A}\}$. Cotorsion pairs were introduced by Salce in his pioneering work [13].

A cotorsion pair $\mathfrak{C}$ is *cogenerated* by a class of modules $\mathcal{C}$ provided that $\mathcal{B} = \{B \in \text{Mod-}R \mid \mathrm{Ext}^1_R(C, B) = 0 \ \forall C \in \mathcal{C}\}$. Moreover, $\mathfrak{C}$ is of *countable type* if $\mathfrak{C}$ is cogenerated by a set of countably presented modules.

1. Generalized Hill Lemma

We start by recalling Hill's notion of a closed subset with respect to a filtration.

Definition 1. Let $\mathcal{M}$ be a filtration $(M_\alpha \mid \alpha \leq \sigma)$ together with a family of modules $(A_\alpha \mid \alpha < \sigma)$ such that $M_{\alpha+1} = M_\alpha + A_\alpha$ for each $\alpha < \sigma$. A subset S of σ is *closed* if every $\beta \in S$ satisfies

$$M_\beta \cap A_\beta \subseteq \sum_{\alpha \in S, \alpha < \beta} A_\alpha.$$

The *height*, $\mathrm{ht}(x)$, of an element $x \in M_\sigma$ is defined as the least ordinal $\alpha < \sigma$ such that $x \in M_{\alpha+1}$. For any subset S of σ, we define

$$M(S) = \sum_{\alpha \in S} A_\alpha.$$

For each ordinal $\alpha \leq \sigma$, we have $M_\alpha = \sum_{\beta<\alpha} A_\beta$, so α $(= \{\beta < \sigma \mid \beta < \alpha\})$ is a closed subset of σ. The following lemma is inspired by the proof of [14, Lemma 1.4]:

Lemma 2. *Let $\mathcal{M}$ be as in Definition 1, S be a closed subset of σ, and $x \in M(S)$. Let $S' = \{\alpha \in S \mid \alpha \leq \mathrm{ht}(x)\}$. Then $x \in M(S')$.*

Proof. Let $x \in M(S)$. Then $x = x_1 + \cdots + x_k$ where $x_i \in A_{\alpha_i}$ for some $\alpha_i \in S$, $1 \leq i \leq k$. W.l.o.g., $\alpha_1 < \cdots < \alpha_k$, and α_k is minimal possible.

If $\alpha_k > \mathrm{ht}(x)$, then $x_k = x - x_1 - \cdots - x_{k-1} \in M_{\alpha_k} \cap A_{\alpha_k} \subseteq \sum_{\alpha \in S, \alpha < \alpha_k} A_\alpha$ since S is closed, in contradiction with the minimality of α_k. □

As an immediate consequence, we get

Corollary 3. *Let $\mathcal{M}$ be as in Definition 1, S be a closed subset of σ, and $x \in M(S)$. Then $\mathrm{ht}(x) \in S$.*

An important implication is the following lemma.

Lemma 4. *Let $\mathcal{M}$ be as in Definition 1, and let S_i, $i \in I$, be a family of closed subsets of σ. Then*

$$M(\bigcap_{i \in I} S_i) = \bigcap_{i \in I} M(S_i)$$

Proof. Let $T = \bigcap_{i \in I} S_i$. Clearly, $M(T) \subseteq \bigcap_{i \in I} M(S_i)$. Suppose there is an $x \in \bigcap_{i \in I} M(S_i)$ such that $x \notin M(T)$, and choose such an x of minimal height. Then $x = y + z$ for some $y \in A_{\mathrm{ht}(x)}$ and $z \in M_{\mathrm{ht}(x)}$. By Corollary 3, $\mathrm{ht}(x) \in S_i$ for all $i \in I$, so $\mathrm{ht}(x) \in T$, and $y \in M(T)$. Then $z \in \bigcap_{i \in I} M(S_i)$, $z \notin M(T)$ and $\mathrm{ht}(z) < \mathrm{ht}(x)$, in contradiction with the minimality. □

Now, we can prove the additional property of closed subsets mentioned in the Introduction:

Proposition 5. *Let $\mathcal{M}$ be as in Definition 1, and let S_i, $i \in I$, be a family of closed subsets of σ. Then both the union and the intersection of this family are again closed in σ. That is, closed subsets of σ form a complete sublattice of 2^σ.*

Proof. As for the union, if $\beta \in S = \bigcup_{i \in I} S_i$, then $\beta \in S_i$ for some $i \in I$, and $M_\beta \cap A_\beta \subseteq \sum_{\alpha \in S_i, \alpha < \beta} A_\alpha \subseteq \sum_{\alpha \in S, \alpha < \beta} A_\alpha$.

For the intersection, let $\beta \in T = \bigcap_{i \in I} S_i$. Then $M_\beta \cap A_\beta \subseteq M(S_i \cap \beta)$ for each $i \in I$. Therefore, Lemma 4 implies that

$$M_\beta \cap A_\beta \subseteq \bigcap_{i \in I} M(S_i \cap \beta) = M(T \cap \beta)$$

which exactly says that T is closed. □

The following is the main result of this section:

Theorem 6. *(Generalized Hill Lemma) Let R be a ring, κ an infinite regular cardinal and $\mathcal{C}$ a set of $< \kappa$-presented modules. Let M be a union of a $\mathcal{C}$-filtration*

$$0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_\alpha \subseteq \cdots \subseteq M_\sigma = M$$

for some ordinal σ. Then there is a family $\mathcal{F}$ of submodules of M such that:

- (H1) *$M_\alpha \in \mathcal{F}$ for all $\alpha \leq \sigma$.*
- (H2) *$\mathcal{F}$ is closed under arbitrary sums and intersections (that is, $\mathcal{F}$ is a complete sublattice of the lattice of submodules of M).*
- (H3) *Let $N, P \in \mathcal{F}$ such that $N \subseteq P$. Then there exists a $\mathcal{C}$-filtration $(\bar{P}_\gamma \mid \gamma \leq \tau)$ of the module $\bar{P} = P/N$ such that $\tau \leq \sigma$, and for each $\gamma < \tau$ there is an $\beta < \sigma$ with $\bar{P}_{\gamma+1}/\bar{P}_\gamma$ isomorphic to $M_{\beta+1}/M_\beta$, and $\bar{P}_{\gamma+1} = \bar{P}_\gamma + (A_\beta + N)/N$.*
- (H4) *Let $N \in \mathcal{F}$ and X be a subset of M of cardinality $< \kappa$. Then there is a $P \in \mathcal{F}$ such that $N \cup X \subseteq P$ and P/N is $< \kappa$-presented.*

Proof. Let $\mathcal{M}$ denote the filtration $(M_\alpha \mid \alpha \leq \sigma)$ together with an arbitrary family of $< \kappa$-generated modules $(A_\alpha \mid \alpha < \sigma)$ such that for each $\alpha < \sigma$:

$$M_{\alpha+1} = M_\alpha + A_\alpha,$$

as in Definition 1. We claim that

$$\mathcal{F} = \{M(S) \mid S \text{ a closed subset of } \sigma\}$$

is the desired family $\mathcal{F}$.

Property (H1) is clear, since each ordinal $\alpha \leq \sigma$ is a closed subset of σ. Property (H2) follows by Proposition 5 and Lemma 4.

Property (H3) is proved as in [14]: we have $N = M(S)$ and $P = M(T)$ for some closed subsets S, T. Since $S \cup T$ is closed, we can assume that $S \subseteq T$. For each $\beta \leq \sigma$, put

$$F_\beta = N + \sum_{\alpha \in T \setminus S, \alpha < \beta} A_\alpha = M(S \cup (T \cap \beta)) \qquad \text{and} \qquad \bar{F}_\beta = F_\beta / N.$$

Clearly, $(\bar{F}_\beta \mid \beta \leq \sigma)$ is a filtration of $\bar{P} = P/N$ such that $\bar{F}_{\beta+1} = \bar{F}_\beta + (A_\beta + N)/N$ for $\beta \in T \setminus S$ and $\bar{F}_{\beta+1} = \bar{F}_\beta$ otherwise. Let $\beta \in T \setminus S$. Then

$$\bar{F}_{\beta+1}/\bar{F}_\beta \cong F_{\beta+1}/F_\beta \cong A_\beta/(F_\beta \cap A_\beta),$$

and

$$F_\beta \cap A_\beta \supseteq (\sum_{\alpha \in T, \alpha < \beta} A_\alpha) \cap A_\beta = M_\beta \cap A_\beta.$$

On the other hand, if $x \in F_\beta \cap A_\beta$ then $\mathrm{ht}(x) \leq \beta$, so $x \in M(T')$ by Lemma 2, where $T' = \{\alpha \in S \cup (T \cap \beta) \mid \alpha \leq \beta\}$. By Proposition 5, we get $x \in M_\beta$ because $\beta \notin S$. Hence $F_\beta \cap A_\beta = M_\beta \cap A_\beta$ and $\bar{F}_{\beta+1}/\bar{F}_\beta \cong A_\beta/(M_\beta \cap A_\beta) \cong M_{\beta+1}/M_\beta$. The filtration $(\bar{P}_\gamma \mid \gamma \leq \tau)$ is obtained from $(\bar{F}_\beta \mid \beta \leq \sigma)$ by removing possible repetitions and (H3) follows. Denote by τ' the ordinal type of the well-ordered set $(T \setminus S, <)$. Notice that the length τ of the filtration can be taken as $1 + \tau'$ (as the ordinal sum, hence $\tau = \tau'$ for τ' infinite).

For property (H4), we first prove that every subset of σ of cardinality $< \kappa$ is contained in a closed subset of cardinality $< \kappa$. Because κ is an infinite regular cardinal, by Proposition 5, it is enough to prove this only for one-element subsets of σ. That is, to prove that every $\beta < \sigma$ is contained in a closed subset of cardinality $< \kappa$. We induct on β. For $\beta < \kappa$, just take $S = \beta + 1$. Otherwise, the short exact sequence

$$0 \to M_\beta \cap A_\beta \to A_\beta \to M_{\beta+1}/M_\beta \to 0$$

shows that $M_\beta \cap A_\beta$ is $< \kappa$ generated. Thus, $M_\beta \cap A_\beta \subseteq \sum_{\alpha \in S_0} A_\alpha$ for a subset $S_0 \subseteq \beta$ of cardinality $< \kappa$. Moreover, we can assume that S_0 is closed in σ by inductive premise, and put $S = S_0 \cup \{\beta\}$. To show that S is closed, it suffices to check the definition for β. But $M_\beta \cap A_\beta \subseteq M(S_0) = \sum_{\alpha \in S, \alpha < \beta} A_\alpha$.

Finally, let $N = M(S)$ where S closed in σ, and let X be a subset of M of cardinality $< \kappa$. Then $X \subseteq \sum_{\alpha \in T} A_\alpha$ for a subset T of σ of cardinality $< \kappa$. By the preceding paragraph, we can assume that T is closed in σ. Let $P = M(S \cup T)$. Then P/N is $\mathcal{C}$-filtered by property (H3), and the filtration can be chosen indexed by 1+ the ordinal type of $T \setminus S$, which is less than κ. In particular, P/N is $< \kappa$-presented. □

Remark. (cf. [7, Remark 2.2]) The proof of property (H3) for the family $\mathcal{F}$ in Theorem 6 has the following additional property: if for $\beta \in T \setminus S$, A_β can be chosen as a complement to M_β in $M_{\beta+1}$, then $(A_\beta + N)/N$ will be a complement of $\bar{P}_\gamma$ in $\bar{P}_{\gamma+1}$ in the filtration of $\bar{P}$. This follows from the fact that in this case (in the proof of (H3)) $F_\beta \cap A_\beta = M_\beta \cap A_\beta = 0$, so $F_\beta/N \cap (A_\beta + N)/N = \bar{0}$.

We will also need a rank version of the Generalized Hill Lemma for torsion-free modules over commutative domains.

Let R be a commutative domain and M a torsion-free module. We define the *rank*, $\operatorname{rk} X$, of a subset $X \subseteq M$ as the torsion-free rank of the submodule $\langle X\rangle$ of M generated by X. Note that $\operatorname{rk} X \leq \operatorname{card}(X)$.

Theorem 7. *(Rank version of the Generalized Hill lemma) Let R be a commutative domain, κ an infinite regular cardinal and $\mathcal{C}$ a set of torsion-free R-modules. Let M be a union of a $\mathcal{C}$-filtration*

$$0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_\alpha \subseteq \cdots \subseteq M_\sigma = M$$

for some ordinal σ. Assume moreover that for each $\alpha < \sigma$ there is a submodule A_α of M of rank $< \kappa$ such that $M_{\alpha+1} = M_\alpha + A_\alpha$. Then there is a family $\mathcal{F}$ of submodules of M such that the properties (H1), (H2), and (H3) from Theorem 6 hold true. Moreover, the following rank version of property (H4) holds:

(H4*) *Let $N \in \mathcal{F}$ and X be a subset of M with $\operatorname{rk} X < \kappa$. Then there are $P \in \mathcal{F}$ and a submodule $A \subseteq M$ of rank $< \kappa$ such that $N \cup X \subseteq P$ and $P = N + A$.*

Proof. Denote $\mathcal{M}$ the filtration $(M_\alpha \mid \alpha \leq \sigma)$ together with the family $(A_\alpha \mid \alpha < \sigma)$ as in Definition 1. Put

$$\mathcal{F} = \{M(S) \mid S \text{ a closed subset of } \sigma\}.$$

The properties (H1), (H2) and (H3) are proved exactly as in Theorem 6. For (H4*), consider $N \in \mathcal{F}$ and $X \subseteq M$ with $\operatorname{rk} X < \kappa$. Note first that we can w.l.o.g. assume that the cardinality of X is $< \kappa$. To see this, take a maximal R-independent subset B of $\langle X\rangle$. Then B has cardinality $< \kappa$ and $\langle B\rangle$ is an essential submodule of $\langle X\rangle$. Then, for a module $P \in \mathcal{F}$ containing B, the inclusion $\langle X\rangle \hookrightarrow M$ induces a map $f : \langle X\rangle/\langle B\rangle \to M/P$. Then $f = 0$ since $\langle X\rangle/\langle B\rangle$ is torsion, but M/P is torsion-free by property (H3). Hence also $X \subseteq P$.

Now, we continue as in the proof of property (H4) in Theorem 6. We prove that every subset of σ of cardinality $< \kappa$ is contained in a closed subset of cardinality $< \kappa$. It is again enough to prove that every $\beta < \sigma$ is contained in a closed subset T of cardinality $< \kappa$. We induct on β. For $\beta < \kappa$, we take $T = \beta + 1$. Otherwise, $A_\beta \cap M_\beta$ has rank $< \kappa$, so we can find by inductive premise a closed subset $T' \subseteq \beta$ of cardinality $< \kappa$ such that $M_\beta \cap A_\beta \subseteq \mathcal{M}(T')$. Then it suffices to take $T = T' \cup \{\beta\}$.

Finally, if $N = M(S)$ and $X \subseteq M(T)$ where S, T are closed and T is of cardinality $< \kappa$, we put $A = M(T \setminus S)$ and $P = N + A$. Clearly, $P = M(S \cup T)$ and A satisfy the claim of (H4*). □

Remark 8. Notice the following difference between the assumptions of the two versions of the Generalized Hill Lemma. The assumption of $\mathcal{C}$ consisting of $< \kappa$-presented modules in Theorem 6 already guarantees existence of a family of $< \kappa$-generated modules $\mathcal{A} = (A_\alpha \mid \alpha < \sigma)$ such that $M_{\alpha+1} = M_\alpha + A_\alpha$ for each $\alpha < \sigma$ (in fact, in the proof of Theorem 6, and in its applications, the particular choice of $\mathcal{A}$ does not really matter).

On the other hand, if we just assume that $M_{\alpha+1}/M_\alpha$ has rank $< \kappa$ for each $\alpha < \sigma$ in Theorem 7, there need not exist any family of modules $\mathcal{A} = (A_\alpha \mid \alpha < \sigma)$ such that $M_{\alpha+1} = M_\alpha + A_\alpha$ and A_α has rank $< \kappa$ for each $\alpha < \sigma$.

Indeed, assume that $\kappa > \aleph_0$ and the minimal number of R-generators of Q is $\lambda \geq \kappa$. So there is an exact sequence $0 \to K \subseteq F \to Q \to 0$ where F is free of rank λ. Since K is torsion-free, there is a filtration $(M_\alpha \mid \alpha \leq \sigma)$ of K such that $M_{\alpha+1}/M_\alpha$ is torsion-free of rank 1 for each $\alpha < \sigma$. Define $M_{\sigma+1} = F$.

Assume that $A_\sigma \subseteq F$ has rank $< \lambda$. Then A_σ is contained in a free direct summand G of F of rank $< \lambda$, so $(A_\sigma + K)/K \subseteq (G + K)/K \subsetneqq F/K$ because $Q \cong F/K$ is not $< \lambda$-generated. So certainly there is no A_σ of rank $< \kappa$ such that $M_{\sigma+1} = M_\sigma + A_\sigma$.

2. Kaplansky Theorem for Cotorsion Pairs

Let R be a ring and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ a cotorsion pair in Mod-R cogenerated by a set $\mathcal{C}$ containing R. Then $\mathcal{A}$ coincides with the class of all direct summands of $\mathcal{C}$-filtered modules (cf. [16, Theorem 2.2]). Our goal is to remove the term 'direct summands' in this characterization of $\mathcal{A}$ on the account of replacing the set $\mathcal{C}$ by a suitable small subset of $\mathcal{A}$.

The following application of Theorem 6 is crucial:

Lemma 9. *Let κ be an uncountable regular cardinal and $\mathcal{C}$ a set of $< \kappa$-presented modules. Denote by $\mathcal{A}$ the class of all direct summands of $\mathcal{C}$-filtered modules. Then every module in $\mathcal{A}$ is $\mathcal{A}^{<\kappa}$-filtered.*

Proof. Let $K \in \mathcal{A}$, so there is a $\mathcal{C}$-filtered module M such that $M = K \oplus L$ for some $L \subseteq M$. Denote by $\pi_K : M \to K$ and $\pi_L : M \to L$ the corresponding projections. Let $\mathcal{F}$ be the family of submodules of M as in Theorem 6. We proceed in two steps:

Step I: By induction, we construct a filtration $(N_\alpha \mid \alpha \leq \tau)$ of M such that

(1) $N_\alpha \in \mathcal{F}$,
(2) $N_\alpha = \pi_K(N_\alpha) + \pi_L(N_\alpha)$, and
(3) $N_{\alpha+1}/N_\alpha$ is $< \kappa$-presented

for all $\alpha < \tau$.

First, $N_0 = 0$, and $N_\beta = \bigcup_{\alpha<\beta} N_\alpha$ for all limit ordinals $\beta \leq \tau$. Suppose we have $N_\alpha \subsetneqq M$ and we wish to construct $N_{\alpha+1}$. Take $x \in M \setminus N_\alpha$; by property (H4), there is $Q_0 \in \mathcal{F}$ such that $N_\alpha \cup \{x\} \subseteq Q_0$ and Q_0/N_α is $< \kappa$-presented. Let X_0 be a subset of Q_0 of cardinality $< \kappa$ such that the set $\{x + N_\alpha \mid x \in X_0\}$ generates Q_0/N_α. Put $Z_0 = \pi_K(Q_0) \oplus \pi_L(Q_0)$. Clearly $Q_0/N_\alpha \subseteq Z_0/N_\alpha$. Since $\pi_K(N_\alpha), \pi_L(N_\alpha) \subseteq N_\alpha$, the module Z_0/N_α is generated by the set

$$\{x + N_\alpha \mid x \in \pi_K(X_0) \cup \pi_L(X_0)\}.$$

Thus, we can find $Q_1 \in \mathcal{F}$ such that $Z_0 \subseteq Q_1$ and Q_1/N_α is $< \kappa$-presented. Similarly, we infer that Z_1/N_α is $< \kappa$-generated for $Z_1 = \pi_K(Q_1) \oplus \pi_L(Q_1)$, and find $Q_2 \in \mathcal{F}$ with $Z_1 \subseteq Q_2$ and Q_2/N_α a $< \kappa$-presented module. In this way, we obtain a chain $Q_0 \subseteq Q_1 \subseteq \dots$ such that for all $i < \omega$: $Q_i \in \mathcal{F}$, Q_i/N_α is $< \kappa$-presented, and $\pi_K(Q_i) + \pi_L(Q_i) \subseteq Q_{i+1}$. It is easy to see that $N_{\alpha+1} = \bigcup_{i<\omega} Q_i$ satisfies the properties (1)–(3).

Step II: By condition (2), we have

$$\pi_K(N_{\alpha+1}) + N_\alpha = \pi_K(N_{\alpha+1}) \oplus \pi_L(N_\alpha)$$

and similarly for L. Hence

$$(\pi_K(N_{\alpha+1})+N_\alpha)\cap(\pi_L(N_{\alpha+1})+N_\alpha) = (\pi_K(N_{\alpha+1})\oplus\pi_L(N_\alpha))\cap(\pi_L(N_{\alpha+1})\oplus\pi_K(N_\alpha))$$
$$= \big(\pi_K(N_{\alpha+1}) \cap (\pi_L(N_{\alpha+1}) \oplus \pi_K(N_\alpha))\big) \oplus \pi_L(N_\alpha) = \pi_K(N_\alpha) \oplus \pi_L(N_\alpha) = N_\alpha$$

and

$$N_{\alpha+1}/N_\alpha = (\pi_K(N_{\alpha+1}) + N_\alpha)/N_\alpha \oplus (\pi_L(N_{\alpha+1}) + N_\alpha)/N_\alpha.$$

By condition (1), $N_{\alpha+1}/N_\alpha$ is $\mathcal{C}$-filtered. Since

$$(\pi_K(N_{\alpha+1}) + N_\alpha)/N_\alpha \cong \pi_K(N_{\alpha+1})/\pi_K(N_\alpha),$$

$\pi_K(N_{\alpha+1})/\pi_K(N_\alpha)$ is isomorphic to a direct summand of a $\mathcal{C}$-filtered module, so $\pi_K(N_{\alpha+1})/\pi_K(N_\alpha) \in \mathcal{A}$. By condition (3), $\pi_K(N_{\alpha+1})/\pi_K(N_\alpha)$ is $< \kappa$-presented. We conclude that $(\pi_K(N_{\alpha+1}) \mid \alpha \leq \tau)$ is the desired $\mathcal{A}^{<\kappa}$-filtration of $K = \pi_K(N_\tau)$. □

Now, we can easily prove the main result of this section:

Theorem 10. *Let R be a ring, κ an uncountable regular cardinal, and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ a cotorsion pair of R-modules. Then the following statements are equivalent:*

(1) *$\mathfrak{C}$ is cogenerated by a class of $< \kappa$-presented modules.*
(2) *Every module in $\mathcal{A}$ is $\mathcal{A}^{<\kappa}$-filtered.*

Proof. (1) $\implies$ (2). Let $\mathcal{C}$ be a class of $< \kappa$-presented modules cogenerating $\mathfrak{C}$. W.l.o.g., $\mathcal{C}$ is a set, and $R \in \mathcal{C}$. Then, by [16, Theorem 2.2], $\mathcal{A}$ consists of all direct summands of $\mathcal{C}$-filtered modules. So statement (2) follows by Lemma 9.

(2) $\implies$ (1). It is well-known that every $\mathcal{A}$-filtered module is again in $\mathcal{A}$ (see e.g. [6, Lemma 1]). Thus, (2) implies that $\mathfrak{C}$ is cogenerated by the class $\mathcal{A}^{<\kappa}$. $\square$

In particular, for $\kappa = \aleph_1$, we get

Corollary 11. *Let R be a ring. A cotorsion pair $(\mathcal{A}, \mathcal{B})$ is of countable type if and only if every module $M \in \mathcal{A}$ is $\mathcal{A}^{\leq\omega}$-filtered.*

As another immediate corollary, for the cotorsion pair (Proj-R, Mod-R) cogenerated by R, we obtain the Kaplansky theorem on the structure of projective modules:

Corollary 12. *Every projective module over an arbitrary ring is a direct sum of countably generated projective modules.*

Remark. In general, it is not possible to extend the results in this section to $\kappa = \aleph_0$, since there are rings which admit countably generated projective modules that are not direct sums of finitely generated projective modules.

3. Strongly flat modules

In this section, R is a commutative domain with the quotient field Q. We denote by $(\mathcal{SF}, \mathcal{MC})$ the cotorsion pair in Mod-R cogenerated by Q. The modules in $\mathcal{SF}$ are called *strongly flat.* They are flat (since Q is flat), hence torsion-free.

A (torsion-free) module M is called *free-by-divisible* provided there exist cardinals κ, λ and an exact sequence $0 \to R^{(\kappa)} \to M \to Q^{(\lambda)} \to 0$. In [16, Proposition 2.8], strongly flat modules were characterized as the direct summands of free-by-divisible modules. Our goal is to remove the term 'direct summand' in this characterization in the case when R is a valuation domain.

First, we need a characterization of free-by-divisible modules:

Lemma 13. *Let R be a domain and M a module. Then M is free-by-divisible if and only if M is $\{R, Q\}$-filtered.*

Proof. The only if-part is clear. For the if-part, let $(M_\alpha \mid \alpha \leq \sigma)$ be an $\{R, Q\}$-filtration of M.

By induction on $\alpha \leq \sigma$, we define ordinals μ_α and ν_α, and a well-ordered direct system of exact sequences $0 \to R^{(\mu_\alpha)} \overset{i_\alpha}{\to} M_\alpha \overset{\pi_\alpha}{\to} Q^{(\nu_\alpha)} \to 0$ and embeddings $(f_\alpha, g_\alpha, h_\alpha)$ $(\alpha \leq \sigma)$, as follows. First, $\mu_0 = \nu_0 = 0$.

If $M_{\alpha+1}/M_\alpha \cong R$ then $M_{\alpha+1} = M_\alpha \oplus x_\alpha R$ where $Ann_R(x_\alpha) = 0$, and we take $\mu_{\alpha+1} = \mu_\alpha + 1$, $\nu_{\alpha+1} = \nu_\alpha$, let $f_\alpha : R^{(\mu_\alpha)} \hookrightarrow R^{(\mu_{\alpha+1})}$ and $g_\alpha : M_\alpha \hookrightarrow M_{\alpha+1}$ be the inclusions, $i_{\alpha+1}$ be the extension of i_α mapping the extra free generator to x_α, and put $h_\alpha = \mathrm{id}$.

If $M_{\alpha+1}/M_\alpha \cong Q$, we consider the pushout of the embedding $g_\alpha : M_\alpha \hookrightarrow M_{\alpha+1}$ and of π_α:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & R^{(\mu_\alpha)} & \xrightarrow{i_\alpha} & M_\alpha & \xrightarrow{\pi_\alpha} & Q^{(\nu_\alpha)} & \to & 0 \\
 & & \| f_\alpha & & \downarrow g_\alpha & & \downarrow h_\alpha & & \\
0 & \to & R^{(\mu_\alpha)} & \xrightarrow{i_{\alpha+1}} & M_{\alpha+1} & \xrightarrow{\pi_{\alpha+1}} & X & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & Q & = & Q & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Since $\operatorname{Ext}^1_R(Q, Q^{(\nu_\alpha)}) = 0$, the right hand column splits, so w.l.o.g. $X = Q^{(\nu_\alpha+1)}$, and we take $\mu_{\alpha+1} = \mu_\alpha$, $\nu_{\alpha+1} = \nu_\alpha + 1$.

If α is a limit ordinal, we take the direct limit of the direct system of exact sequences $0 \to R^{(\mu_\beta)} \overset{i_\beta}{\to} M_\beta \to Q^{(\nu_\beta)} \to 0$ with the embeddings $(f_\beta, g_\beta, h_\beta)$ $(\beta < \alpha)$, so $\mu_\alpha = \sup_{\beta<\alpha} \mu_\beta$ and $\nu_\alpha = \sup_{\beta<\alpha} \nu_\beta$.

Finally, the sequence $0 \to R^{(\mu_\sigma)} \overset{i_\sigma}{\to} M_\sigma \overset{\pi_\sigma}{\to} Q^{(\nu_\sigma)} \to 0$ shows that $M = M_\sigma$ is free-by-divisible. □

Lemma 13 does not guarantee validity of the assumptions of the rank version of the Generalized Hill Lemma (see Remark 8). However, in our particular setting, we have:

Lemma 14. *Let R be a valuation domain and P be a free-by-divisible module. Then there are an $\{R, Q\}$-filtration $\mathcal{P} = (P_\alpha \mid \alpha \le \sigma)$ of P, and a sequence of submodules $(A_\alpha \mid \alpha < \sigma)$ of P, such that A_α has countable rank and $P_{\alpha+1} = P_\alpha + A_\alpha$ for each $\alpha < \sigma$.*

Proof. We will prove the lemma in three steps:

Step I: By assumption, there is an exact sequence $0 \to R^{(\kappa)} \overset{\subseteq}{\to} P \to Q^{(\lambda)} \to 0$ for some cardinals κ and λ. We put $\sigma = \kappa + \lambda$ (the ordinal sum). By induction on α, we will construct the sequence $(A_\alpha \mid \alpha < \sigma)$ together with the filtration $\mathcal{P}$ – the latter simply by taking $P_\alpha = \sum_{\beta<\alpha} A_\beta$. This is easy in case $\kappa = 0$ or $\lambda = 0$, so we will assume that $\kappa > 0$ and $\lambda > 0$.

For $\alpha < \kappa$, we take A_α as the α'th copy of R in the canonical direct sum decomposition of $R^{(\kappa)}$. For $\alpha \ge \kappa$, we need some preparation first.

Step II: Take any submodule $R^{(\kappa)} \subseteq N \subseteq P$ such that $N/R^{(\kappa)} \cong Q$. We claim that there is a countable rank submodule $A \subseteq N$ such that $R^{(\kappa)} + A = N$. Consider the pushout of the inclusions $i : R^{(\kappa)} \hookrightarrow N$ and and $j : R^{(\kappa)} \hookrightarrow Q^{(\kappa)}$:

$$\begin{array}{ccccccccc}
0 & \to & R^{(\kappa)} & \xrightarrow{i} & N & \xrightarrow{p \restriction N} & Q & \to & 0 \\
 & & \downarrow j & & \downarrow \subseteq & & \| & & \\
0 & \to & Q^{(\kappa)} & \xrightarrow{\subseteq} & X & \xrightarrow{p} & Q & \to & 0
\end{array}$$

Since $\operatorname{Ext}^1_R(Q, Q^{(\kappa)}) = 0$, the second row splits. Let $k : Q \to X$ be the splitting monomorphism with $pk = \mathrm{id}_Q$. Let $Y = \operatorname{Im}(k)$. Then $X = Q^{(\kappa)} \oplus Y$.

If Q is countably generated, we take any countable subset S of N such that $R^{(\kappa)} + \langle S \rangle = N$ and put $A = \langle S \rangle$.

If Q is not countably generated, then – since R is a valuation domain – there are a regular uncountable cardinal ρ and a set $\{r_\gamma \mid \gamma < \rho\} \subseteq R$ with the following two properties:

(1) $\{r_\gamma^{-1} \mid \gamma < \rho\}$ generates Q as an R-module, and
(2) r_γ is divisible by r_δ, but r_γ does not divide r_δ, for each $\delta < \gamma$.

That is, $(r_\gamma R \mid \gamma < \rho)$ is a strictly descending chain of principal right ideals with zero intersection.

For each $\gamma < \rho$, let $n_\gamma \in N \subseteq X$ be such that $p(n_\gamma) = r_\gamma^{-1}$. Then $n_\gamma \in X$ decomposes as $n_\gamma = q_\gamma + k(r_\gamma^{-1})$ where $q_\gamma \in Q^{(\kappa)} = \operatorname{Ker}(p)$. By property (1), $R^{(\kappa)} + \langle n_\gamma \mid \gamma < \rho \rangle = N$. Since $R^{(\kappa)} \subseteq \operatorname{Ker}(p \restriction N)$, we can w.l.o.g. assume that

$(*)$ all the (finitely many) non-zero components of q_γ in the direct sum $Q^{(\kappa)}$ belong to $Q \setminus R$.

Denote by I_γ ($\subseteq \kappa$) the support of q_γ. By property (2), for each $\delta < \gamma$, there is a (non-invertible) element $r_{\gamma\delta} \in R$ such that $r_{\gamma\delta} \cdot r_\gamma^{-1} = r_\delta^{-1}$, and hence $r_{\gamma\delta} q_\gamma - q_\delta \in \operatorname{Ker}(p \restriction N) = R^{(\kappa)}$. By $(*)$, it follows that $I_\delta \subseteq I_\gamma$.

We claim that there is a finite set $I \subseteq \kappa$ such that $I_\gamma \subseteq I$ for all $\gamma < \rho$. If not, there is a countably infinite set $\{x_n \mid n < \omega\} \subseteq \kappa$ such that for each $n < \omega$ there is $\gamma_n < \rho$ with $x_n \in I_{\gamma_n}$. Since ρ is regular and uncountable, there exists $\gamma < \rho$ such that $\gamma_n < \gamma$ for all $n < \omega$. But then $I_\gamma \supseteq \bigcup_{n<\omega} I_{\gamma_n} \supseteq \{x_n \mid n < \omega\}$ is infinite, a contradiction.

This proves that $n_\gamma \in Q^{(I)} \oplus Y$ for each $\gamma < \rho$. Let $A = \langle n_\gamma \mid \gamma < \rho \rangle$. Then A is a submodule of N of finite rank, and $R^{(\kappa)} + A = N$.

Step III: We enumerate the copies of Q in $Q^{(\lambda)} = P/R^{(\kappa)}$ by ordinals $< \lambda$. Then for each $\tau < \lambda$, there is a unique module N_τ such that $R^{(\kappa)} \subseteq N_\tau \subseteq P$ and $N_\tau/R^{(\kappa)}$ is the τ'th copy of Q in $P/R^{(\kappa)}$. The modules $A_{\kappa+\tau}$ ($\tau < \lambda$) are defined by induction on $\tau < \lambda$ as follows.

First, for $\tau = 0$, we take $N = N_0$, construct A as in Step II for this choice of N, and put $A_\kappa = A$. Then $R^{(\kappa)} + A_\kappa = N_0$.

If $\alpha = \kappa + \tau$ for an ordinal $0 < \tau < \lambda$ then, by induction, we already have an exact sequence $0 \to R^{(\kappa)} \overset{\subseteq}{\to} P_\alpha \to Q^{(\tau)} \to 0$ where $P_\alpha = \sum_{\beta<\alpha} A_\beta$. Moreover, $N_\tau \cap P_\alpha = R^{(\kappa)}$. We take $N = N_\tau$, construct A as in Step II for this choice of N, and put $A_\alpha = A$. Then $R^{(\kappa)} + A_\alpha = N_\tau$, and $P_{\alpha+1} = P_\alpha + N_\tau$. So $P_{\alpha+1}/P_\alpha \cong N/(N \cap P_\alpha) = N/R^{(\kappa)} \cong Q$ and we have the exact sequence $0 \to R^{(\kappa)} \to P_{\alpha+1} \to Q^{(\tau+1)} \to 0$.

Finally, by construction, $P = \bigcup_{\alpha<\sigma} P_\alpha$. □

The following result was proved in [3, Theorem 3.13]:

Lemma 15. *Let R be a valuation domain and M a module of countable rank. Then M is strongly flat if and only if M is free-by-divisible.*

Before characterizing strongly flat modules of any rank over valuation domains, we will apply the rank version of the Generalized Hill Lemma in order to obtain a rank version of Lemma 9:

Lemma 16. *Let R be a commutative domain, κ be an uncountable regular cardinal and $\mathcal{C}$ a set of torsion-free R-modules. Denote by $\mathcal{A}$ the class of all direct summands of the modules M satisfying:*

(†) *there is a $\mathcal{C}$-filtration $(M_\alpha \mid \alpha \leq \sigma)$ of M and a family of modules $(A_\alpha \mid \alpha < \sigma)$ of rank $< \kappa$ such that $M_{\alpha+1} = M_\alpha + A_\alpha$ for each $\alpha < \sigma$.*

Then every module in $\mathcal{A}$ is filtered by modules from $\mathcal{A}$ of rank $< \kappa$.

Proof. The proof is very similar to the one for Lemma 9. Let $K \in \mathcal{A}$; that is, there is a module $M = K \oplus L$ with a $\mathcal{C}$-filtration $(M_\alpha \mid \alpha \leq \sigma)$ and a family of modules $(A_\alpha \mid \alpha < \sigma)$ as above. Denote by $\pi_K : M \to K$ and $\pi_L : M \to L$ the projections.

We will construct a filtration $(N_\alpha \mid \alpha \leq \tau)$ of M such that

(1) $N_\alpha \in \mathcal{F}$,
(2) $N_\alpha = \pi_K(N_\alpha) + \pi_L(N_\alpha)$, and
(3) $N_{\alpha+1}/N_\alpha$ has rank $< \kappa$

for all $\alpha < \tau$; the rest of the proof then follows as in Step II of Lemma 9.

Let $\mathcal{F}$ be a family of submodules of M given by Theorem 7. We will construct the filtration by induction. By definition, $N_0 = 0$ and $N_\beta = \bigcup_{\alpha<\beta} N_\alpha$ for limit ordinals β. Suppose we have constructed $N_\alpha \subsetneqq M$ for some α and let $x \in M \setminus M_\alpha$. Let $A_0 \subseteq M$ be a submodule of rank $< \kappa$ such that $A_0 \in \mathcal{F}$ and $x \in A_0$. Then the module $\pi_K(A_0) + \pi_L(A_0)$ has also rank $< \kappa$, so there is a module $A_1 \in \mathcal{F}$ of rank $< \kappa$ such that $\pi_K(A_0) + \pi_L(A_0) \subseteq A_1$. Iterating this process, we obtain a chain

$$x \in A_0 \subseteq A_1 \subseteq A_2 \subseteq \dots$$

of submodules of M with rank $< \kappa$ such that $\pi_K(A_i) + \pi_L(A_i) \subseteq A_{i+1}$ for $i < \omega$. Put $A = \bigcup_{i<\omega} A_i$. Then clearly A has rank $< \kappa$ and $A = \pi_K(A) + \pi_L(A)$. Hence $N_{\alpha+1} = N_\alpha + A$ has the required properties. □

Now, we can extend Lemma 15 to modules of arbitrary rank, giving a positive answer to the problem of Bazzoni and Salce:

Theorem 17. *Let R be a valuation ring and M be a module. Then M is strongly flat if and only if M is free-by-divisible.*

Proof. The if-part is clear since both R and Q are strongly flat, and strongly flat modules are closed under arbitrary direct sums and extensions.

For the only-if part, let M be strongly flat. By [16, Proposition 2.8], M is a direct summand in a free-by-divisible module P. By Lemma 14, strongly flat modules form a class $\mathcal{A}$ as in Lemma 16 for $\mathcal{C} = \{R, Q\}$ and $\kappa = \aleph_1$. Thus, M is filtered by countable rank strongly flat modules. But such modules are $\{R, Q\}$-filtered by Lemma 15. Hence, M is free-by-divisible by Lemma 13. □

4. Matlis localizations of commutative rings

In this section, R denotes a commutative ring, S a multiplicative subset in R consisting of regular elements (= non-zero-divisors), and Q the localization RS^{-1}.

Q is a *Matlis localization* provided that Q has projective dimension ≤ 1 (as R-module). For example, if R is a domain and $S = R \setminus \{0\}$, then the quotient field $Q = RS^{-1}$ is a Matlis localization if and only if R is a Matlis domain in the sense of [9, IV.§4].

Our goal here is to apply the Generalized Hill Lemma to a simple proof of a characterization of Matlis localizations given in [1].

We will first need some preliminary definitions and results. We start with Hamsher's notion of a restriction, and Griffith's of a $G(\aleph_0)$-family:

A submodule N of a module M is a *restriction* if for each prime (equivalently, maximal) ideal p of R, the localization N_p of N at p satisfies $N_p = 0$ or $N_p = M_p$.

A family $\mathcal{S}$ of submodules of a module M is a *$G(\aleph_0)$-family* provided that $0, M \in \mathcal{S}$, $\mathcal{S}$ is closed under unions of chains, and if $N \in \mathcal{S}$ and X is a countable subset of M then there exists $N' \in \mathcal{S}$ such that $N \cup X \subseteq N'$ and N'/N is countably generated.

Lemma 18. *Let R be a commutative ring, S a multiplicative subset in R consisting of regular elements, and $Q = RS^{-1}$.*

(1) *The set $\mathcal{S}$ of all restrictions of the R-module Q/R is a $G(\aleph_0)$-family of submodules of Q/R.*
(2) *If N is a restriction of Q/R such that $(Q/R)/N$ has projective dimension ≤ 1, then N is a direct summand in Q/R.*

Proof. (1) is proved in [1, p.543], and (2) in [1, Proposition 3.10]. □

Another ingredient is the notion of an (infinitely generated) tilting module. Recall that an R-module T is *tilting* provided that

(T1) T has projective dimension ≤ 1,
(T2) $\mathrm{Ext}^1_R(T, T^{(\kappa)}) = 0$ for all cardinals κ, and
(T3) there is an exact sequence $0 \to R \to T_0 \to T_1 \to 0$ such that T_0 and T_1 are direct summands in (possibly infinite) direct sums of copies of T.

We arrive at the main result of this section:

Theorem 19. *([1, Theorem 1.1]) Let R be a commutative ring, S a multiplicative subset in R consisting of regular elements, and $Q = RS^{-1}$. Then the following conditions are equivalent:*

1. *Q is a Matlis localization.*
2. *$T = Q \oplus Q/R$ is a tilting R-module.*
3. *Q/R decomposes into a direct sum of countably presented R-submodules.*

Proof. Assume (1). We will verify conditions (T1)–(T3) for T. First, the projective dimension of Q, Q/R, and hence of T, is ≤ 1 by the assumption, so (T1) holds. (T3) holds since there is the exact sequence $0 \to R \to Q \to Q/R \to 0$. In order to prove (T2), in view of (T1), it suffices to show that $\mathrm{Ext}^1_R(Q/R, Q^{(\kappa)}) = 0$ for each cardinal κ. However, $\mathrm{Ext}^1_R(Q, Q^{(\kappa)}) \cong \mathrm{Ext}^1_Q(Q, Q^{(\kappa)}) = 0$ since Q is a localization of R. So in order to prove that $\mathrm{Ext}^1_R(Q/R, Q^{(\kappa)}) = 0$, it suffices to show that any $f \in \mathrm{Hom}_R(R, Q^{(\kappa)})$ extends to some $g \in \mathrm{Hom}_R(Q, Q^{(\kappa)}) = \mathrm{Hom}_Q(Q, Q^{(\kappa)})$. But we can simply define $g(q) = f(1)q$ for all $q \in Q$.

Assume (2). Consider the cotorsion pair $(\mathcal{A}, \mathcal{B})$ cogenerated by T. By [2], each module in $\mathcal{A}$ is $\mathcal{A}^{\leq\omega}$-filtered. In particular, this holds for $Q/R \in \mathcal{A}$. Let $\mathcal{F}$ be a family corresponding to a $\mathcal{A}^{\leq\omega}$-filtration of Q/R by Theorem 6. Let $\mathcal{G}$ be the intersection of $\mathcal{F}$ with the $G(\aleph_0)$-family of restrictions of Q/R coming from Lemma 18.1.

Then there is a filtration $(G_\alpha \mid \alpha \leq \sigma)$ of Q/R such that $G_\alpha \in \mathcal{G}$ for all $\alpha \leq \sigma$ and $G_{\alpha+1}/G_\alpha$ is countably presented. In particular, G_α is a restriction of $Q/R = G_\sigma$ such that $(Q/R)/G_\alpha \in \mathcal{A}$, so $(Q/R)/G_\alpha$ has projective dimension ≤ 1. By Lemma 18.2, G_α is a direct summand in Q/R, and hence in $G_{\alpha+1}$, for each $\alpha < \sigma$. This yields a decomposition of Q/R into a direct sum of copies of the countably presented modules $G_{\alpha+1}/G_\alpha$ $(\alpha < \sigma)$.

The implication (3) ⇒ (1) is well known, cf. [1, Proposition 7.1]. □

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