Smashing localization for rings of weak global dimension 1

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Outline

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1. Localization of derived categories
2. Relation to homological epimorphisms
3. A classification for valuation domains

The derived category of a ring

- Object of interest: the derived category $\mathrm{D}(\mathrm{Mod}R) = \mathrm{C}(\mathrm{Mod}R)[\text{quasi-iso}^{-1}]$ of a ring R.
- $\mathrm{D}(\mathrm{Mod}R)$ is triangulated, the suspension functor $\Sigma\colon \mathrm{D}(\mathrm{Mod}R) \to \mathrm{D}(\mathrm{Mod}R)$ shifts complexes

$$X\colon \qquad \cdots \to X^{-1} \to X^0 \to X^1 \to \dots$$

  to the left and changes sings of the differentials.
- $\mathrm{D}(\mathrm{Mod}R)$ is compactly generated. There is a set $\mathcal{C}$ of objects of $\mathrm{D}(\mathrm{Mod}R)$ such that each $C \in \mathcal{C}$ is compact (that is, $\mathrm{Hom}(C, -)\colon \mathrm{D}(\mathrm{Mod}R) \to \mathrm{Ab}$ preserves coproducts) and for each $0 \neq X \in \mathrm{D}(\mathrm{Mod}R)$ there exists $0 \neq f\colon C \to X$ with $C \in \mathcal{C}$. For instance $\mathcal{S} = \{R[n] \mid n \in \mathbb{Z}\}$.
- If R is commutative, $(\mathrm{D}(\mathrm{Mod}R), \otimes^L_R, R)$ is a symmetric monoidal category, where $\otimes^L_R$ denotes the left derived functor of the tensor product. The functor $- \otimes^L_R -$ is exact in each variable.

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- We would like to understand the structure of $\mathrm{D}(\mathrm{Mod}R)$. It is hopeless to classify objects, but we may classify kernels of various triangulated functors.
- We have

$$\left\{\begin{array}{c}\text{Coproduct preserving Verdier localizations}\\ \mathrm{D}(\mathrm{Mod}R)\to\mathrm{D}(\mathrm{Mod}R)/\mathcal{L}\end{array}\right\}$$

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Thomason's classification of finite localizations

Theorem (Thomason, 1997)

Let R be a commutative ring. Then there is a bijection between

1. *compactly generated localizations $L\colon \mathrm{D}(\mathrm{Mod}R) \to \mathrm{D}(\mathrm{Mod}R)$;*
2. *Thomason subsets of $\operatorname{Spec} R$.*

Definition

A subset $U \subseteq \operatorname{Spec} R$ is a Thomason set if U is a union of Zariski closed sets of $\operatorname{Spec} R$ with quasi-compact complements.

Example

Let R be a valuation domain with $\operatorname{Spec} R\colon 0 = \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{m}$. Then the Thomason sets are simply upper sets with respect to $\subseteq$. The corresponding localization for $\{\mathfrak{p}_j, \mathfrak{p}_{j+1}, \dots, \mathfrak{p}_n\} \subseteq \operatorname{Spec} R$ with $j \geq 1$ is

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A functor $L\colon \mathrm{D}(\mathrm{Mod}R) \to \mathrm{D}(\mathrm{Mod}R)$ is a smashing localization if it preserves coproducts.

Facts

1. If R is commutative, this is equivalent to $L \cong S \otimes_R^L -$ for some complex S.
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Remark

The term smashing comes from the stable homotopy category, where the role of $\otimes_R^L$ is taken by the smash product $\wedge$.

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- A ring homomorphism $f : R \to S$ is an epimorphism if and only if $\mu : S \otimes_R S \to S$ is an isomorphism.
- For a homological epimorphism we further require that $\mathrm{Tor}_i^R(S, S) = 0$ for all $i \geq 1$ (due to Geigle and Lenzing). We can write that compactly as $S \otimes_R^L S \cong S$.
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Outline

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Corollary

Suppose that R is a valuation domain, $f : R \to S$ is a homological epimorphism (so that w.gl.dim. $S \leq 1$) and $\mathfrak{m} \in \operatorname{mSpec} S$.

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Example

- Let R be a valuation domain with $\operatorname{Spec} R = \{0, \mathfrak{m}\}$ and $\mathfrak{m} = \mathfrak{m}^2$. For instance

$$R = \left\{ \sum_{i=0}^{\infty} a_{\frac{i}{\ell}} x^{\frac{i}{\ell}} \right\}$$

  (Puiseux series ring).
- Let Q be the quotient field and $k = R/\mathfrak{m}$ the residue field.
- Then there are 5 homological epimorphisms:
  1. $R \to 0$,
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- Let R be a valuation domain with Spec $R = \{0, \mathfrak{m}\}$ and $\mathfrak{m} = \mathfrak{m}^2$. For instance

$$R = \left\{ \sum_{i=0}^{\infty} a_{\frac{i}{\ell}} x^{\frac{i}{\ell}} \right\}$$

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A hint for the telescope conjecture?

- There are valuation domains R with Zariski spectrum

$$\operatorname{Spec} R: \quad 0 \subsetneqq \mathfrak{p}_1 \subsetneqq \mathfrak{p}_2 \subsetneqq \mathfrak{p}_3 \subsetneqq \cdots \subsetneqq \mathfrak{m}.$$

For instance those with the value group $(\mathbb{Z}^{(\omega)}, \leq_{\text{lex}})$.

- We do not know which $\mathfrak{p}_i$'s are idempotent, but $\mathfrak{m}$ must be such! In particular, there certainly are smashing localizations of $\mathrm{D}(\mathrm{Mod}\, R)$ which are not compactly generated (the telescope conjecture fails for $\mathrm{D}(\mathrm{Mod}\, R)$).
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