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Noncommutative algebraic geometry based on quantum flag manifolds

Part II.

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Homogeneous coordinate rings

- Let $V \subseteq \mathbb{P}^n_{\mathbb{C}}$ and

$$S(V) = \mathbb{C}[x_0, x_1, \ldots, x_n]/(f \text{ homogeneous}, f|_V \equiv 0)$$

be its homogeneous coordinate ring. Then

$$S(V) = \bigoplus_{n=0}^{\infty} S(V)_n$$

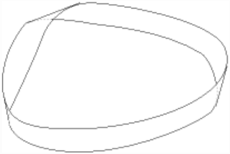
is naturally $\mathbb{Z}$-graded.

- Question: We know that the elements of $S(V)$ are not functions on V. What are they?
- The homogeneous parts $S(V)_n$, $n \geq 0$ are global sections of certain line bundles $\mathscr{L}_n$.
- So every projective variety is the set of zeros of sections in line bundles.

The tautological bundle

- There is an important line bundle over $\mathbb{P}^n_{\mathbb{C}}$, the tautological bundle $\mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(1)$.
- It is dual to $\mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(-1) \subseteq \mathcal{O}^{n+1}_{\mathbb{P}^n_{\mathbb{C}}}$, whose the fiber over $(a_0 : a_1 : \cdots : a_n)$ is the line $\langle a_0, a_1, \ldots, a_n \rangle \subseteq \mathbb{C}^{n+1}$.

$\mathcal{O}_{\mathbb{P}^1_{\mathbb{R}}}(1)$:



- If $\iota\colon V \subseteq \mathbb{P}^n_{\mathbb{C}}$, consider the restricted line bundle $\mathscr{L} := \iota^*\mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(1)$. This is an example of what is called ample (algebraic geometry) or positive (in the context of Kähler manifolds) line bundle.
- Fact: $S(V) \cong \bigoplus_{n=0}^{\infty} \Gamma(V, \mathscr{L}^{\otimes n})$. The homogeneous coordinate ring is the direct sum of global sections of tensor powers of $\mathscr{L}$.

- The twist functor: If $\mathscr{F} \in \operatorname{Qcoh} V$ and $n \in \mathbb{Z}$, put

$$\mathscr{F}(n) := \mathscr{F} \otimes_{\mathcal{O}_V} \mathscr{L}^{\otimes n}$$

(that is, $\mathscr{F}(n)(U) := \mathscr{F}(U) \otimes_{\mathcal{O}_V(U)} \mathscr{L}(U)^{\otimes n}$ on U in an open basis of V).

- The graded module associated to a sheaf: If $\mathscr{F} \in \operatorname{Qcoh} V$, we put

$$\Gamma_*(\mathscr{F}) := \bigoplus_{n\in\mathbb{Z}} \Gamma(\mathscr{F}(n)) = \bigoplus_{n\in\mathbb{Z}} \operatorname{Hom}\big((\mathscr{L}^*)^{\otimes n}, \mathscr{F}\big).$$

Example: $\Gamma_*(\mathcal{O}_V) \cong S(V)$.

Theorem (Serre, 1955)

1. *The functor* $\Gamma_*\colon \operatorname{Qcoh} V \to \operatorname{Mod}^{\mathbb{Z}} S(V)$ *is fully faithful.*
2. Γ_* *has an exact left adjoint* $Q\colon \operatorname{Mod}^{\mathbb{Z}} S(V) \to \operatorname{Qcoh} V$ *which satisfies a universal property:* $\operatorname{Qcoh} V = \operatorname{Mod}^{\mathbb{Z}} S(V)/\operatorname{Mod}^{\mathbb{Z}}_0 S(V)$ *(Serre quotient).*
3. *Similarly,* $\operatorname{coh} V = \operatorname{mod}^{\mathbb{Z}} S(V)/\operatorname{mod}^{\mathbb{Z}}_0 S(V)$.

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- We have $SL_n/P \cong \mathrm{Gr}_{n,r}$, where

$$P = \left(\begin{array}{c|c} P_r & Q \\ \hline 0 & P_{n-r} \end{array}\right),$$

where $P_r \in M_r(\mathbb{C})$ and $P_{n-r} \in M_{n-r}(\mathbb{C})$. The bijection sends the coset UP, $U = (u_{ij})_{i,j=1}^n \in SL_n$ to the linear hull of the first r columns of U.

- If we view $\mathrm{Gr}_{n,r} \subseteq \mathbb{P}_{\mathbb{C}}^{\binom{n}{r}-1}$ via the Plücker embedding, the quotient map $SL_n \twoheadrightarrow \mathrm{Gr}_{n,r}$ sends $U = (u_{ij})_{i,j=1}^n$ to a point with homogeneous coordinates $\sum_\sigma (-1)^{\mathrm{sgn}(\sigma)} u_{\sigma(i_1),1} u_{\sigma(i_2),2} \cdots u_{\sigma(i_r),r}$, one for each sequence $i_1 < i_2 < \cdots < i_r$.
- In terms of coordinate rings, this shows that $S(\mathrm{Gr}_{n,r})$ coincides with the subring

$$\mathbb{C}\left[\sum_\sigma (-1)^{\mathrm{sgn}(\sigma)} u_{\sigma(i_1),1} u_{\sigma(i_2),2} \cdots u_{\sigma(i_r),r} \mid i_1 < i_2 < \cdots < i_r\right] \subseteq \mathbb{C}[SL_n].$$

Quantum Grassmannians

- In terms of coordinate rings, this shows that $S(\mathrm{Gr}_{n,r})$ coincides with the subring

$$\mathbb{C}\left[\sum_{\sigma}(-1)^{\mathrm{sgn}(\sigma)}u_{\sigma(i_1),1}u_{\sigma(i_2),2}\cdots u_{\sigma(i_r),r} \mid i_1 < i_2 < \cdots < i_r\right] \subseteq \mathbb{C}[SL_n].$$

- We have $\mathbb{C}[SL_n] = U(\mathfrak{sl}_n)^\circ$ ($(-)^\circ$ is the Hopf dual).
- Quantum deformation: We can deform $\mathbb{C}[SL_n]$ to $U_q(\mathfrak{sl}_n)^\circ$ and define $S_q[\mathrm{Gr_{n,r}}]$ as the subring

$$\mathbb{C}\left[\sum_{\sigma}(-q)^{\ell(\sigma)}u_{\sigma(i_1),1}u_{\sigma(i_2),2}\cdots u_{\sigma(i_r),r} \mid i_1 < i_2 < \cdots < i_r\right] \subseteq U_q(\mathfrak{sl}_n)^\circ.$$

- Representation-theoretic perspective: Again $S_q[\mathrm{Gr_{n,r}}] \cong \bigoplus_{k=0}^{\infty} V(k\varpi_r)^*$, where $V(k\varpi_r)$ is the corresponding representation of $U_q(\mathfrak{sl}_n)$.

Quantized homogeneous coordinate rings of flags

- One can do the same for all flags (Soibelman 1992, Taft and Towber 1991, Lakshmibai and Reshetikin 1992, Braveman 1994, ...).
- Let $\mathfrak{g}$ be a complex semisimple Lie algebra, G the corresponding complex simply connected algebraic group and P a parabolic subgroup. Then the flag $F = G/P$ is a projective variety and

$$\bigoplus_{k=0}^{\infty} V(k\lambda)^* \cong S_q(F) \subseteq U_q(\mathfrak{g})^\circ,$$

  where λ is the sum of fundamental weights for F and $V(k\lambda)$ are the corresponding finite dimensional representations of $U_q(\mathfrak{g})$.
- One can define a quantization for the category of coherent sheaves: $\mathrm{coh}_q F := \mathrm{mod}^{\mathbb{Z}} S_q(F)/\mathrm{mod}_0^{\mathbb{Z}} S_q(F)$.
- This is an abelian category and we can, for instance, define and study the analogue of the sheaf cohomology as well as other algebraic properties.

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Compact Lie versus algebraic groups

- Aim: Relate the quantized algebraic and differential geometry.
- We have $SU_n \subseteq SL_n$, where
  1. SL_n is a complex affine algebraic group,
  2. SU_n is a real compact Lie group but it is also a real algebraic group!
- Rings of functions in place:
  1. For SL_n we have the complex coordinate ring $\mathbb{C}[SL_n]$,
  2. For SU_n we have the hierarchy

$$C(SU_n) \supseteq C^\infty(SU_n) \supseteq \mathcal{O}(SU_n)$$

     where $\mathcal{O}(SU_n)$ is the ring of polynomial functions $s\colon SU_n \to \mathbb{C}$ of real algebraic varieties.
  3. The magic here: $\mathbb{C}[SL_n] \cong \mathcal{O}(SU_n)$ (via the restriction of $s\colon SL_n \to \mathbb{C}$ to SU_n).

A pocket dictionary: algebraic to differential geometry

- A “cultural” problem:
  1. In differential geometry, a compact complex manifold is a real manifold with an extra structure (flat connection $\overline{\partial}\colon C^\infty(V) \to \Omega^{(0,1)}$).
  2. In algebraic geometry, one usually encounters only polynomial or rational (so holomorphic) functions.
- To relate the two, we need a meeting point of (1) and (2).
- We have $\mathrm{Gr}_{n,r} \cong SL_n/P \cong SU_n/L$, where

$$P = \left(\begin{array}{c|c} P_r & Q \\ \hline 0 & P_{n-r} \end{array}\right) \quad \textit{and} \quad L = P \cap SU_n = \left(\begin{array}{c|c} L_r & 0 \\ \hline 0 & L_{n-r} \end{array}\right).$$

- Now:
  1. The expression $\mathrm{Gr}_{n,r} \cong SL_n/P$ allows to view the Grassmannian as a projective complex algebraic variety.
  2. The expression $\mathrm{Gr}_{n,r} \cong SU_n/L$ allows to view the Grassmannian as a affine real algebraic variety.
- The meeting point: Try to view a complex algebraic variety V as a real algebraic variety with a “complex structure” (a flat connection $\overline{\partial}\colon \mathcal{O}(V) \to \Omega^{(0,1)}$).

Dolbeault dg algebra

- If V is a complex manifold, we have the Dolbeault complex:

$$0 \longrightarrow C^\infty(V) \xrightarrow{\overline{\partial}} \Omega^{(0,1)} \xrightarrow{\overline{\partial}} \Omega^{(0,2)} \xrightarrow{\overline{\partial}} \cdots$$

- We can wedge forms ($\wedge\colon \Omega^{(0,i)} \otimes \Omega^{(0,j)} \longrightarrow \Omega^{(0,i+j)}$). Then $\Omega^{(0,\bullet)} = \bigoplus_i \Omega^{(0,i)}$ is a $\mathbb{Z}$-graded associative algebra over $\mathbb{C}$.
- Moreover, we have the graded Leibniz rule: $\overline{\partial}(\omega_i \wedge \omega_j) = \overline{\partial}(\omega_i) \wedge \omega_j + (-1)^i \omega_i \wedge \overline{\partial}(\omega_j)$ for each $\omega_i \in \Omega^{(0,i)}$ and $\omega_j \in \Omega^{(0,j)}$. In other words, $(\Omega^{(0,\bullet)}(V), \wedge, \overline{\partial})$ is a differential graded (dg) algebra.
- If $V = \mathrm{Gr}_{n,r} = SU_n/L$, then

$$\mathcal{O}(\mathrm{Gr}_{n,r}) \subseteq C^\infty(\mathrm{Gr}_{n,r}) \subseteq C(\mathrm{Gr}_{n,r})$$

  and $\mathcal{O}(\mathrm{Gr}_{n,r})$ is dense with respect to $|| - ||_\infty$.
- Now, the Dolbeault dg algebra for $\mathrm{Gr}_{n,r}$ does restrict to real algebraic sections:

$$0 \longrightarrow \mathcal{O}(\mathrm{Gr}_{n,r}) \xrightarrow{\overline{\partial}} \Omega^{(0,1)}_{\mathrm{alg}} \xrightarrow{\overline{\partial}} \Omega^{(0,2)}_{\mathrm{alg}} \xrightarrow{\overline{\partial}} \cdots$$

The differential calculus of Heckenberger and Kolb

- The Dolbeault dg algebra for $\mathrm{Gr}_{n,r}$ does restrict to real algebraic sections:

$$0 \longrightarrow \mathcal{O}(\mathrm{Gr}_{n,r}) \xrightarrow{\overline{\partial}} \Omega^{(0,1)}_{\mathrm{alg}} \xrightarrow{\overline{\partial}} \Omega^{(0,2)}_{\mathrm{alg}} \xrightarrow{\overline{\partial}} \cdots$$

- and can be quantized:

$$0 \longrightarrow \mathcal{O}_q(\mathrm{Gr}_{n,r}) \xrightarrow{\overline{\partial}} \Omega^{(0,1)}_{q} \xrightarrow{\overline{\partial}} \Omega^{(0,2)}_{q} \xrightarrow{\overline{\partial}} \cdots$$

  ($\big(\Omega_q^{(0,\bullet)}(\mathrm{Gr}_{n,r}), \wedge, \overline{\partial}\big)$ is a dg algebra again).
- If we impose some more natural conditions on $\Omega_q^{(0,\bullet)}(\mathrm{Gr}_{n,r})$, it is unique (Heckenberger and Kolb, 2006)!
- In fact, Heckenberger and Kolb quantized the Dolbeault dg algebra for all compact Hermitian symmetric flags.

The Koszul-Malgrange theorem

Theorem (Koszul and Malgrange, 1958)

Let V be a compact complex manifold. Then there is a bijective correspondence between

1. *holomorphic vector bundles $p\colon E \to V$ and*
2. *smooth complex vector bundles equipped with a flat connection $\nabla_E\colon \Gamma^\infty(E) \to \Gamma^\infty(E) \otimes_{C^\infty(V)} \Omega^{(0,1)}$, where*

$$\Gamma^\infty(E) = \{s\colon V \to E \text{ smooth map} \mid p \circ s = 1_V\}.$$

The holomorphic sections of E are precisely ∇_E.

- By a version of the Serre-Swan theorem, $\Gamma^\infty(E)$ is a finitely generated projective $C^\infty(V)$-module.
- Define quantized algebraic vector bundles over $\mathrm{Gr}_{n,r}$ as flat connections $\nabla\colon P \to P \otimes_{\mathcal{O}_q(\mathrm{Gr}_{n,r})} \Omega_q^{(0,1)}$, where P is a finitely generated projective $\mathcal{O}_q(\mathrm{Gr}_{n,r})$-module.

The first match (quantized alg. vs. diff. geometry)

- Recall: On $\mathrm{Gr}_{n,r} = SU_{n+1}/L$, we have only one reasonable quantized Dolbeault dg algebra $(\Omega_q^{(0,\bullet)}, \wedge, \overline{\partial})$.
- Since $\mathrm{Gr}_{n,r}$ is homogeneous, one can use representation theory of $\mathfrak{l}$ to construct quantum deformations $\mathscr{L}_{n,q}$ of tensor powers $\mathscr{L}^{\otimes n}$ of the tautological bundle $\mathscr{L}$.
- That is, there are finitely generated projective $L_{n,q}$ are finitely generated projective $\mathcal{O}_q(\mathrm{Gr}_{n,r})$-modules and certain flat connections, unique by Ó Buachalla and Mrozinski,

$$\nabla_{\mathscr{L}_{n,q}} \colon L_{n,q} \longrightarrow L_{n,q} \otimes_{\mathcal{O}_q(\mathrm{Gr}_{n,r})} \Omega_q^{(0,1)}.$$

Theorem (Ó Buachalla and Mrozinski, 2017)

For each $n \geq 0$, We have $S_q(\mathrm{Gr}_{n,r})_n \cong \ker \nabla_{\mathscr{L}_{n,q}}$.
So the holomorphic sections of line bundles based on the Heckenberger-Kolb calculus and the Koszul-Malgrange theorem agree with the older “naive” construction of the quantized coordinate ring.