CHARLES UNIVERSITY PRAGUE

faculty of mathematics and physics

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Representability of functors and Gorenstein homological algebra

(joint with Ivo Dell'Ambrogio and Greg Stevenson)

ECI Workshop, Třešť

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- Question: How much do we learn about $\mathscr{T}$ from $\mathscr{M}od\,\mathscr{C}$?

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- Aim: Develop methods to establish the UCT and the representability of functors satisfying the above condition in our setup.

▸ to the setup ▸ to fin.hom.dim

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- Now just apply $\mathscr{T}(-, Y)$ to this triangle. □

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The category $\mathscr{M}od\,\mathscr{C}$ is Gorenstein if

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Examples

Rings, for which $\mathscr{M}od\, R$ is Gorenstein:

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[Bass, ’63]: On the ubiquity of Gorenstein rings.

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$$\begin{aligned} \mathrm{FD}\,\mathscr{C} &= \{X \in \mathscr{M}od\,\mathscr{C} \mid \text{proj. dim. } X < \infty\} \\ &= \{X \in \mathscr{M}od\,\mathscr{C} \mid \text{inj. dim. } X < \infty\}. \end{aligned}$$

▸ to fin.hom.dim.

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- Observation: Given $X\colon \mathscr{C}^{\mathrm{op}} \to \mathscr{A}b$, then $\operatorname{Ext}^1_{\mathscr{T}}(G, X) = 0$ iff

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- The latter is true for any X in the image of $h\colon \mathscr{T} \to \mathscr{M}od\,\mathscr{C}$!

Corollary

If $\mathscr{M}od\,\mathscr{C}$ is Gorenstein and locally coherent, the following are equivalent for $X \in \mathscr{M}od\,\mathscr{C}$:

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- Thus, to answer Q2, it suffices to decide whether we have enough triangles in $\operatorname{add}\mathscr{C}$.

▸ to questions ▸ to representability

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Let $\mathscr{C} = \Sigma\mathscr{C}$ *be a set of compact objects in* $\mathscr{T}$ *such that*

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- *either* $\operatorname{proj.dim.} F \leq 1$ *and* $F \cong hX$ *for some* X *in the localizing class generated by* $\mathscr{C}$,
- *or* $\operatorname{proj.dim.} F = \infty$ *and* F *is not of the form* hX *for any* $X \in \mathscr{T}$.

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- Kasparov's bivariant K-theory KK—a monoidal triangulated category with $\Sigma^2 \cong \mathrm{id}$

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Example

$\mathscr{T} = \mathrm{KK}^{C(p)}$ ($C(p)$ = cyclic group of prime order) contains a 1-Gorenstein and Gorenstein closed subcategory generated by the following quiver